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Open problems in deformation theory of discontinuous groups acting on homogeneous spaces

Ali Baklouti and Imed Kedim
Sfax University, Department of Mathematics,
Faculty of Sciences at Sfax, Route de Soukra, 3000 Sfax, Tunisia.
University of Carthage, Department of Mathematics,
Faculty of Sciences of Bizerte, 7021 Jarzouna, Tunisia.
e-mail: Ali.Baklouti@fss.rnu.tn
e-mail: Imed.Kedim@fsg.rnu.tn

Abstract

Let G be a Lie group and H a connected Lie subgroup of G. Given any discontinuous subgroup Γ for the homogeneous space $\mathscr{M} = G/H$ and a discrete subgroup Γ' of G isomorphic to Γ, the action of Γ' may fail to be properly discontinuous on $\mathscr{M}$ (for instance, in the case where H is not compact). To understand this issue, we consider the set $\mathscr{R}(\Gamma, G, H)$ of deformation parameters consisting of all injective homomorphisms of Γ in G, which transform Γ to a discontinuous subgroup of $\mathscr{M}$, so that the related Clifford-Klein forms become manifolds. The group G acts on $\mathscr{R}(\Gamma, G, H)$ by conjugation and the subsequent quotient space $\mathscr{T}(\Gamma, G, H)$ is called the deformation space of the action of Γ on $\mathscr{M}$. The study of these spaces from topological and geometrical points of view, raises many problems of different nature. The main hurdles is to understand the structures of these spaces and some of their topological features. This note aims to record some recent results in the setup of solvable Lie groups and present some open problems in this framework.

Keywords: *Exponential Lie groups, discontinuous subgroups, deformation space, rigidity, stability.*

1 Introduction

Let G be a Lie group, H a closed subgroup of G and Γ a finitely generated discrete subgroup of G. The group Γ does not in general act properly discontinuously on G/H when H is not compact. The problem of deformation consists in seeking how to deform Γ by means of homomorphisms from Γ to G (thus to consider the set $\mathrm{Hom}(\Gamma, G)$ of all these homomorphisms) in a way such that the deformed discrete subgroup acts properly on G/H. The problem of describing deformations was first advocated by T. Kobayashi in [21] for the general non-Riemannian setting and precisely determines as proposed in [5], the set of deformation parameters that allow Γ to deform in a way to guarantee the proper discontinuity on G/H. The following parameter space

$$\mathscr{R}(\Gamma, G, H) := \left\{ \varphi \in \mathrm{Hom}(\Gamma, G) \;\middle|\; \begin{array}{l} \varphi \text{ is injective, } \varphi(\Gamma) \text{ discrete and} \\ \text{acts properly and fixed point} \\ \text{freely on } G/H \end{array} \right\} \tag{1}$$

(endowed with the point wise convergence topology), rather than $\mathrm{Hom}(\Gamma, G)$, plays a crucial role in these problems. In order to be precise on parameters, our main goal is to investigate the deformation space $\mathscr{T}(\Gamma, G, H)$ which is merely the quotient space of the parameters space given above through the equivalence relation arising inner automorphisms.

Getting comprehensive information about the structure of the deformation space helps to understand the local geometric structures as many examples reveal:

1. Let M_g be a Reimann surface of genus $g \geq 2$. For $G = PSL_2(\mathbb{R})$, $H = SO_2$ and $\Gamma = \pi_1(M_g)$, G/H is the Poincaré disk, $M_g = \Gamma\backslash G/H$ and $\mathscr{T}(\Gamma, G, H)$ is the *Teichmüller space of* M_g.
2. When $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $G = SL_2(\mathbb{K}) \times SL_2(\mathbb{K})$, $H = diag(G)$, Γ is a lattice in $SL_2(\mathbb{K})$, $\mathscr{T}(\Gamma, G, H)$ is identified to the deformation space of the *Lorentz structures* in the real case and to the deformation space of the *complex structures on a 3-dimensional manifolds* otherwise.

We refer the reader to the following expository papers ([13, 22, 26] and to some references therein) where many settings have been considered. We also focus attention on the study of the concepts of stability and local rigidity of discontinuous groups for homogeneous spaces. These concepts, which basically repose on the topological features of the deformation spaces, provide accurate information on the geometric nature of the subsequent Clifford-Klein forms. The study of rigidity problems started by a result of Selberg and Weil [29](and generalized later by T. Kobayashi) proving that for an irreducible Riemannian symmetric space G/H of dimension ≥ 3 with a compact subgroup H and Γ a uniform lattice of G/H, there does not exist any essential deformation of Γ. This result claims that the deformation space is discrete in this context and can be regarded as the original model for various kinds of rigidity theorems in Riemannian geometry. An analogous result in the framework of exponential Lie groups (the non-Riemannian case) is also obtained in [3] stating that there is no essential deformation outside the setup of the group $ax + b$ of affine transformations of the real line.

Many other results on rigidity will be recorded in section 4, and some related problems are posed, namely the rigidity conjecture posed in [6] in the setting of nilpotent Lie groups.

The notion of stability is introduced in [23] and discusses the fact that the properness of the action of $\varphi(\Gamma)$ on G/H is preserved in a small neighborhood of φ in $\mathscr{R}(\Gamma, G, H)$. Stability is studied in many cases by now, and the notion of stable subgroups is introduced. Some related open problems are therefore presented. A new purely geometric motivation to the study of deformation is also studied. This is subject of section 5.

The next section is devoted to fix some notation, to define different kinds of group actions and give some links between them. Section 3 aims to present the deformation space of $(G, H)-$structures and to relate it to the deformation space $\mathscr{T}(\Gamma, G, H)$. Some open problems are also exposed.

2 Discontinuous actions

We begin this section with fixing some notation, terminologies and recording some basic facts about deformations. The readers could consult the references [17, 18, 20, 21, 22] and some references therein for broader information about the subject. Concerning the entire subject, we strongly recommend the papers [17] and [22].

Let X be a locally compact space and K a locally compact topological group. The continuous action of the group K on X is said:

(1) To be *proper* if, for each compact subset $S \subset X$ the set $K_S = \{k \in K : k \cdot S \cap S \neq \emptyset\}$ is compact.

(2) To be *fixed point free* (or *free*) if, for each $x \in X$, the isotropy group $K_x = \{k \in K : \ k \cdot x = x\}$ is trivial.

(3) To be *properly discontinuous* if, K is discrete and the action of K on X is proper and free.

(4) To satisfy the *compact intersection property* (CI for abbreviation), if for every $x \in X$ the isotropy group $K_x = \{k \in K : k \cdot x = x\}$ is compact.

We focus attention in the setup where G is a locally compact group and H and K are closed subgroups of G. Then K acts on the homogeneous space $X = G/H$ by a left multiplication. In this case, it is well known that the action of K on X is proper if and only if $SHS^{-1} \cap K$ is compact for any compact set S in G. Likewise the action of K on X is free if for every $g \in G$, $K \cap gHg^{-1} = \{e\}$.

In this context, the subgroup K is said to be a *discontinuous group* for the homogeneous space X, if the action of K is properly discontinuous.

Both proper and free actions induce the (CI) property and if K has no non-trivial compact subgroups then the (CI) property is equivalent to the fact that the action is free. It is also clear that if K is compact the action is always (CI) and proper, and if H is compact, then the action of K is proper. In the situation where the triple (K, G, H) is reductive, Kobayashi [20] proved that K acts properly on G/H if and only if the triple (K, G, H) is (CI). In [24] Lipsmann conjectured that if G is nilpotent, connected and simply connected, then the proper action of K on G/H is equivalent to the (CI) property. In the setup of solvable Lie groups, such an equivalence was shown to hold in the following situations:

1. If G is a nilpotent Lie group of step $N \leq 3$ an affirmative answer to this conjecture was given separately by Nasrine [25] (for $N = 2$), Baklouti and Khlif [11] ($N \leq 3$) and Yoshino [31], ($N = 3$).
2. When G is a connected simply connected solvable Lie group, H and K are connected Lie subgroups of G such that one of them is normal or maximal (cf. [11]).
3. When G is a special connected simply connected solvable Lie group, which means that G admits an abelian normal subgroup of codimension one (cf. [11]).

In [30], Yoshino presented a counter example for a family of nilpotent Lie groups where $N = 4$. In a recent work, Nasrine constructed a family of nilpotent triples (K, G, H) for which, the (CI) property is equivalent to the proper action.

We now pose the following questions:

Question 2.1 *(T. Kobayashi). Characterize the triples (K, G, H) in a connected solvable Lie group, for which the proper action is equivalent to the (CI) property.*

Question 2.2 *Give a simple criterion for the action of K on G/H to be proper in the context of solvable Lie groups.*

When the action of K on G/H is proper, the double cosets space $K\backslash G/H$ is a Hausdorff space. A good motivation to investigate these questions is the study of Clifford-Klein forms. For a given discontinuous subgroup Γ for the homogeneous space $X = G/H$, the quotient space $\Gamma\backslash X$ is said to be a *Clifford-Klein form* for the homogeneous space X. Any Clifford-Klein form is endowed with a smooth manifold structure for which the quotient map $\pi : X \to \Gamma\backslash X$ turns out to be an open covering and particularly a local diffeomorphism. On the other hand, any Clifford-Klein form $\Gamma\backslash X$ inherits any G-invariant local geometric structure (e.g. complex structures, pseudo-Riemannian structures, conformal structures, affine structures, symplectic structures,...) from the homogeneous space X through the covering map π. In fact, in many cases, a smooth manifold endowed with a local structure turns out to be a Clifford-Klein form. A standard reference for more details on this subject is [19].

In [10], the notion of weak and finite proper action was singled out. It has been shown that these notions are equivalent to the fact that K acts freely on G/H when G is solvable.

Definition 2.3 *Let K be a locally compact group and X be a K-locally compact space. We say that the action of K on X is:*

(1) weakly proper if for every compact set S in X, the set $K_{x,S} = \{k \in K : k \cdot x \in S\}$ is compact for every $x \in X$,

(2) finitely proper if for every finite set S in X, the set $K_S = \{k \in K : k \cdot S \cap S \neq \emptyset\}$ is finite.

We now pose the following question:

Question 2.4 *Give a simple criterion for the weak proper action of K on G/H, for a given Lie group G.*

In [10], the following result is obtained and gives a complete answer to Question 2.4 in the solvable setting:

Theorem 2.5 *Let G be a connected simply connected solvable Lie group, H and K be closed connected subgroups of G. Then the following assertions are equivalent*

(i) The action of K on G/H is finitely proper.

(ii) The action of K on G/H is weakly proper.

(iii) The action of K on G/H is free.

(iv) The triple (G, H, K) has the (CI) property.

3 The Deformation space of a discontinuous subgroup

Let Γ be a discrete subgroup of a Lie group G and $\mathscr{R}(\Gamma, G, H)$ the parameter space of the discontinuous actions of Γ on G/H defined as in (1). The Lie group G acts on $\mathrm{Hom}(\Gamma, G)$ by

$$(g \cdot \varphi)(\gamma) = g\varphi(\gamma)g^{-1}, g \in G, \gamma \in \Gamma$$

and the parameter space is a $G-$stable subset. According to this definition and as earlier, for each $\varphi \in \mathscr{R}(\Gamma, G, H)$, the space $\varphi(\Gamma)\backslash G/H$ is a Clifford-Klein form which is a Hausdorff topological space and even equipped with a structure of a manifold for which, the quotient canonical map is an open covering. Let now $\varphi \in \mathscr{R}(\Gamma, G, H)$ and $g \in G$, we consider the element $\varphi^g := g^{-1} \cdot \varphi \cdot g$ of $\mathrm{Hom}(\Gamma, G)$ defined by $\varphi^g(\gamma) = g^{-1}\varphi(\gamma)g, \gamma \in \Gamma$. It is then clear that the element $\varphi^g \in \mathscr{R}(\Gamma, G, H)$ and that the map:

$$\varphi(\Gamma)\backslash G/H \longrightarrow \varphi^g(\Gamma)\backslash G/H, \quad \varphi(\Gamma)xH \mapsto \varphi^g(\Gamma)g^{-1}xH$$

is a natural diffeomorphism. We consider then the space of orbits:

$$\mathscr{T}(\Gamma, G, H) = \mathscr{R}(\Gamma, G, H)/G$$

instead of $\mathscr{R}(\Gamma, G, H)$ in order to avoid the unessential part of deformations arising inner automorphisms and to be quite precise on parameters. We call the set $\mathscr{T}(\Gamma, G, H)$ the deformation space of the action of Γ on the homogeneous space G/H.

We now introduce the Clifford-Klein space

$$CK(\Gamma, G, H) = \left\{ \Gamma'\backslash G/H \;\middle|\; \begin{array}{l} \Gamma' \text{ is isomorphic to}\Gamma \\ \Gamma' \text{ is discontinuous for } G/H \end{array} \right\}.$$

By a deformation of the Clifford-Klein form $\Gamma\backslash G/H$, we mean any element of the related Clifford-Klein space $CK(\Gamma, G, H)$.

There is a natural surjective map $\Psi : \mathscr{R}(\Gamma, G, H) \to CK(\Gamma, G, H)$, $\varphi \mapsto \varphi(\Gamma)\backslash G/H$. Now the group G itself acts on $\mathscr{R}(\Gamma, G, H)$ and $CK(\Gamma, G, H)$ and the map Ψ is G-equivariant. In sum, $\Psi(\mathscr{T}(\Gamma, G, H)) = CK(\Gamma, G, H)/G$ and the deformation space determines therefore all possible deformations of the related Clifford-Klein form modulo the $G-$action. Let $\Gamma_i,\ i = 1, 2$ be discontinuous subgroups for G/H, then $\Gamma_1\backslash G/H = \Gamma_2\backslash G/H$ if and only if $\Gamma_1 gHg^{-1} = \Gamma_2 gHg^{-1}$ for all $g \in G$. We define an equivalence relation $\curlywedge$ on $\mathscr{R}(\Gamma, G, H)$ as follows: $\varphi \curlywedge \varphi'$ if and only if $\varphi(\Gamma)gHg^{-1} = \varphi'(\Gamma)gHg^{-1}$ for any $g \in G$. We then introduce the space $\widehat{R}(\Gamma, G, H)$ as being the quotient subsequent space. Clearly the map Ψ factors to a bijection from $\widehat{R}(\Gamma, G, H)$ to $CK(\Gamma, G, H)$ and to a topological homeomorphism when these spaces are endowed with adequate topologies. Now the action of G on $\mathscr{R}(\Gamma, G, H)$ induces an associated action on $\widehat{R}(\Gamma, G, H)$ which commutes with the relation $\curlywedge$. The space $\widehat{\mathscr{T}}(\Gamma, G, H) := \widehat{R}(\Gamma, G, H)/G$ is called the *refined deformation space* and is identified to the space $CK(\Gamma, G, H)/G$.

Our interest to these spaces comes from the deformation theory of the (G, X)-structures, where G is a Lie group and X is a homogeneous space. Let M be a smooth manifold such that $\dim X = \dim M$. A (G, X)-atlas on M is a collection $(U_\alpha, \phi_\alpha)_{\alpha \in I}$, where $\{U_\alpha, \alpha \in I\}$ is an open covering of M and $\{\phi_\alpha : U_\alpha \to X, , \alpha \in I\}$ is a family of local coordinates charts such that, on a connected component C of $U_\alpha \cap U_\beta$, there exists $g_{C,\alpha,\beta} \in G$ satisfying

$$g_{C,\alpha,\beta} \circ \phi_\alpha = \phi_\beta.$$

A (G, X)-structure on M is a maximal (G, X)-atlas on M and a (G, X)-manifold is a manifold endowed with a (G, X)-structure. Let Σ be a smooth manifold, a marked (G, X)-structure on Σ is a pair (M, f) where M is a (G, X)-manifold and $f : \Sigma \to M$ is a diffeomorphism. Let $D_{(G,X)}(\Sigma)$ be the space of the all marked (G, X)-structures on Σ. The group $\mathrm{Diff}_0(\Sigma)$ (the subgroup of the group of diffeomorphisms of Σ isotopic to the identity) acts on $D_{(G,X)}(\Sigma)$ through the law:

$$\psi \star (M, f) = (M, f \circ \psi^{-1}), \psi \in \mathrm{Diff}_0(\Sigma).$$

The deformation space of the (G, X)-structures on Σ is the quotient space

$$\mathrm{Def}_{(G,X)}(\Sigma) = D_{(G,X)}(\Sigma)/\mathrm{Diff}_0(\Sigma).$$

Assume Σ is compact. By the deformation Theorem of Thurston, the *holonomy* map is a local homeomorphism between the deformation space of marked (G, X)-structures on Σ and the quotient space $\mathrm{Hom}(\pi_1(\Sigma), G)/G$, (cf. [14]). If Γ is a discontinuous subgroup for X, the Clifford-Klein form $\Gamma\backslash X$ is a (G, X)-manifold. If there is a diffeomorphism $f : \Sigma \to \Gamma\backslash X$, then the marked (G, X)-structure $(\Gamma\backslash X, f)$ is said to be complete. The set $D^c_{(G,X)}(\Sigma)$ of the complete (G, X)-structures on Σ, is invariant under the action of $\mathrm{Diff}_0(\Sigma)$. The deformation space of complete (G, X)-structures on Σ is defined as

$$\mathrm{Def}^c_{(G,X)}(\Sigma) = D^c_{(G,X)}(\Sigma)/\mathrm{Diff}_0(\Sigma).$$

Then the deformation space $\mathscr{T}(\Gamma, G, H)$ of the discontinuous actions of Γ on $X = G/H$ contains the image of $\mathrm{Def}^c_{(G,X)}(\Sigma)$ by the holonomy map. Furthermore, if all the forms $\varphi(\Gamma)\backslash G/H$ are diffeomorphic for $\varphi \in \mathscr{R}(\Gamma, G, H)$, then the deformation space $\mathscr{T}(\Gamma, G, H)$ coincides with the image of $\mathrm{Def}^c_{(G,X)}(\Sigma)$.

Any information concerning the spaces $\mathrm{Hom}(\Gamma, G)$, $\mathscr{R}(\Gamma, G, H)$, $\mathscr{T}(\Gamma, G, H)$ and $\mathrm{Hom}(\Gamma, G)/G$ may help to understand the properties of $\mathrm{Def}^c_{(G,X)}(\Sigma)$ and $\mathrm{Def}_{(G,X)}(\Gamma\backslash X)$. We are therefore interested to the study of the topological, geometric, algebraic, or others, local or global properties of the aforementioned spaces. It is well know that the deformation space may enjoy with several pathological phenomena. The analytic variety $\mathrm{Hom}(\Gamma, G)$ may have some singularities and there is no clear raison, to say that the parameter space $\mathscr{R}(\Gamma, G, H)$ is an analytic or algebraic or smooth manifold. It has been shown in [5] and [4] that the set $\mathrm{Hom}^0(\Gamma, G)/G$, (where $\mathrm{Hom}^0(\Gamma, G)$ designates the set of injective homomorphisms) is endowed with a smooth manifold structure in some nilpotent contexts. In general, the action of G on the space $\mathrm{Hom}(\Gamma, G)$ is not proper, then the quotient spaces $\mathrm{Hom}(\Gamma, G)/G$ and $\mathscr{T}(\Gamma, G, H)$ may fail to be Hausdorff spaces. It is therefore natural to pose the following questions:

Question 3.1 *When is the deformation space $\mathscr{T}(\Gamma, G, H)$ a Hausdorff space?*

Question 3.2 *When is the deformation space $\mathscr{T}(\Gamma, G, H)$ a smooth manifold?*

These questions have been investigated by many authors and the program of the classification of these spaces, in all generality was initiated by T. Kobayashi in [22].

We now give some answers to Questions 3.1 and 3.2 in some contexts of connected simply connected exponential Lie groups. To do so, we need an algebraic interpretation of both the deformation and parameter spaces. Let Γ be a discrete subgroup of G. Let L be the syndetic hull of Γ which is the smallest (and hence the unique) connected Lie subgroup of G which contains Γ cocompactly (see [7]). Recall that the Lie subalgebra $\mathfrak{l}$ of L is the real span of the lattice $\log \Gamma$, which is generated by $\{\log \gamma_1, \dots, \log \gamma_k\}$ where $\{\gamma_1, \dots, \gamma_k\}$ is a set of generators of Γ. The group G also acts on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ by:

$$g \cdot \psi = \mathrm{Ad}_g \circ \psi. \tag{2}$$

Our first observation is that the parameter space only depends on the structure of the syndetic hull of Γ when the basis group G is completely solvable. Recall that any continuous homomorphism of a connected Lie groups is smooth and its derivative is a homomorphism of Lie algebras. We consider the smooth map $d : \mathrm{Hom}_c(L, G) \longrightarrow \mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$, $\varphi \mapsto d\varphi_{|e}$ where $\mathfrak{l}$ is the Lie algebras of L. In the case of exponential Lie groups $d\varphi_{|e}(X) = \log \circ \varphi \circ \exp(X)$ for any $X \in \mathfrak{g}$. The group G acts on the spaces $\mathrm{Hom}(\Gamma, G)$, $\mathrm{Hom}(L, G)$ and $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ respectively through the following laws:

$$(g \cdot \varphi)(\gamma) = g\varphi(\gamma)g^{-1} : \ \gamma \in L, \ \varphi \in \mathrm{Hom}(L, G), g \in G$$

$$g \cdot \psi = \mathrm{Ad}_g \circ \psi, \ \psi \in \mathrm{Hom}(\mathfrak{l}, \mathfrak{g}), g \in G.$$

The following useful result was originated in [23] and obtained in [7].

Theorem 3.3 *Let $G = \exp \mathfrak{g}$ be an exponential solvable Lie group, $H = \exp \mathfrak{h}$ a closed connected subgroup of G, Γ a discontinuous abelian subgroup for* *the homogeneous space G/H and $L = \exp \mathfrak{l}$ its syndetic hull. Then, up to a homeomorphism, the parameter space $\mathscr{R}(\Gamma, G, H)$ is given by:*

$$\mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \left\{ \psi \in Hom(\mathfrak{l}, \mathfrak{g}) \;\middle|\; \begin{array}{l} \dim \psi(\mathfrak{l}) = \dim \mathfrak{l}, \\ \exp(\psi(\mathfrak{l})) \text{ acts properly on } G/H \end{array} \right\}.$$

The deformation space $\mathscr{T}(\Gamma, G, H)$ is likewise homeomorphic to the space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h}) = \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})/Ad$, where the action Ad of G is given as in (2). Furthermore, when G is completely solvable, the assumption on Γ to be abelian can be removed.

3.1 Case of Heisenberg groups.

Let $\mathfrak{g} := \mathfrak{h}_{2n+1}$ designate the Heisenberg Lie algebra of dimension $2n + 1$ and $G := H_{2n+1}$ the corresponding Lie group. $\mathfrak{g}$ can be defined as a real vector space endowed with a skew-symmetric bilinear form b of rank $2n$ and a fixed generator Z belonging to the kernel of b. The center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is then the kernel of b and it is the one dimensional subspace $[\mathfrak{g}, \mathfrak{g}]$. For any $X, Y \in \mathfrak{g}$, the Lie bracket is given by

$$[X, Y] = b(X, Y)Z.$$

The following theorem provides a necessary and sufficient condition for the deformation space to be a smooth manifold.

Theorem 3.4 *(cf.[5]). Let $H = \exp(\mathfrak{h})$ be a connected subgroup of the Heisenberg group $G = \exp(\mathfrak{g})$ and Γ a discontinuous subgroup of G for the homogeneous space G/H with a syndetic hull $L = \exp(\mathfrak{l})$. Then the following assertions are equivalent:*

1. *The space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is equipped with a smooth manifold structure.*
2. *The space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ is a Hausdorff space.*
3. *$\dim G \cdot \psi$ is constant for any $\psi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$.*
4. *$\mathfrak{z}(\mathfrak{g}) \subset \psi(\mathfrak{l}) + \mathfrak{h}$ for any $\psi \in \mathscr{R}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$.*

More generally, the space $\mathscr{T}(\mathfrak{l}, \mathfrak{g}, \mathfrak{h})$ admits a smooth manifold as its dense open subset whose pre-image consists of topologically stable and maximal dimensional orbit points.

Definition 3.5 *Let $\mathfrak{g}$ be a Lie algebra. A maximal abelian subalgebra of $\mathfrak{g}$ is an abelian subalgebra of $\mathfrak{g}$ of maximal dimension. Maximal subalgebras are not unique and contain obviously the center of $\mathfrak{g}$.*

When G is a connected simply connected two step nilpotent Lie group, the following result is obtained in [2].

Theorem 3.6 *Let $\mathfrak{g}$ be a two-step nilpotent Lie algebra, if one of the following holds:*

1. All G-orbits in $\mathscr{R}(\Gamma, G, H)$ have a common dimension

2. $\mathfrak{l}$ is a maximal abelian subalgebra of $\mathfrak{g}$,

then the deformation space $\mathscr{T}(\Gamma, G, H)$ is a Hausdorff space.

3.2 Case of the $n-$step Threadlike groups.

Let now $\Gamma \subset G$ be a discontinuous subgroup acting on a *threadlike* homogeneous space G/H. Threadlike means here that the Lie algebra $\mathfrak{g}$ of the $n-$step basis group G admits a stratified basis $\mathscr{B} = \{X, Y_1, \ldots, Y_n\}$ with non-trivial Lie brackets:

$$[X, Y_i] = Y_{i+1}, \ i \in \{1, \ldots, n-1\}. \tag{3}$$

The following result concerning the class of threadlike Lie groups is is proved in [4]:

Theorem 3.7 *Let G be a threadlike Lie group, H a connected Lie subgroup of G and Γ a non-abelian discontinuous subgroup for G/H. Then the parameter space $\mathscr{R}(\Gamma, G, H)$ is semi-algebraic smooth manifold of dimension $n+k$ if $k > 3$ and $n+4$ otherwise. Furthermore, we have:*

1. The deformation space $\mathscr{T}(\Gamma, G, H)$ is a Hausdorff space.

2. For $k > 3$, $\mathscr{T}(\Gamma, G, H)$ is endowed with a smooth manifold structure.

3. For $k = 3$, $\mathscr{T}(\Gamma, G, H)$ is a disjoint union of an open dense smooth manifold and a closed smooth manifold.

More significantly, the phenomenon of Hausdorffness of the deformation space is strongly linked to the feature of adjoint orbits of the basis group G on $\mathscr{R}(\Gamma, G, H)$, specifically to their dimensions. We have (cf. [4]):

Theorem 3.8 *Let G be a threadlike Lie group, H a closed connected subgroup of G and Γ a discontinuous subgroup for G/H. If G acts on the parameter space $\mathscr{R}(\Gamma, G, H)$ with constant dimension orbits, then the deformation space $\mathscr{T}(\Gamma, G, H)$ is a Hausdorff space. When the Clifford-Klein form $\Gamma\backslash G/H$ is compact, this implication becomes an equivalence.*

4 On the concept of rigidity

4.1 The terminology of rigidity

We keep the same notations and assumptions. A. Weil [29] introduced the notion of local rigidity of homomorphisms in the case where the subgroup H was compact. T. Kobayashi [18] generalized it in the case where H is not compact. For non-Riemannian setting G/H with H non-compact, the local rigidity does not hold in general. In the reductive case, T. Kobayashi first proved in [16] that local rigidity may fail even for irreducible symmetric space of high dimensions. For non-compact setting, the local rigidity does not hold in general in the non-Riemannian case and has been studied in [7, 17, 22, 23]. We briefly recall here some details. For a comprehensible information, we refer the readers to the references [7, 15, 16, 17, 18, 20, 21, 22, 23]. For $\varphi \in \mathscr{R}(\Gamma, G, H)$, the discontinuous subgroup $\varphi(\Gamma)$ for the homogeneous space G/H is said to be *locally rigid* (resp. *rigid*) ([18]) as a discontinuous group of G/H if the orbit of φ under the inner conjugation is open in $\mathscr{R}(\Gamma, G, H)$ (resp. in $\mathrm{Hom}(\Gamma, G)$). This means equivalently that any point sufficiently close to φ should be conjugate to φ under an inner automorphism of G. So, the homomorphisms which are locally rigid are those which correspond to those which are open points in the deformation space $\mathscr{T}(\Gamma, G, H)$. When every point in $\mathscr{R}(\Gamma, G, H)$ is locally rigid, the deformation space turns out to be discrete and the Clifford-Klein form $\Gamma\backslash G/H$ does not admit *continuous deformations*. If a given $\varphi \in \mathscr{R}(\Gamma, G, H)$ is not locally rigid, it admits continuous deformations and the related Clifford-Klein form is continuously deformable.

In the context of connected simply connected nilpotent Lie groups, the following Conjecture was substantiated in [6]:

Conjecture 4.1 *Let G be a connected simply connected nilpotent Lie group, H a connected subgroup of G and Γ a non-trivial discontinuous subgroup for G/H. Then, the local rigidity globally fails to hold on the parameter space.*

A positive solution to conjecture 4.1, has been given in the following settings:

1. $\Gamma \simeq \mathbb{Z}^k$ acting properly discontinuously on $G/H \simeq \mathbb{R}^{k+1}$ by affine transformations. (cf [23]).
2. G is the Heisenberg groups, (cf. [9] and [7]).
3. G is treadlike, (cf. [8] and [4]).
4. Γ is abelian, (cf. [3]).
5. G is two-step, (cf. [2]).
6. The Lie algebra $\mathfrak{l}$ of the syndetic hull of Γ is not characteristically nilpotent. This is the case for instance when $\mathfrak{l}$ is a graded algebra. And also the case $\dim G \leq 7$, (cf. [1]).

We now focus on the exponential case. Let first $\mathfrak{g} = \mathrm{Aff}(\mathbb{R}) := \mathbb{R}\text{-span}(X, Y)$ be the Lie algebra of the affine group of the real line, $ax + b$ say, with the Lie bracket $[X, Y] = Y$. For $\mathfrak{h} = \mathbb{R}X$ a maximal subalgebra of $\mathfrak{g}$ and Γ any discontinuous subgroup for $\exp(\mathfrak{g})/\exp(\mathfrak{h})$, the local rigidity property holds. Indeed, if Γ is non-trivial, it is isomorphic to $\exp(\mathbb{Z}Y)$. The corresponding parameter space is then homeomorphic to $\mathbb{R}^{\times}Y$. For $\varphi = aY \in \mathscr{R}(\Gamma, G, H)$ with $a \in \mathbb{R}^{\times}$, we have

$$G \cdot \varphi = \{ae^{b}Y,\ b \in \mathbb{R}\}.$$

This means that $\mathscr{R}(\Gamma, G, H)$ only admits two open orbits.

In ([3], Theorem 5), we have proved the following result, referred to as the Selberg-Weil-Kobayashi theorem:

Theorem 4.1 *Let G be an exponential Lie group, H connected maximal non normal subgroup of G and Γ a discontinuous subgroup for G/H. Then the three following assertions are equivalents :*
$\imath$) There exist a locally rigid homomorphism in $\mathscr{R}(\Gamma, G, H)$.
$\imath\imath$) Every homomorphism in $\mathscr{R}(\Gamma, G, H)$ is locally rigid.
$\imath\imath\imath$) G is isomorphic to Aff($\mathbb{R}$).

This leads us to pose the following question:

Question 4.2 *Let G be an exponential Lie group, H a connected Lie subgroup of G and Γ a abelian discontinuous subgroup for G/H. Is it true that $\mathscr{R}(\Gamma, G, H)$ admits a locally rigid homomorphism if and only if G is isomorphic to the group Aff($\mathbb{R}$) and H is maximal and non-normal in G.*

When Γ is abelian, we answered positively this question. This was the subject of Corollary 3.12 and Theorem 3.13 in [1]. More precisely, the local rigidity property fails to hold in this context if and only if $\dim G \neq 2$ or otherwise H is normal in G.

5 On the concept of stability

5.1 The terminology of stability in the sense of Kobayashi-Nasrin

Let us come back to the general setting for a while. The homomorphism $\varphi \in \mathscr{R}(\Gamma, G, H)$ is said to be *topologically stable* or merely *stable* in the sense of Kobayashi-Nasrin [23], if there is an open set in $\mathrm{Hom}(\Gamma, G)$ which contains φ and is contained in $\mathscr{R}(\Gamma, G, H)$. When the set $\mathscr{R}(\Gamma, G, H)$ is an open subset of $\mathrm{Hom}(\Gamma, G)$, then, obviously each of its elements is stable which is the case for any irreducible Riemannian symmetric space with the assumption that Γ is a torsion free uniform lattice of G ([23] and [29]). Furthermore, we point out in this setting that the concept of stability may be one fundamental concept to understand the local structure of the deformation space.

5.2 Stability of discrete subgroups

Let G be a locally compact group and Γ a closed subgroup of G. In ([20], (5.2.1)), T. Kobayashi defines the set $\pitchfork (\Gamma : G)$ consisting of subsets H for which $SHS^{-1} \cap \Gamma$ is compact for any compact set S in G. Let $\pitchfork_{gp} (\Gamma : G)$ be the set of all closed connected subgroups belonging to $\pitchfork (\Gamma : G)$.

The following questions have been posed in [6].

Question 5.1 *(cf. [6]) For a given discrete subgroup Γ of G, is it possible to characterize all the subgroups $H \in \pitchfork_{gp}(\Gamma : G)$ for which the parameter space $\mathscr{R}(\Gamma, G, H)$ is open. (Any deformation parameter is stable).*

Similarly, let H be a connected subgroup of G and $\pitchfork_d(H : G)$ the set of the discrete subgroups of G belonging to $\pitchfork(H : G)$.

Question 5.2 *Let H be a connected subgroup of G, is it possible to characterize all the subgroups $\Gamma \in \pitchfork_d(H : G)$ for which the parameter space $\mathscr{R}(\Gamma, G, H)$ is open?*

Let Γ be a discrete subgroup of G and $\mathrm{Stab}(\Gamma : G)$ the set of all subgroups $H \in \pitchfork_{gp}(\Gamma : G)$ for which the parameter space $\mathscr{R}(\Gamma, G, H)$ is open. The subgroup Γ is said to be *stable*, if $\mathrm{Stab}(\Gamma : G) = \pitchfork_{gp}(\Gamma : G)$.

Question 5.3 *(cf. [6]) Is it possible to characterize all stable discrete subgroups of connected simply connected solvable Lie groups?*

Similarly, for a connected Lie subgroup H of G we denote by $\mathrm{Stab}(H : G)$ the set of $\Gamma \in \pitchfork_d(\Gamma : G)$ such that $\mathscr{R}(\Gamma, G, H)$ is open. The subgroup H is said to be stable if $\mathrm{Stab}(H : G) = \pitchfork_d(H : G)$.

Question 5.4 *Is it possible to classify stable connected subgroups of connected simply connected solvable Lie groups?*

Some answers to these questions are already provided for some restrictive cases of exponential Lie groups. In the setting of nilpotent Lie groups, the following result (see [3]) is a partial answer to the questions 5.1 and 5.2.

Theorem 5.5 *(cf. [3]). Let G be a connected simply connected nilpotent Lie group, H be a connected subgroup of G, and Γ be a discontinuous subgroup for G/H. If the Clifford-Klein form $\Gamma\backslash G/H$ is compact, then the stability holds everywhere; that is, the parameter space $\mathscr{R}(\Gamma, G, H)$ is semi-algebraic and open.*

In the setting of Heisenberg groups, we got in [5] an answer to question 5.3. We proved the following:

Theorem 5.6 *(cf. [5]). Let Γ be a discontinuous subgroup of the Heisenberg group $G = \exp(\mathfrak{g})$ and $\exp(\mathfrak{l})$ its syndetic hull. Then Γ is stable if and only if Γ is non-abelian or $\mathfrak{l}$ is abelian and maximal in $\mathfrak{g}$.*

When G is two-step nilpotent, we have the following result concerning the case where $\mathfrak{l}$ is a maximal subalgebra of $\mathfrak{g}$, see [2].

Proposition 5.7 *(cf. [2]). If $\mathfrak{l}$ is a maximal abelian subalgebra of $\mathfrak{g}$, then the stability property holds.*

As for the threadlike setting, we proved in [4] the following, as a partial answer to question 5.3:

Theorem 5.8 *(cf. [4]). Let Γ be a discontinuous subgroup of the Threadlike group G and $\exp(\mathfrak{l})$ its syndetic hull. If Γ is non-abelian or $\mathfrak{l}$ is abelian and maximal in $\mathfrak{g}$, then Γ is stable.*

More generally when G is a exponential Lie group, the following result (Proposition 5.1 in [7]) is a partial answer to questions 5.3 and 5.4. Consider first the natural continuous action of $\mathrm{Aut}(\mathfrak{l})$ on $\mathrm{Hom}(\mathfrak{l}, \mathfrak{g})$ which respects $\mathscr{R}(\Gamma, G, H)$. Our upshot in this section is the following:

Theorem 5.9 *(cf. [7]). Let G be a completely solvable Lie group, H a connected subgroup of G and Γ a discontinuous subgroup for G/H such that $[\mathfrak{l}, \mathfrak{l}] = [\mathfrak{g}, \mathfrak{g}]$. Then $\mathscr{R}(\Gamma, G, H)$ is an open set in $Hom(\Gamma, G)$ and semi-algebraic. Moreover for $\varphi \in \mathscr{R}(\Gamma, G, H)$ the following assertions are equivalent:*

ı) φ is rigid.

ıı) φ is locally rigid.

ııı) The orbit $\varphi Aut(\mathfrak{l})$ is open in $Hom(\mathfrak{l}, \mathfrak{g})$ and

$$\dim Aut(\mathfrak{l}) + \dim \varphi(\mathfrak{l})^{\perp} = \dim \mathfrak{g},$$

where $\varphi(\mathfrak{l})^{\perp} = \{Y \in \mathfrak{g},\ [X, Y] = 0 \text{ for all } X \in \varphi(\mathfrak{l})\}$.

Remark 5.10 *We close the paper with the following important remark. Assume that the deformation space $\mathscr{T}(\Gamma, G, H)$ of the discontinuous actions of Γ on $X = G/H$ coincides with the image of $Def^{c}_{(G,X)}(\Sigma)$ by the holonomy map hol and that the stability holds. Then the restriction*

$$hol : Def^{c}_{(G,X)}(\Sigma) \to \mathscr{T}(\Gamma, G, H)$$

is a local homeomorphism. Indeed, $\mathscr{T}(\Gamma, G, H)$ is an open set of $Hom(\Gamma, G)/G$, therefore $Def^{c}_{(G,X)}(\Sigma)$ is an open set of $Def_{(G,X)}(\Sigma)$. As hol is a local homeomorphism, we are done.

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