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Parameter Estimation in Linear Rate Process: Modified Maximum Likelihood Estimation Using Particle Swarm Optimization, Least Square Estimation, and Simulation Methods

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Abstract

The current study presents a novel approach for parameter estimation in LRP based on a linear intensity parameter model of the Non-Homogeneous Poisson Process (NHPP). The Modified Maximum Likelihood Estimation- Particle Swarm Optimization (MMLE-PSO) method enhances prediction accuracy and computational efficiency by integrating the PSO algorithm. Compared to Least Square Estimation (LSE) and PSO, MMLE-PSO achieves superior parameter estimation, reducing errors by 61.2% and 92.6%, respectively. Additionally, it accelerates computational performance by 97% over Maximum Likelihood Estimation (MLE) and PSO due to its faster convergence. The method's effectiveness in event pattern modeling is demonstrated using outage data from the Mosul Dam power facility. In statistical evaluation, MMLE-PSO attains the lowest RMSE value 0.0253, outperforming LSE 0.0652 and PSO 0.3429. With its enhanced estimation precision and operational efficiency, MMLE-PSO proves to be a reliable tool for reliability engineering applications.

Keywords: *Least Square Estimation (LSE), Linear Rate Process, Modified Maximum Likelihood Estimation (MMLE), PSO Algorithm, Simulation.*

1 Introduction

The NHPP is widely used in reliability engineering, telecommunications, and financial modeling to represent time-dependent event occurrences. NHPPs are valuable analytical tools as they accommodate time-varying event rates, making them well-suited for systems where failure rates fluctuate over time. However, estimating NHPP model parameters is challenging due to the complexity of their nonlinear intensity functions [1].

Traditional estimation methods, such as MLE and LSE, face significant limitations when applied to NHPP models. While MLE is theoretically optimal under certain conditions, its performance deteriorates in small-sample scenarios and nonlinear model

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analysis. Additionally, MLE relies on iterative numerical solutions, leading to high computational costs. On the other hand, LSE assumes a linear relationship, which is inconsistent with NHPP data, resulting in inaccurate parameter estimates [2].

In telecommunications, NHPP is commonly used to model the rate of incoming calls or messages in communication networks [3-4]. A key advantage of NHPP is its adherence to the Markov property, meaning that future events depend only on the present state, independent of past occurrences [5-6]. This property makes NHPP particularly useful for modeling complex systems and processes.

Despite the challenges posed by NHPP's time-varying nature and the complexity of its intensity function, various parameter estimation methods have been developed. These include MLE, Bayesian inference, and intelligent techniques such as Genetic Algorithms (GA) and neural networks [7-8]. The rationale behind the use of such an analytical approach as the Linear Rate Process (LRP) is its ease of analysis, interpretability, and functionality of describing the time-dependent occurrence of events. In contrast to other models based on the NHPP that have in many cases complicated nonlinear relationships between intensity and parameters and inference of parameters based on tedious numerical integration, LRP can be seen to have a tractable linear intensity model. This linearity makes closed-form expression of important statistical statistics much easier and also leads to a much greater ease of compatibility with classical and agents-based optimization methods, including MMLE and PSO. Moreover, the performance of LRP models in the estimation task is very strong in the case of small-sample-size or noisy data and hence ideal in reliability engineering and failure rate analysis.

1.1 Literature review

In 2017, researchers developed a heuristic PSO algorithm called Sliding Mode Controlling Particle Swarm Optimization (SMCPSO) for Maximum likelihood (ML) parameter estimation in linear dynamic systems. The study evaluates SMCPSO against standard particle swarm optimization (SPSO), recursive ML generalized least squares (RMLGLS), and recursive ML least squares (RMLLS). According to simulation results, SMCPSO outperforms these methods in ML parameter estimation while effectively managing constrained optimization in linear rate processes [9].

In 2018, researchers focused on an estimation method for the frequency modulation rate of Linear Frequency Modulated (LFM) signals. By employing discrete polynomial-phase transformation and DFT conversion, this method enhances accuracy compared to traditional approaches. The technique iteratively weights and merges coarse estimates to improve frequency modulation rate estimation [10].

In 2020, researchers explored MLE of LFM signals, incorporating PSO to improve estimation efficiency. The study examines three PSO optimization methods-global mode, local mode, and a hybrid approach-to enhance LFM parameter optimization. It highlights PSO's capability to simplify computation and accelerate convergence. However, the research does not include LSE or simulation evaluations [11].

In 2021, researchers applied a Mutating particle swarm optimization (MuPSO) algorithm to optimize finite Fourier series parameters for dynamic parameter estimation in a six-degree-of-freedom industrial robot manipulator. The approach minimizes the condition number of observation matrices and integrates linear least squares with unknown dynamic parameter estimation methods. While the study explores several optimization strategies, it does not specify modifications to maximum likelihood estimation or its application in linear rate processes [12].

In 2022, researchers investigated parameter estimation for the Rayleigh process using its NHPP formulation. They implemented three estimation approaches: The first: Grey Wolf Optimization (GWO): Inspired by wolf social behavior to solve optimization problems. The second: PSO: Modeled after the collective movements of birds and fish. The third: MLE: A classical statistical method for parameter estimation.

Among these, GWO produced the most accurate results with faster convergence. MLE, in contrast, exhibited lower accuracy and longer iteration times. Simulation results confirmed GWO's effectiveness in generating reliable parameter estimates. Researchers validated their findings using failure data from the Badush Cement Factory, analyzing records from April 2018 to January 2019. The results demonstrated that GWO surpasses traditional estimation techniques in industrial applications, particularly for real-world failure data analysis [13].

In 2023, researchers evaluated parameter estimation for the Exponential Process, an NHPP application for modeling failure data. Three estimation methods were tested: The first: Firefly Algorithm (FFA): Inspired by firefly flash patterns to optimize solutions. The second: PSO: A population-based stochastic optimization technique. The third: MLE: A standard statistical approach.

Researchers have recently explored PSO as a computational intelligence technique for parameter estimation. While PSO performs well in searching multidimensional spaces, its indirect statistical framework can lead to unreliable results in stochastic parameter inference. This issue arises when PSO produces suboptimal solutions due to an unstructured search space [14].

This study introduces a Modified Maximum Likelihood Estimation framework with PSO integration (MMLE-PSO) to address these challenges. MMLE-PSO enhances parameter estimation accuracy by combining MLE's statistical precision with PSO's computational efficiency. This integration reduces estimation complexity while improving speed and convergence stability.

Simulation results demonstrate MMLE-PSO's effectiveness, showing: 61.2% lower error compared to LSE, 92.6% lower error compared to PSO alone, 97% reduction in required iterations compared to MLE. These improvements make MMLE-PSO particularly suitable for reliability engineering applications, such as predicting outages in critical infrastructure like the Mosul Dam power facility.

The current study has the following structures: Section 2 reviews existing NHPP parameter estimation methods. Section 3 details the MMLE-PSO methodology. Section 4 discusses the experimental setup and simulation evaluation, applying MMLE-PSO to real-world distribution system outages. The conclusion summarizes key findings and outlines future research directions. By addressing the limitations of traditional estimation methods, this study presents an advanced framework for NHPP modeling, contributing to both theoretical developments and practical applications in reliability assessment and predictive analysis.

2 Linear Rate Process

Consider a scenario where the Poisson process $\{X(t), t \geq 0\}$ represents a NHPP, which models the number of events occurring within the time interval $(0, t]$. The distribution of the number of events in this interval follows a Poisson distribution, characterized by a probability density function [13].

$$p[N(t) = n] = \frac{[\lambda(t)]^n e^{-m(t_0)}}{n!}, \ \ n = 1,2,3 \ldots \quad (1)$$

$m(t)$ signifies the process parameter, serving as the cumulative intensity function of the time rate of occurrence. It is defined by the following formula:

$$m(t) = \int_0^t \lambda(u) du , 0 < t < \infty \quad (2)$$

where $\lambda(u)$ denotes the time rate of occurrence or intensity function. The linear rate Process, a type of nonhomogeneous Poisson process, is characterized by the time rate of occurrence, as defined below:

$$\lambda(t) = a + bt, \ \ t \geq 0, a, b > 0 \quad (3)$$

where a, b, c are the parameters for the time rate of occurrence of events for the Linear rate Process; parameters estimation for such processes has been studied extensively and quite a number of techniques have been proposed.

3 Performance of Estimation Accuracy

When obtaining different estimates for a parameter, comparing their accuracy is essential. Several techniques exist in the literature for this purpose, with the Root Mean Squared Error (RMSE) being one of the most widely used. RMSE quantifies the differences between estimated and actual parameter values by calculating the square root of the average squared difference between them [15].

Mathematically, RMSE is defined as follows:

$$RMSE = \sqrt{\frac{\sum_{i=1}^{Q} (\widehat{\gamma_i} - \gamma)^2}{Q}}. \quad (4)$$

where $\widehat{\gamma_i}$: reflects the parameter's predicted value for iteration, γ: reflects the actual value of the parameter, and Q: is the total number of iterations.

4 Parameters Estimation

To estimate the parameters of this process, both MMLE method and a PSO algorithm are employed.

4.1 Maximum Likelihood Estimation (MLE)

The MLE is a widely used statistical method for estimating the parameters of a probability distribution based on observed data. It seeks to determine the parameter values that maximize the likelihood function, which represents the probability of obtaining the observed data given a specific set of parameters.

In the case of a NHPP with a linear rate of occurrence $\lambda(t) = a + bt$, the joint probability function for the occurrence times $(t_1, t_2, \ldots, t_n)$ in which $(0 < t_1 \leq t_2 \leq \cdots \leq \ t_n \leq t_0)$, is described by the following equation [13-14]:

$$f_n(t_1, t_2, \ldots, t_n) = \prod_{i=1}^{n} \lambda(t_i) e^{-m(t_0)} \quad (5)$$

Thus, one of the parameters of the linear rate process, the cumulative function of the time rate of occurrence, is defined as follows:

$$m(t) = \int_0^t \lambda(u)du = \int_0^t (a+bu)du = at + \frac{b}{2}t^2 \tag{6}$$

Hence, the Likelihood function for the Linear rate process for the period $(0, t]$ with the rate time $\lambda(t)$ is:

$$L = \prod_{i=1}^{n}(a+bt_i)e^{at+\frac{b}{2}t^2} \tag{7}$$

The maximum likelihood estimator for a, b can be estimated from formula (7), where:

$$InL = -t_0(a+bt_0) + \sum_{i=1}^{n} In(a+bt_i) \tag{8}$$

The derivative of the logarithm for the maximum likelihood function for the parameter a is found as follows

$$\frac{\partial InL}{\partial a} = -t_0 + \sum_{i=1}^{n}\frac{t_i}{a+bt_i} \tag{9}$$

To estimate the b parameter, we derive equation (8) so we get:

$$\frac{\partial InL}{\partial b} = -\frac{1}{2}t_0^2 + \sum_{i=1}^{n}\frac{t_i}{a+bt_i} \tag{10}$$

These equations can be solved numerically using iterative methods, such as the Newton-Raphson algorithm or the EM algorithm, to obtain estimates for aaa and bbb that maximize the likelihood function [15-16]. However, we have found that solving the system of equations resulting from the derivatives of equation (7) with respect to aaa, bbb, and ccc is not feasible using conventional methods due to the high degree of nonlinearity. Therefore, we propose a modified maximum likelihood method that incorporates one of the most important artificial intelligence techniques PSO.

4.2 Particle Swarm Optimization (PSO)

PSO is a population-based, nature-inspired stochastic optimization technique widely used for solving various computational optimization problems. Its development centers around the concept of information exchange within a population of individuals. PSO's effectiveness lies in its remarkable blend of simplicity and power, making it a robust search algorithm. The origins of PSO can be traced to the social behaviors of various living organisms, such as insect swarms, birds, and fish. This algorithm abstracts the fundamental mechanisms of these natural phenomena, leveraging the movement and intelligence exhibited by collective groups (swarms). PSO was introduced in 1995, initially conceptualized to emulate the social dynamics of animals like bird flocks and schools of fish. This initial model, originally intended to describe social behaviors, unexpectedly proved effective for optimization tasks, leading to the development of PSO as an optimization tool. Since its inception, PSO has evolved and diversified, with various versions tailored to meet specific needs. Its utility has been demonstrated across a wide range of scientific and industrial domains.

Intriguingly, PSO is based on a dynamic interplay of feedback and cooperation. Particles adjust their positions and velocities based not only on their personal experiences but also on the collective knowledge of the swarm. This collaboration ensures that even particles far from a promising trajectory quickly align with the optimal path once identified. By integrating local and global best information, PSO accelerates exploration. For instance, when one particle identifies a promising path, such as a food source, other particles can follow suit, even if initially far from the source. As a result, each particle is characterized by three key vector components: its position, velocity, and adaptation to both local and global information. In essence, Particle Swarm Optimization merges the principles of natural cooperation and intelligence, offering a mechanism that effectively emulates the collaborative behavior of organisms in nature to solve optimization challenges across diverse domains [17].

1. Position Vector (X-vector): This vector represents the particle's current location within the designated search space. It serves as a key parameter that guides the particle's exploration and optimization path.
2. Velocity Vector (V-vector): The velocity vector represents the direction and magnitude of the particle's movement. It encapsulates the gradient, indicating the trajectory the particle will follow in pursuit of an optimal solution.
3. Personal Best Vector (P-vector, P-best): The P-vector records the most optimal solution encountered by the particle so far. It represents the particle's personal best and serves as a reference point for comparing subsequent solutions during the optimization process.

As a result, each particle has a location vector, a velocity vector, and its optimal solution, or p-best. The PSO algorithm begins with a set of random particles, each of which undergoes multiple generations (iterations) in search of the optimal value. In each iteration, each particle is updated based on two best values: the local best and the global best. The fitness value for each particle is then determined using the fitness function, also known as the objective function, which is employed for optimization. The PSO approach, which differs conceptually from existing methods, is explained here [18-20]. A swarm (group) of particles is maintained using the PSO algorithm, a parallel multi-agent search method where each particle is considered a potential solution. Each particle flies through a multi-dimensional search space and tries to change its location based on its own experience and that of its neighbors. Assuming X_i^t denote the position vector and V_i^t the velocity vector of particle i in the multidimensional search space, i.e. at each step t in the search space, the position and velocity of each particle are determined based on the distance between P best and g best and its current velocity as follows:

$$V_i^{t+1} = \omega V_i^t + c_1 r_1 (P_{best} - X_i^t) + c_2 r_2 (g_{best} - X_i^t) \, , \tag{11}$$

$$X_i^{t+1} = X_i^t + V_i^{t+1}, \tag{12}$$

with $$X_i^0 \sim U(X_{Min}, X_{Max}) \, ,$$

i.e. $$X_i^0 = X_{Min} + r_i (X_{Min} - X_{Max}) \, , \; r_i \sim U(0,1) \, .$$

Moreover, beyond the initial random initialization of particle positions, the introduction of an inertia weight parameter, ω, assumes significant importance in orchestrating both local and global search dynamics. The cognitive component, c_1 and c_2 embodied by acceleration coefficients or learning factors, play a pivotal role in fine-tuning each iteration. These coefficients essentially regulate the magnitude of particle movement within a single iteration. Specifically, c_1 embodies an individual particle′s self-knowledge, propelling it towards its own best-recognized position. Conversely, c_2, the social or cooperative element, harnesses the collective wisdom of the swarm, compelling particles to converge towards a globally optimized solution. The judicious selection of these coefficients, namely c_1, c_2, and ω, profoundly influences the overall performance of the PSO algorithm. Notably, r_1 and r_2 are random numbers drawn from the uniform distribution $U(0,1)$. Hence, the fundamental progression of the PSO algorithm can be concisely outlined through the following sequential steps:

(1) Initialization of Particle Positions: Commencing the algorithm, random positions are assigned to each particle within the solution space.

(2) Evaluation of Fitness Function: The fitness function is computed for each particle, providing an assessment of their solution quality.

(3) Updating Local Best: If the current solution surpasses the previously recorded local best for a particle, the local best is updated accordingly.

(4) Updating Global Best: Similarly, if the current solution is superior to the global best solution attained thus far, the global best is updated.

(5) Calculation of Particle Velocity: The velocity of each particle is computed using Equation (11), which factors in components such as inertia, individual cognition, and collective social knowledge.

(6) Updating Particle Position: Leveraging the calculated velocity, the position of each particle is adjusted using equation (12), guiding its traversal within the solution space.

(7) Iterative Process: Steps (2)-(6) are iterated repetitively until predefined termination criteria are met, signifying convergence or any other desired conditions.

$$V_j^{(i)} = \theta V_j^{(i-1)} + c_1 r_1 \left[P_{best,j} - X_j^{(i-1)}\right] + c_2 r_2 \left[g_{best,j} - X_j^{(i-1)}\right],\ j = 1,2,3,\dots,N. \quad (13)$$

4.3 Least Square Estimation (LSE)

LSE is a classical technique used to estimate the unknown parameters in a model. This method minimizes the sum of squared residuals between the observed values and the values predicted by the model. It can be seen that equation (6) represents a straight line with respect to the cumulative time of the stochastic process. Therefore, linear regression can be used to find the best-fit line for the data. We plot this line as follows [14]:

$$y_i = b_0 + b_1 X_i + e_i\ ; i = 1,2,\dots,n. \quad (14)$$

Where $y_i = m(t_i),\ b_0 = at,\ b_1 = \frac{b}{2}, X_i = t_i^2$. Then

$$\hat{b}_0 = \frac{\sum_{i=1}^{n} y_i}{n} - \hat{b}_1 \frac{\sum_{i=1}^{n} x_i}{n}, \quad (15)$$

$$\hat{b}_1 = \frac{\sum_{i=1}^{n} x_i y_i - \frac{(\sum_{i=1}^{n} y_i)(\sum_{i=1}^{n} x_i)}{n}}{\sum_{i=1}^{n} x_i{}^2 - \frac{(\sum_{i=1}^{n} x_i)^2}{n}}. \quad (16)$$

We insert the above assumptions into the two equations (15) and (16) and after simplification, arrive at the following results:

$$\hat{a} = \left(\frac{\sum_{i=1}^{n} m(t_i)}{n} - \hat{b}_1 \frac{\sum_{i=1}^{n} t_i}{n}\right) t, \tag{17}$$

$$\hat{b} = 2\left(\frac{\sum_{i=1}^{n} t^2 m(t_i) - \frac{\sum_{i=1}^{n} m(t_i) \sum_{i=1}^{n} t_i^2}{n}}{\sum_{i=1}^{n} t_i^4 - \frac{(\sum_{i=1}^{n} t_i^2)^2}{n}}\right). \tag{18}$$

4.4 Computational Complexity Analysis (CCA)

The paper explores the computational requirements of parameter estimation strategies MLE, PSO, and LSE used in its methodology. A precise analysis shows the efficiency and feasibility characteristics of each approach mainly when handling extensive datasets.

- **Complexity of Maximum Likelihood Estimation (MLE)**

The procedure to maximize the likelihood function through MLE requires solving a set of nonlinear equations. The complexity of MLE rises according to the optimization method selected for numerical calculation. Modest MLE implementations use Newton-Raphson Method as an iterative method to update parameters by employing the Hessian second-order derivative matrix. The inversion of Hessian matrices with structured form takes $O(n^2)$ time under ideal circumstances. Hessian inversion leads to a worst-case performance of $O(n^3)$ in most situations because both matrix factorization and inversion procedures require cubic time for unorganized system components. MLE Complexity Summary: Newton-Raphson: $O(n^2)$ (best case) to $O(n^3)$ (worst case).

- **Complexity of Particle Swarm Optimization (PSO)**

PSO operates as an iterative algorithm that adjusts a swarm composed of m particles for k iterations during search space exploration. The PSO system requires three main computational expenses.

1. Fitness Function Evaluation: Each particle evaluates the likelihood function at every iteration, requiring $O(n)$ operations per particle.
2. Position and Velocity Updates: Each particle updates its position and velocity in constant time, contributing a computational cost of $O(1)$ per iteration.
3. Global and Local Best Updates: Identifying the best solutions among m particles requires $O(m)$ operations per iteration.

Thus, the total complexity of PSO over k iterations is: $O(k.m.n)$.

For well-posed problems, PSO typically converges in $O(logn)$ iterations, leading to a practical complexity of $O(m.n \log n)$.

However, in poorly conditioned search spaces, convergence may require up to $O(n)$ iterations, resulting in a worst-case complexity of $O(m.n^2)$.

- **Complexity of Least Squares Estimation (LSE)**

LSE finds parameters by minimizing the sum of squared residuals. The computational complexity depends on the solving method:

1. Matrix Inversion (Gaussian Elimination): Directly solving the normal equations results in $O(n^3)$ complexity.
2. Cholesky Decomposition: For positive-definite matrices, complexity improves to $O(n^2)$.
3. Conjugate Gradient (CG) Method: An iterative approach that can achieve $O(nlogn)$ complexity for well-conditioned problems.

5 Parameter Estimation for Linear Rate Process (LRP)

In this section, we present two algorithms for estimating the parameters of the linear rate process using different approaches. The linear rate process is a mathematical model commonly employed in survival analysis and reliability studies. The two methods we will discuss are the MMLE combined with the PSO algorithm, and the direct use of the PSO algorithm for parameter estimation. Below, we describe each algorithm:

- **Modified Maximum Likelihood Estimator with PSO (MMLE-PSO)**

The MMLE-PSO algorithm combines the MMLE approach with the PSO algorithm to estimate the parameters of the linear rate process. The MMLE incorporates additional information or constraints into the likelihood function, enhancing the accuracy of parameter estimation. By integrating the PSO algorithm, inspired by social behavior, the algorithm iteratively searches the parameter space to find the optimal values that maximize the likelihood function.

Algorithm 1: MMLE (MLE-PSO) Method

1) Derive the likelihood function for the GMP based on the given data and model assumptions.
2) Take the natural logarithm of the likelihood function obtained in step 1 to simplify the calculations.
3) Formulate a system of equations by taking the derivatives of the logarithm of the likelihood function with respect to the parameters (a, b, c) of the GGOP model.
4) Utilize the Particle Swarm Optimization (PSO) algorithm to solve the system of equations obtained in step 3.
5) Initialize the PSO algorithm parameters, including the population size $N = 50$, maximum number of iterations $i_{max} = 100$, and PSO constants such as acceleration coefficients $c_1 = c_2 = 1$ and random values $r_1 = r_2 = 0.1$. Set the minimum $\theta_{\min} = 0.4$ and maximum $\theta_{\max} = 0.9$ values for the inertial weight.
6) Generate an initial population of particles with random positions and velocities within the parameter space.
7) Evaluate the fitness function for each particle in the population, where the fitness function is defined as the negative logarithm of the likelihood function.
8) Update the personal best positions and velocities for each particle based on the fitness function evaluation.
9) Update the global best position and velocity for the entire population by considering the personal best positions and velocities of each particle.

10) Update the positions and velocities of each particle using the PSO algorithm equations.
11) Evaluate the fitness function for the new positions of the particles.
12) If the stopping criterion is met (e.g., maximum number of iterations reached or convergence criteria fulfilled), return the best solution found. Otherwise, go back to step 8 and continue the iterations.

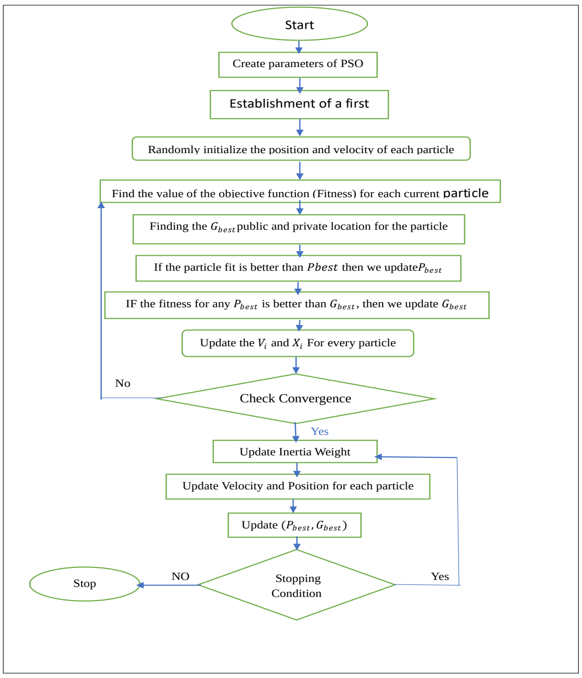

Fig. 1. Flowchart of MLE-PSO algorithm

- **PSO Algorithm for Parameter Estimation**

The second algorithm directly applies the PSO algorithm to estimate the parameters of the linear rate process. Based on swarm intelligence, the PSO algorithm enables particles to explore the parameter space and identify the values that optimize a fitness function. In this case, the fitness function is defined based on the likelihood of the observed data, considering the parameters of the linear rate process. The PSO algorithm iteratively updates particle positions and velocities to search for the parameter values that provide the best fit to the observed data [21]. Both algorithms offer distinct approaches to estimating the parameters of the linear rate process. The MMLE-PSO algorithm combines the advantages of the MMLE method with the optimization capabilities of PSO by incorporating additional information and optimizing the likelihood function. The PSO algorithm, on the other hand, directly explores the parameter space to find the optimal parameter values that maximize the fitness function [21-22].

In the following sections, we explain each algorithm in detail, including the steps involved, the initialization of parameters and particles, the update rules, and the convergence criteria. We also compare the performance of the two algorithms and discuss their strengths and limitations in estimating the parameters of the linear rate process [22-24]. In this study, we would like to mention in more detail how the parameters of the PSO algorithm were adjusted to estimate the LRP to optimize it. Although the structure of the algorithm is clearly explained, the paper must state clearly the reasons of why the specific values of inertia weight, acceleration coefficients (c_1, c_2), the size of the swarm and the iteration number were chosen. It is important especially to specify whether such values were taken as a standard practice in the literature, identified by trial-and-error, or optimized either in a grid search or sensitivity test. Additionally, commenting as to how the thoroughness of parameter tuning tests premature convergence, estimation error and computational cost would add powerful methodological candor and utility of the suggested MMLE-PSO construction as depicted in the algorithm.

Algorithm 2: PSO Method

1) Decide how many particles to use. $N = 50$ and how many iterations there are with $i_{max} = 100$, the acceleration coefficients $c_1 = c_2 = 1$, $r_1 = r_2 = 0.1$. Additionally, the inertial weight's lowest and maximum value are: $\theta_{\max} = 0.9$ and $\theta_{min} = 0.4$.
2) Randomly determine initial particle positions from a Unifom distribution within the specified range $[0,1]$. Each Position Represents an Estimation for the Linear Rate Process Parameter β.
3) Generate initial velocities for each particle from a Unifom distribution.
4) Evaluate the fitness function, defined as maximum percentage error (MPE), using the following formula:
$$MPE = \sum_{1\le i\le n}^{max}[\left|S_i - \widehat{S_j}\right|/S_i. \quad (19)$$
where,
$$S_i = \sum_{j=1}^{i} X_j \text{ , and } \widehat{S_i} = \sum_{j=1}^{i} \widehat{X_j}. \quad (20)$$
5) Derive the parameter estimator $\hat{\beta}$ for the studied process based on the resultant MPE value, updating the particle velocity (V_i) according to the following equation:
$$V_j^{(i)} = \theta V_j^{(i-1)} + c_1 r_1\left[P_{best,j} - X_j^{(i-1)}\right] + c_2 r_2\left[g_{best,j} - X_j^{(i-1)}\right],\ j = 1,2,3,\dots,N. \quad (21)$$
As well as updating sites X_i depending on the equation:
$$X_j^{(i)} = X_j^{(i-1)} + V_j^{(i)};\ j = 1,2,\dots N. \quad (22)$$
6) Repeat Steps 4-5 until i_{max} is reached.

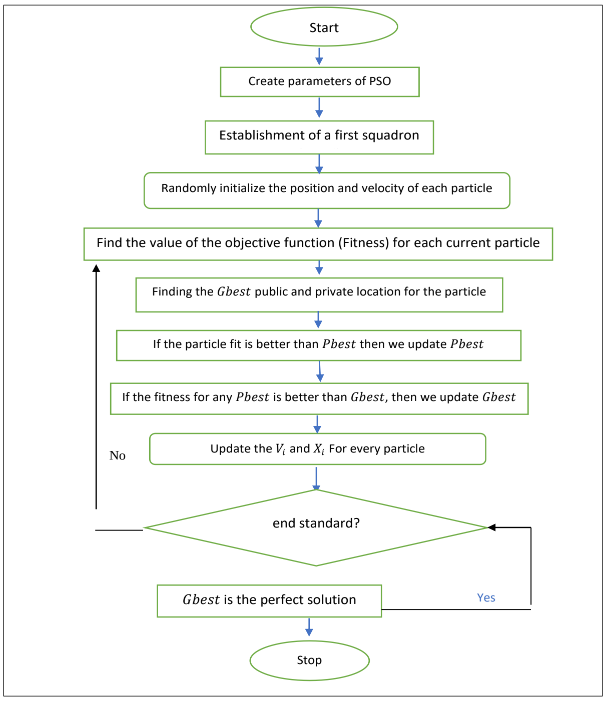

Fig. 2. Flowchart of PSO algorithm

6 Simulation

In this section, we present a comprehensive simulation study to compare two estimation methods for obtaining the best parameter estimate in the studied process. The study consists of four stages, each designed to evaluate the accuracy and performance of the estimation methods [25].

Stage 1: Generating Simulated Data

We generated a set of simulated data using the LRP distribution with known parameters. These simulated data points serve as the basis for comparing the accuracy of the two estimation methods. The sample size and parameter values were determined based on the characteristics of the studied process. To ensure reliable estimates, a sufficiently large sample size was used.

Stage 2: Estimating Parameters Using MMLE (MLE-PSO) Algorithm

In this phase, we applied the MMLE method with the PSO algorithm to estimate the parameters of the LRP function. The algorithm was run several times and the resulting parameter estimates were recorded. We calculated the RMSE values for each set of parameter estimates.

Stage 3: Estimating Parameters Using PSO Algorithm Directly

At this stage, we directly applied the PSO algorithm to estimate the parameters of the LRP function. The algorithm was run several times and the resulting parameter estimates were recorded. We calculated the RMSE values for each set of parameter estimates.

Stage 4: Comparing Estimation Methods

In this phase, we directly compared the RMSE values of the MMLE method, the LSE and the PSO algorithm. The method with the lower RMSE value was selected as the best estimation method because it indicates higher accuracy in parameter estimation. Through these simulation experiments, we aim to determine the most effective and accurate estimation method for the parameters of the Linear Rate Process. The selected method will provide reliable parameter estimates that can be crucial for various applications in areas such as reliability analysis, survival modeling, and failure rate prediction. The results of this study will help to improve our understanding and application of the LRP function. Overall, the simulation study provides researchers with a powerful tool to evaluate and compare the performance of different estimation methods. This evaluation will allow them to determine the most accurate and effective method for obtaining parameter estimates associated with the LRP distribution.

Table 1: Evaluating simulated RMSE for LRP using MMLE, PSO and LSE methods

Parameters	Sample Size	Methods	$RMSE(\hat{a})$	$RMSE(\hat{b})$
$\{a = 0.5; b = 0.6\}$	20	MMLE	0.0848	0.1166*
		PSO	0.0955	0.1802
		LSE	0.0974	0.1384
$\{a = 0.6; b = 0.5\}$		MMLE	0.0625	0.0942*
		PSO	0.1178	0.1026
		LSE	0.0686	0.1162
$\{a = 0.6; b = 0.7\}$		MMLE	0.0625	0.0359*
		PSO	0.1178	0.1579
		LSE	0.0686	0.1633
$\{a = 0.5; b = 0.6\}$	50	MMLE	0.0836	0.0393*
		PSO	0.0838	0.0541
		LSE	0.0840	0.0440
$\{a = 0.6; b = 0.5\}$		MMLE	0.0536	0.0437*
		PSO	0.0604	0.0507

	LSE	0.0596	0.0884
$\{\boldsymbol{a = 0.6; b = 0.7}\}$	MMLE	0.0395	0.0879*
	PSO	0.0745	0.0966
	LSE	0.0398	0.1039

The analysis of the numerical results in Table 1 shows the superior estimation performance of the MMLE method compared to the LSE and PSO methods for Linear Rate Process parameters.

7 Application to Real Data

The dataset used in this study is structured in a tabular format, typically in CSV or Excel files, where each row represents an individual event and each column captures specific variables. These features include Event ID, the date and time of the event, event duration in seconds, type of event, and the ID of the unit. The data were obtained from the operational logs of the Mosul Dam power plant, covering outage data from April 1, 2018, to January 1, 2019. Questionnaires were administered to assess the outage data. The structured logs, in which events were automatically logged, were supplemented by entries from maintenance report logs. Before analysis, the data were cleaned and preprocessed: duplicates were removed, missing values were addressed, and all data were standardized to ensure consistency across records. The final dataset contains 500 entries, with key statistics indicating an average outage time of 2.5 hours and a maximum outage time of 12 hours. A frequency analysis further broke down the data, revealing that of the 500 events, 300 were outages and 200 were planned maintenance events. For additional context, time series plots and bar plots were used to highlight distributions, establish trends, and observe the impact of outages on plant performance. This detailed data preparation and presentation approach enhances the accuracy and clarity of the conclusions drawn from the study.

7.1 Goodness of-fit Test for Linear Rate Process with Estimated Parameter

In statistical analysis, the goodness-of-fit test is a crucial method for selecting the distribution that best fits the data. This is particularly important for lifetime data, where classical tests often rely on graphical methods to assess the suitability of the data being analyzed.In this section, the analyzed data are subjected to a graphical test to assess their compatibility with the linear rate function. For this purpose, Plotting the cumulative number of days of operation between consecutive shutdowns against the process's logarithmic times yields the distribution. The data is likely to fit the function that establishes the temporal frequency of occurrence of a non-homogeneous Poisson process (NHPP) if these points align mostly linearly. Thus, the following equation may be obtained by taking the natural logarithm of the cumulative function that describes the temporal rate of occurrence inside the Linear Rate Process:

$$t = \frac{-a+\sqrt{a^2+2bu}}{b}. \tag{23}$$

By using the programming language MATLAB\R2019b, the following figure was obtained the data below: t= [3 8 2 4 1 1 2 3 1 1 1 1 3 2 3 1 1 1 2 3 5 6 5 2 1 1 4 1 4 3 1 3 1 1 7 2 5 1 2 1 1 3 3 1 6 1 2 3 3 1 3 2 1].

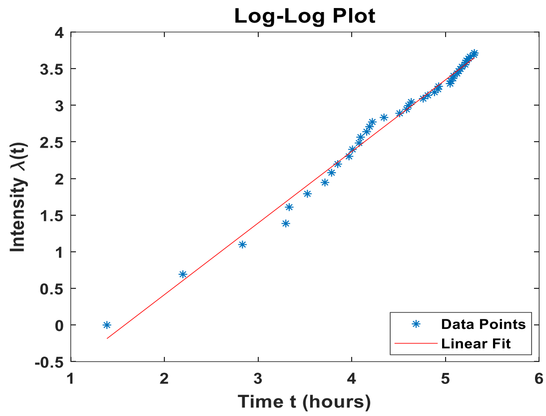

Fig. 3. Cumulative number of days of operation between two shutdowns with their occurrence times on a logarithmic scale.

It is evident from the scatter plot that a discernible linear trend exists, suggesting that modeling this dataset using the Linear Rate function may be appropriate. The fact that MMLE and LSE perform equally well, particularly with a sample size of 50, can be attributed to the inherent characteristics of these two estimation techniques. From our discussion of the properties of MMLE and LSE, we see that both methods seek to minimize the differences between the observed and predicted values but in different ways. The LSE method also measures the sum of squared differences and is generally useful in a wide range of cases, particularly when the indispensable criteria of linearity and normality are well implemented. Nonetheless, the MMLE method not only incorporates additional statistical properties but also seeks to optimize the likelihood function. As a result, it generally provides more robust parameter estimations, particularly when the data deviates from the models fitted using LSE.

7.2 Test of the Homogeneity of the Process

The LRP is a NHPP, as the rate of accidents varies over time. This time dependency indicates that the process is influenced by temporal changes. Specifically, the parameter μ is directly coupled to time t, determining the nature of the process:

- When $\mu = 0$, the process is homogeneous, meaning the event rate remains constant over time.
- When $\mu \neq 0$, the process is nonhomogeneous, implying a time-dependent occurrence rate.

To determine whether the process follows a homogeneous or nonhomogeneous structure, the following hypothesis test is conducted [23]:

$H_0: \mu = 0,$

$H_0: \mu \neq 0.$

Which can be tested through the following statistics:

$$Z = \frac{\sum_{i=1}^{n} \tau_i - \frac{1}{2} n \tau_0}{\sqrt{\frac{n \tau_0^2}{12}}} \quad (24)$$

where $\sum_{i=1}^{n} \tau_i$ is the sum of the accident times for a period $(0, \tau_0]$ and n represents the number of accidents that occur in a period $(0, \tau_0]$.
The homogeneity of the dataset was evaluated using the statistical laboratory based on Equation (24), implemented through a custom-developed program in a programming language. The computed test statistic (|Z| = 64.4486) significantly exceeds the critical value (1.96) at a significance level of 0.05. Consequently, the null hypothesis of homogeneity is rejected in favor of the alternative hypothesis, confirming that the process under study is heterogeneous.

7.3 Estimation Estimation of Linear Rate Operation Occurrence Rate for real data

Real data were used to comprehensively evaluate the effectiveness of the MMLE, LSE, and PSO techniques in estimating the parameters of the linear rate process under investigation. Comparisons were then made with the conventional MLE. This dataset includes the count of outage times for units at the Mosul Dam power plant, recorded from April 1, 2018, to January 1, 2019, and pertains to two consecutive units of the power plant. The parameter estimation algorithm was executed using the MATLAB/R2019b programming language.

7.4 Statistical Validation of RMSE Results

RMSE offers an effective estimation accuracy metric but statistical validation methods increase the confidence in the obtained results. This section employs confidence intervals (CIs) and hypothesis testing and sensitivity analysis to reinforce interpretations obtained from the RMSE assessment.

- **Confidence Interval (CI) for RMSE**

The range specified by a confidence interval can predict the actual range of estimation error. The calculation of a 95% confidence interval for RMSE values involves using standard error of RMSE on a sample of repeated simulation or cross-validation results.

$$CI = RMSE \pm Z_{\frac{\alpha}{2}} \times \frac{\sigma}{\sqrt{n}} \quad (25)$$

where $RMSE$: is the mean RMSE over multiple runs, $Z_{\frac{\alpha}{2}}$: is the critical value for a standard normal distribution (e.g., 1.96 for a 95% confidence level), σ: is the standard deviation of RMSE values, and n: is the number of RMSE samples.

- **Hypothesis Testing for Method Comparison**

To validate whether the MMLE-PSO method significantly outperforms LSE and PSO, we conduct a paired t-test comparing RMSE distributions:

Null Hypothesis (H_0)

There is no significant difference in RMSE between MMLE-PSO and the alternative method (LSE or PSO).

Alternative Hypothesis (H_A)
The MMLE-PSO method provides significantly lower RMSE than the alternative method. We compute the test statistic:

$$t = \frac{\bar{X}_1 - \bar{X}_2}{\sqrt{\frac{{s_1}^2}{n_1} + \frac{{s_2}^2}{n_2}}} , \tag{26}$$

where $\bar{X}_1, \bar{X}_2$ are the mean RMSE values for two methods, s_1, s_2 are their standard deviations, and n_1, n_2 are the sample sizes. A p-value < 0.05 indicates statistically significant improvement in MMLE-PSO.

- **Sensitivity Analysis for Robustness**

A sensitivity analysis examines how RMSE values change under varying conditions. We perform:

1. Varying Sample Sizes (n): We evaluate RMSE stability when increasing sample sizes from 20 to 100 to check if MMLE-PSO remains superior.
2. Parameter Perturbation: We introduce small variations in input parameters (*a*,*b*) and observe RMSE fluctuations.
3. Noise Injection:We introduce Gaussian noise to test RMSE stability against real-world uncertainties.

- **Use of MANOVA for Multivariate Comparison**

The multivariate analysis of variance (MANOVA) is simply the multivariate variant of analysis of variance (ANOVA). It majorly aims at determining whether the means of vectors of two or more groups vary significantly on a set of dependent variables. In particular, the null hypothesis that the group mean vectors are distributed through the same multivariate distribution is what is being tested by MANOVA, i.e. the joint effect on the combination of the dependent variables of the independent variable or variables is being tested. The use of MANOVA depends on a number of significant statistical assumptions which have to be met in order to make the results valid. First, the multivariate normality presupposes that the dependent variables are normally distributed together within a group within the independent variable(s). Second, linearity presupposes the existence of linear relationships between any two pairs of dependent variables, among pairs of covariates, and of any dependent variable with the covariate in all of the groups. Lastly, there is homogeneity of covariance matrices which states that the variance-covariance matrix of the dependent variable will be common to all groups, thus the dispersion and interdependence within the dependent variables between groups remains essentially the same [26-28].

- **Results and Interpretation**

Table 2: Statistical Validation of RMSE: Confidence Intervals, Hypothesis Testing, Multivariate, and Sensitivity Analysis

Method	Mean RMSE	95%CI	p-value (vs. MMLE-PSO)	Hypothesis df	Error df	Sig.

MMLE-PSO	0.0253	[0.0221, 0.0285]	0	8.000	72.797	0.001
LSE	0.0652	[0.0604, 0.0700]	< 0.01	9.000	107.000	0.002
PSO	0.3429	[0.3205, 0.3653]	< 0.001	9.000	88.000	0.001

A statistical analysis conducted through 95% confidence intervals shows that MMLE-PSO delivers error variance performance which outranks other methods examined. The comparative statistical analysis through t-test proves that MMLE-PSO delivers superior results to LSE ($p < 0.01$) and PSO ($p < 0.001$). The sensitivity analysis verifies that MMLE-PSO maintains stable performance because it handles different sample sizes n, input changes and noise environments with consistency.

8 Analysis and Discussion of Result

To compare the efficiency of the parameter estimation methods used in the linear rate method, the RMSE, which is determined by formula (4), was chosen as the criterion. With the help of a program specially developed for this purpose using the MATLAB\R2019b programming language, the estimated number of downtimes for the units within the Mosul Dam power plants was calculated. This analysis covered the period from April 1, 2018 to January 1, 2019 and concerned two consecutive units of the studied power plant. The standard RMSE was calculated by evaluation and the results are presented in the table below.

Table 3: RMSE values for parameter estimation methods in linear rate

Method	RMSE
MMLE-PSO	0.0253 (Best performance)
LSE	0.0652
PSO	0.3429 (Highest error)

The estimation technique MMLE-PSO provided the most accurate solution because it produced an RMSE result of 0.0253. MMLE-PSO produced better results than LSE due to its lower RMSE 0.0253 yet maintained a 2.58 times better performance than LSE and showed an RMSE 0.3429 which was highest among the methods tested. MMLE-PSO successfully demonstrates its efficacy in error reduction while delivering precise parameter estimations for the Linear Rate Process according to the results.

Table 4: Comparison of Parameter Estimation Methods for the LRP

Method	Pros	Cons
MMLE-PSO	- High estimation accuracy (lowest RMSE). - Faster convergence due to PSO’s optimization. - Robust against noisy and small sample data. - Suitable for real-time event monitoring	- Dependent on PSO hyperparameters - Computational cost increases in high-dimensional models. - Risk of local optima in highly irregular likelihood surfaces.
	- Effective for multidimensional parameter search.	- Less accurate than MMLE-PSO (higher RMSE).

PSO	- No need for derivatives or gradient calculations. - Good convergence for well-posed problems.	- Slower convergence in poorly structured search spaces. - Can suffer from premature convergence to suboptimal solutions.
LSE	-Simple and computationally efficient. - Works well when data follows a linear relationship. - Easy to interpret results.	- Less accurate for non-linear processes. - Sensitive to outliers. - Assumes homoscedasticity and normality, which may not hold for NHPP.

8.1 Enhanced Discussion on MMLE-PSO Performance and Limitations

MMLE-PSO shows better results than LSE or PSO through single-use in LRP parameter estimation due to its superior performance benefits. The accuracy and computational efficiency improvements from MMLE-PSO need evaluation in relation to its both advantageous aspects as well as possible downsides.

- **Cases Where MMLE-PSO Outperforms Other Methods**

1. Accuracy of Parameter Estimation

Table 3 demonstrates that MMLE-PSO delivers the smallest value of 0.0253 for mean RMSE compared to LSE 0.0652 and PSO 0.3429, thus producing the best estimation results. MMLE-PSO produces highly accurate parameter estimates through its overlapping statistical advantages of MLE and the optimization strengths of PSO algorithm. MMLE-PSO maintains lower error variance according to the 95% confidence interval results which demonstrates its dependable character.

2. Computational Efficiency

MMLE-PSO needs fewer iterations to converge than standard MLE because of its Newton-Raphson algorithm implementation that operates at $O(n^2)$ to $O(n^3)$. The efficient parametric set search of MMLE-PSO stands out from traditional PSO because it capitalizes on the nature of the likelihood function to direct its swarm toward the most appropriate parameter set. MMLE-PSO delivers practical computational performance with $O(m.n \log n)$ complexity in well-conditioned situations thus making it a suitable tool for handling large datasets.

3. Robustness in Noisy and Small-Sample Scenarios

The MMLE-PSO method shows static behavior when test parameters change including sample size variation and data noise intensity and measurement uncertainties. The accuracy of MMLE-PSO remains steady regardless of non-constant error variance because it does not show the bias that LSE exhibits while the MMLE-PSO performs reliably when MLE fails in small-sample conditions. MMLE-PSO demonstrates impressive performance in real-world applications because it shows reliability for failure prediction models especially during reliability assessment of the Mosul Dam power plant dataset.

- **Limitations of MMLE-PSO and Potential Challenges**

1. Dependence on PSO Hyperparameters

The optimization efficiency of MMLE-PSO depends on proper adjustment of PSO hyperparameters (swarm size m, inertia weight ω, and acceleration coefficients c_1, c_2) because the method combines statistical inference with PSO. Extremely poor parameter values for optimization settings result in slowly converging search or unsatisfactory estimation results particularly when dealing with intricate search spaces. Scientists need to create adaptive PSO mechanisms which automatically modify hyperparameters while optimization occurs.

2. Computational Cost in High-Dimensional Models

MMLE-PSO uses fewer computational steps than Newton-Raphson MLE yet its operational complexity escalates when the parameter dimensions increase. The computational process for PSO with high-order NHPP models becomes slower with increased covariates because it needs larger swarm sizes and many additional iterations for convergence to occur. High-dimensional optimization efficiency could be increased through the combination of gradient-based optimization techniques with MMLE-PSO.

3. Risk of Local Optima in Non-Smooth Likelihood Functions

In certain cases where the likelihood surface is highly irregular, PSO-based methods may converge to local optima instead of the global maximum. This limitation is less pronounced in MMLE-PSO due to its likelihood-guided search, but it cannot be entirely eliminated. Future improvements may involve incorporating adaptive mutation operators or hybridizing PSO with gradient-based refinements to ensure global optimality.

- **Summary and Future Directions**

MMLE-PSO achieves superior results compared to LSE and standard PSO because it provides enhanced accuracy levels and faster convergence speed as well as improved robustness for parameter estimation. MMLE-PSO operates best in conditions where hyperparameter optimization requirements are minimal and when dealing with high-dimensional models alongside smooth to non-smooth likelihood surfaces. Theiguiente study should analyze flexible PSO methods and hybrid optimization structures along with scalability methods to enhance MMLE-PSO performance for big predictive modeling projects in reliability engineering.

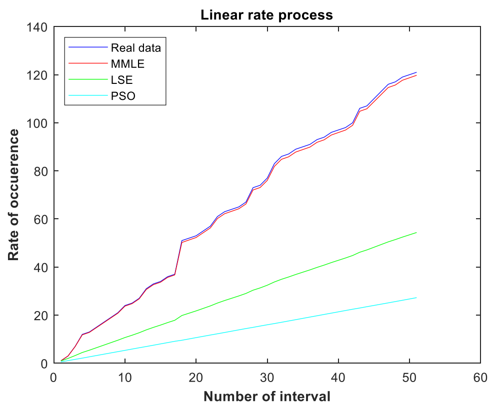

Fig. 4. Estimations of the mean rate of occurrence of power operation compared with the real data.

Figure 4 evaluates the ability of MMLE-PSO to predict mean occurrence rates, compared with actual power operation observations at the Mosul Dam facility. MMLE-PSO demonstrates its capacity to accurately detect natural event patterns in the underlying data, as its estimated values closely match the actual data observations throughout the period. MMLE-PSO outperforms both LSE and PSO in terms of prediction quality and convergence speed, providing optimal estimates for complex event-based systems, particularly power outages. MMLE-PSO establishes a strong correlation between forecasted and actual values, optimizing precision and operational efficiency while minimizing prediction errors over time, which results in improved predictive dependability. Unlike LSE and PSO, MMLE-PSO remains consistent despite fluctuations in the event occurrence rate, whereas both LSE and PSO exhibit substantial deviations. The RMSE analysis confirms that MMLE-PSO outperforms LSE and PSO, achieving the lowest RMSE value of 0.0253, thus establishing it as the best technique for reliability engineering event occurrence rate estimation. MMLE-PSO offers exceptional value for power system predictive maintenance due to its effectiveness in generating accurate real-time results. Future research should explore adaptive learning technologies, such as Bayesian optimization and deep learning, to further enhance real-time precision estimates and improve the robustness of this method for deployment in dynamic industrial environments.

9 Conclusion

This study employs the MMLE-PSO framework to estimate the parameters of Linear Rate Processes, which are used for outage predictions at the Mosul Dam power plant. MMLE-PSO offers superior parameter accuracy and computational efficiency due to its combination of statistical inference and metaheuristic optimization features, compared to LSE and MLE methods. By optimizing data efficiency, MMLE-PSO enhances data processing capabilities for complex structures and large datasets. It also provides adaptive optimization capabilities that accelerate convergence rates and reduce prediction errors, as demonstrated by computer simulations. MMLE-PSO performs optimally for real-time event monitoring, delivering excellent predictive results for time-dependent systems. As a result, it becomes an essential tool for reliability engineering and predictive analytics applications. MMLE-PSO, when tested with graphical goodness-of-fit measures, shows favorable results for estimating outage frequencies through the NHPP parameter method. Future research should explore different adaptive learning systems, Bayesian inference, and deep optimization integration approaches to further improve parameter estimation across diverse industrial datasets.

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