An answer to a Dobbs conjecture about treed domains

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(joint work with Ahmed Ayache)

Problem

Let (R, S) be a pair of rings. Find necessary and sufficient conditions for (R, S) to provide a $\mathcal{P}-$ pair (that is each ring T between R and S satisfies $\mathcal{P}$) for a given ring-theoretic property $\mathcal{P}$.

Some Definitions

- $Spec(R)$: **The set of prime ideals of the ring R**
- R_P: **The localisation of R at the prime ideal P of R, that is**

$$R_P = \{\frac{a}{b} | a \in R, b \in R \setminus P\}$$

- $qf(R)$: **The quotient field of an integral domain R.**
- **Valuation domain: an integral domain R such that for each $x \in qf(R)$, either $x \in R$ or $x^{-1} \in R$.**
- **Prüfer domain: An integral domain R such that R_P is a valuation domain for each $P \in Spec(R)$**

• **Overring**: of an integral domain R is a ring T such that $R \subseteq T \subseteq qf(R)$

• **Saturated chain of primes of length n**

$$P_0 \subset P_1 \subset ... \subset P_n$$

(at each step $P_i \subset P_{i+1}$ are consecutive)

• **Height $ht(P)$**, of a prime P of R: The supremum of the lengths of saturated chains of primes of R arising at P

• **Krull dimension $dim(R)$** of a ring

$$dim(R) = Sup\{ht(P)|P \in Spec(R)\}$$

• $R[n]$: The polynomial ring in n indeterminates

Jaffard domains

From **A. Seidenberg**, in the 50's, it is known that

$$n + dim(R) \leq dim(R[n]) \leq (n+1)dim(R) + n$$

The lower bound holds for **Noetherian** rings and **Prüfer** rings.

Definition

The valuative dimension $dim_v(R)$ of a ring R is the limit:

$$dim_v(R) = \lim_{n \to +\infty} (dim(R[n]) - n)$$

For a domain, the valuative dimension is also the upper bound of the Krull dimension of the overrings of R.

In fact, it is enough to consider the valuation overrings of R [Paul Jaffard 1958]

Twenty years ago, in honor of **Paul Jaffard**.

David F. Anderson, Alain Bouvier, David E. Dobbs, Marco Fontana and Salah Kabbaj defined the Jaffard domains.

Similarly, for a ring with zero divisors:

Definition

A **Jaffard ring** is a (finite dimensional) ring R such that

$$dim(R) = dim_v(R)$$

Equivalently $dim(R[n]) = dim(R) + n$, for each integer n.

If R is Jaffard, so (clearly!) is $R[n]$ for all n.

Jaffard rings are not stable under localization.

Definition

R is locally Jaffard ring if R_P is Jaffard for every prime P.

Universally catenarian domains

Definition

A ring R is catenarian if, for every pair of primes $P \subset Q$ all saturated chains between P and Q have same length.

Equivalently, for a domain R, if $P \subset Q$ are consecutive, then $ht(Q) = ht(P) + 1$

Catenarity is stable under localization and quotient.

Catenarity is not stable under polynomial extensions (Example [M. Nagata] of a (2 dimensional local Noetherian) catenarian domains such that $R[X]$ is not catenarian.

Definition (continued)

- R is $n-$ catenarian if $R[n]$ is catenarian.
- R is universally catenarian if $n-$ catenarian for all n.

Prüfer domains are universally catenarian domains.

Proposition

$n-$ catenarian $\Rightarrow (n-1)-$ catenarian.

A famous question

Theorem (L. J. Ratliff Jr. 1970)

For Noetherian rings

$1-$ catenarian $\Leftrightarrow$ universally catenarian.

Question

$1-$ catenarian $\neq 2-$ catenarian.

There is Now an answer.

Example (M. Ben Nasr and N. Jarboui, 2002)

Let p be a prime number, $\mathbf{Z}[[t]]$ be the ring of power series with coefficients in $\mathbf{Z}$, and R be the pullback

$$R = \mathbf{Z} + p\mathbf{Z}[[t]]$$

Then

- $R[X]$ is catenarian: R is $1-$ catenarian.
- R is not $2-$ catenarian.

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Dobbs Conjecture

In 1987, D. E. Dobbs has constructed a non integrally closed, two dimensional domain R which is not going-down, although (it and) each of its overrings is treed. It has since been an open question whether an integrally closed, quasi local treed domain (R, M) of valuative dimension two such that R/M is an algebraically closed field and each overring of R is treed is necessarily a going-down domain.

• A ring extension $R \subseteq T$ satisfies the going-down property (for short GD property) if for each pair of primes of R, $P \subset Q$, if Q lifts to a prime Q' of T (that is $Q' \cap R = Q$), then P lifts to a prime P' of R such that $P' \subset Q'$.

• An integral domain R is called a going-down domain (GD domain) if $R \subseteq T$ satisfies the GD property for each domain T containing R.

• A treed domain is an integral domain R, whose incomparable prime ideals are coprime.

Theorem (A. Ayache and N. Jarboui 2008)

Let R be a locally Jaffard, integrally closed domain. Then the following statements are equivalent:

(i) each overring of R is treed.

(ii) R is a Prüfer domain.

Theorem (A. Ayache and N. Jarboui 2008)

Let R be a quasi local integrally closed Jaffard domain. Then the following statements are equivalent:

(i) each overring of R is treed.

(ii) R is a valuation domain.

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Corollary

Let R be an integrally closed domain such that $\dim_v(R) = 2$. If each overring of R is treed, then R is a going-down domain.

Corollary

Let R be an integral domain such that $\dim_v(R) = 2$. If each overring of R is treed and $R[X]$ is catenarian, then R is a going-down universally catenarian domain.

Example

We provide an example of a domain R that satisfies the following properties:

(i) R is a Jaffard but not a locally Jaffard domain with Krull dimension two.

(ii) There exist two primes $q \subset q'$ of an overring T lying above the same prime p of R.

(iii) R is not a Prüfer domain.

(iv) R is integrally closed.

(v) Every overring of R is treed.

(vi) R is going-down.

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