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On some Čebyšev type inequalities for functions whose second derivatives are co-ordinated logarithmically convex

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Abstract

The aim of this paper is to establish some new Čebyšev type inequalities involving functions of two independante variable whose second derivatives are co-ordinated logarithmically convex.

Keywords: *Čebyšev inequality, co-ordinated* log*-convex, integral inequality.*

2010 Mathematics Subject Classification: 26D15, 26D20.

1 Introduction

In 1882, Čebyšev [4] gave the following inequality

$$|T(f,g)| \leq \frac{1}{12}(b-a)^2 \|f'\|_\infty \|g'\|_\infty, \tag{1}$$

for $f, g : [a,b] \to \mathbb{R}$ are absolutely continuous functions, whose first derivatives f' and $g'$are bounded, where

$$T(f,g) = \frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx - \left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right), \tag{2}$$

and $\|.\|_\infty$ denotes the norm in $L_\infty[a,b]$ defined as $\|f\|_\infty = ess\sup\limits_{t\in[a,b]}|f(t)|$.

During the past few years, many researchers have given considerable attention to the inequality (1.1). Various generalizations, extensions and variants of this inequality have appeared in the literature, we can mention the works [1, 3, 6, 8, 9, 10, 11, 12, 14], and references cited therein.

Recently, Guezane-Lakoud and Aissaoui [6] gave the analogue of the functional (1.2) for functions of two variables and established the following Čebyšev type inequalities for functions whose mixed derivatives are bounded

$$|T(f,g)| \leq \frac{49}{3600}k^2 \|f_{\lambda\alpha}\|_\infty \|g_{\lambda\alpha}\|_\infty, \tag{3}$$

and

$$\begin{aligned}|T(f,g)| \leq\ & \frac{1}{8k^2}\int_a^b\int_c^d \left[\left(|g(x,y)|\,\|f_{\lambda\alpha}\|_\infty + |f(x,y)|\,\|g_{\lambda\alpha}\|_\infty\right)\right. \\ & \times \left.\left[\left((x-a)^2+(b-x)^2\right)\left((y-c)^2+(d-y)^2\right)\right]\right] dydx, \end{aligned}\tag{4}$$

where

$$\begin{aligned} T(f,g) =\ & \frac{1}{k}\int_a^b\int_c^d f(x,y)\,g(x,y)\,dydx - \frac{(d-c)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_a^b f(t,y)\,dt\right)dydx \\ & -\frac{(b-a)}{k^2}\int_a^b\int_c^d g(x,y)\left(\int_c^d f(x,v)\,dv\right)dydx \\ & +\frac{1}{k^2}\left(\int_a^b\int_c^d f(x,y)\,dydx\right)\left(\int_a^b\int_c^d g(t,v)\,dvdt\right). \end{aligned}\tag{5}$$

Sarikaya et al. [16] have discussed the case where the mixed derivatives are convex on the co-ordinates. Meftah et al. [10, 9, 8] have treated the cases where the mixed derivatives are on co-ordinated (h_1,h_2)- convex, (s,r)- convex in the first sense and (s_1,m_1)-(s_2,m_2)-convex.

Motivated by the above results, in this paper, we establish some new Čebyšev type inequalities whose mixed derivatives are logarithmically-convex functions on the co-ordinates.

2 Preliminaries

In this section, we recall some definitions and lemmas.

Throughout this paper we denote by Δ the bidimensional interval in $[0,\infty)^2$, $\Delta =: [a,b]\times[c,d]$ with $a<b$ and $c<d$, $k=(b-a)(d-c)$ and $\frac{\partial^2 f}{\partial\lambda\partial\alpha}$ by $f_{\lambda\alpha}$.

Definition 2.1 ([7]) *A function $f : \Delta \to \mathbb{R}$ is said to be convex on the co-ordinates on Δ, if the following inequality:*

$$\begin{aligned} f\left(tx+(1-t)\,u,\lambda y+(1-\lambda)\,v\right) &\leq t\lambda f(x,y)+t\,(1-\lambda)\,f(x,v) \\ &\quad +(1-t)\,\lambda f(u,y)+(1-t)\,(1-\lambda)\,f(u,v) \end{aligned} \tag{6}$$

holds for all $t,s\in[0,1]$ and $(x,y),(x,v),(u,y),(u,v)\in\Delta$.

Clearly, every convex mapping $f : \Delta \to \mathbb{R}$ is convex on the co-ordinates. Furthermore, there exists co-ordinated convex function which is not convex see [5].

Definition 2.2 ([2]) *A positive function $f : \Delta \to \mathbb{R}$ is called* log-*convex on the co-ordinates on Δ, if the following inequality*

$$\begin{aligned} & f\left(tx+(1-t)\,u,\lambda y+(1-\lambda)\,v\right) \\ \leq\; & f^{t\lambda}(x,y)f^{t(1-\lambda)}(x,v)f^{(1-t)\lambda}(u,y)f^{(1-t)(1-\lambda)}(u,v) \end{aligned} \tag{7}$$

holds for all $t,\lambda\in[0,1]$ and $(x,y),(x,v),(u,y),(u,v)\in\Delta$.

Lemma 2.3 (Lemma 1. [13]) *Let $f : \Delta \to \mathbb{R}$ be a partially differentiable mapping on Δ in $\mathbb{R}^2$. If $f_{\lambda\alpha}\in L_1(\Delta)$, then for any $(x,y)\in\Delta$, we have the equality:*

$$\begin{aligned} f(x,y) &= \frac{1}{b-a}\int_a^b f(t,y)dt+\frac{1}{d-c}\int_c^d f(x,s)ds-\frac{1}{k}\int_a^b\int_c^d f(t,s)dsdt \\ &\quad +\frac{1}{k}\int_a^b\int_c^d (x-t)\,(y-s) \\ &\quad \times\left(\int_0^1\int_0^1 f_{\lambda\alpha}\left(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s\right)d\alpha d\lambda\right)dsdt. \end{aligned} \tag{8}$$

3 Main result

Theorem 3.1 *Let $f,g : \Delta \to \mathbb{R}$ be partially differentiable functions, such that their second derivatives $f_{\lambda\alpha}$ and $g_{\lambda\alpha}$ are integrable on Δ. If $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$ are co-ordinated logarithmically convex, then*

$$|T(f,g)|\leq\frac{49}{3600}k^2MN, \tag{9}$$

where $T(f,g)$ is define as in (1.5), $k=(b-a)(d-c)$,
$M=\underset{x,t\in[a,b],y,s\in[c,d]}{ess\sup}\left[|f_{\lambda\alpha}(x,y)|+|f_{\lambda\alpha}(x,s)|+|f_{\lambda\alpha}(t,y)|+|f_{\lambda\alpha}(t,s)|\right]$, *and*
$N=\underset{x,t\in[a,b],y,s\in[c,d]}{ess\sup}\left[|g_{\lambda\alpha}(x,y)|+|g_{\lambda\alpha}(x,s)|+|g_{\lambda\alpha}(t,y)|+|g_{\lambda\alpha}(t,s)|\right].$

Proof 3.2 *From Lemma 1, we have*

$$\begin{aligned}
&f(x,y)-\frac{1}{b-a}\int_a^b f(t,y)dt-\frac{1}{d-c}\int_c^d f(x,s)ds+\frac{1}{k}\int_a^b\int_c^d f(t,s)dsdt\\
=\ &\frac{1}{k}\int_a^b\int_c^d (x-t)(y-s)\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\,d\alpha d\lambda\right)dsdt,
\end{aligned}\tag{10}$$

and

$$\begin{aligned}
&g(x,y)-\frac{1}{b-a}\int_a^b g(t,y)dt-\frac{1}{d-c}\int_c^d g(x,s)ds+\frac{1}{k}\int_a^b\int_c^d g(t,s)dsdt\\
=\ &\frac{1}{k}\int_a^b\int_c^d (x-t)(y-s)\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\,d\alpha d\lambda\right)dsdt.
\end{aligned}\tag{11}$$

Multipliying (3.2) by (3.3), then integrating the resultant equality with respect to x and y over Δ, and multiplying the result by $\frac{1}{k}$, using properties of modulus, and Fubini's Theoerm, we get

$$\begin{aligned}
|T(f,g)|\leq\ &\frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\\
&\times\left.\left(\int_0^1\int_0^1 |f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)v)|\,d\alpha d\lambda\right)dvdt\right]\\
&\times\left[\int_a^b\int_c^d |x-t|\,|y-v|\right.\\
&\times\left.\left(\int_0^1\int_0^1 |g_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)v)|\,d\alpha d\lambda\right)dvdt\right]\Bigg]dydx.
\end{aligned}\tag{12}$$

Since $|f_{\lambda\alpha}|$ *and* $|g_{\lambda\alpha}|$ *are co-ordinated* log-*convex, (3.4) gives*

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\left[\int_0^1\int_0^1 |f_{\lambda\alpha}(x,y)|^{\alpha\lambda}\,|f_{\lambda\alpha}(x,s)|^{\alpha(1-\lambda)}\right.\right. \\
&\quad \times \left.\left.|f_{\lambda\alpha}(t,y)|^{(1-\alpha)\lambda}\,|f_{\lambda\alpha}(t,s)|^{(1-\alpha)(1-\lambda)}\,d\alpha d\lambda\right] dsdt\right] \\
&\quad \times \int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\left[\int_0^1\int_0^1 |g_{\lambda\alpha}(x,y)|^{\alpha\lambda}\,|g_{\lambda\alpha}(x,s)|^{\alpha(1-\lambda)}\right.\right. \\
&\quad \times \left.\left.|g_{\lambda\alpha}(t,y)|^{(1-\alpha)\lambda}\,|g_{\lambda\alpha}(t,s)|^{(1-\alpha)(1-\lambda)}\,d\alpha d\lambda\right] dsdt\right] dydx \\
&\leq \frac{1}{k^3}\int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\right. \\
&\quad \times \left.\left[\int_0^1\int_0^1 M^{\alpha\lambda+\alpha(1-\lambda)+(1-\alpha)\lambda+(1-\alpha)(1-\lambda)}d\alpha d\lambda\right] dsdt\right] \\
&\quad \times \int_a^b\int_c^d\left[\int_a^b\int_c^d |x-t|\,|y-s|\right. \\
&\quad \times \left.\left[\int_0^1\int_0^1 N^{\alpha\lambda+\alpha(1-\lambda)+(1-\alpha)\lambda+(1-\alpha)(1-\lambda)}d\alpha d\lambda\right] dsdt\right] dydx \\
&= \frac{1}{k^3}MN\int_a^b\int_c^d\left(\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt\right)^2 dydx \\
&= \frac{49}{3600}k^2MN, \qquad (13)
\end{aligned}$$

where we have use the fact that

$$\int_a^b\int_c^d\left(\int_a^b\int_c^d |x-t|\,|y-v|\,dvdt\right)^2 dydx = \frac{49}{3600}k^5.$$

The proof is thus achieved.

Theorem 3.3 *Under the assumptions of Theorem 1, we have*

$$\begin{aligned}
|T(f,g)| &\leq \frac{1}{8k^2}\int_a^b\int_c^d (M\,|g(x,y)|+N\,|f(x,y)|) \\
&\quad \times \left[(x-a)^2+(b-x)^2\right]\left[(y-c)^2+(d-y)^2\right] dydx, \qquad (14)
\end{aligned}$$

where $T(f,g)$ $M,N,$ and k are defined as in Theorem 1.

Proof 3.4 *Clearly (3.2) and (3.3) are valid. Let $G(x,y)=\frac{1}{2k}g(x,y)$ and $F(x,y)=\frac{1}{2k}f(x,y)$.*

Multiplying (3.2) by $G(x,y)$ and (3.3) by $F(x,y)$, summing the resultant equalities, then integrating the result with respect to x and y over Δ, we have

$$\begin{aligned} T(f,g) &= \frac{1}{2k^2}\int_a^b\int_c^d \Bigg[g(x,y)\Bigg[\int_a^b\int_c^d (x-t)(y-s) \\ &\times\left(\int_0^1\int_0^1 f_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\,d\alpha d\lambda\right)dsdt\Bigg] \\ &+f(x,y)\Bigg[\int_a^b\int_c^d (x-t)(y-s) \\ &\times\left(\int_0^1\int_0^1 g_{\lambda\alpha}(\lambda x+(1-\lambda)t,\alpha y+(1-\alpha)s)\,d\alpha d\lambda\right)dsdt\Bigg]dydx. \end{aligned} \tag{15}$$

Using the modulus and the log*-convexity of $|f_{\lambda\alpha}|$ and $|g_{\lambda\alpha}|$, we get*

$$\begin{aligned} |T(f,g)| &\le \frac{1}{2k^2}\int_a^b\int_c^d \Bigg[|g(x,y)|\Bigg[\int_a^b\int_c^d |x-t|\,|y-s| \\ &\times\Bigg[\int_0^1\int_0^1 |f_{\lambda\alpha}(x,y)|^{\alpha\lambda}\,|f_{\lambda\alpha}(x,s)|^{\alpha(1-\lambda)}\,|f_{\lambda\alpha}(t,y)|^{(1-\alpha)\lambda} \\ &+|f_{\lambda\alpha}(t,s)|^{(1-\alpha)(1-\lambda)}\,d\alpha d\lambda\Bigg]dsdt\Bigg] \\ &+|f(x,y)|\Bigg[\int_a^b\int_c^d |x-t|\,|y-s| \\ &\times\Bigg[\int_0^1\int_0^1 |g_{\lambda\alpha}(x,y)|^{\alpha\lambda}\,|g_{\lambda\alpha}(x,s)|^{\alpha(1-\lambda)}\,|g_{\lambda\alpha}(t,y)|^{(1-\alpha)\lambda} \\ &+|g_{\lambda\alpha}(t,s)|^{(1-\alpha)(1-\lambda)}\,d\alpha d\lambda\Bigg]dsdt\Bigg]dydx \end{aligned} \tag{16}$$

$$\leq \frac{1}{2k^2}\int_a^b\int_c^d \left[M\,|g(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt \right] \right.$$

$$\left. + N\,|f(x,y)| \left[\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt \right] \right] dydx$$

$$= \frac{1}{2k^2}\int_a^b\int_c^d (M\,|g(x,y)| + N\,|f(x,y)|) \left[\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt \right] dxdy$$

$$= \frac{1}{8k^2}\int_a^b\int_c^d (M\,|g(x,y)| + N\,|f(x,y)|)$$

$$\times \left[(x-a)^2 + (b-x)^2 \right] \left[(y-c)^2 + (d-y)^2 \right] dydx, \tag{17}$$

where we have use the fact that

$$\int_a^b\int_c^d |x-t|\,|y-s|\,dsdt$$

$$= \frac{1}{4}\left[(x-a)^2 + (b-x)^2 \right] \left[(y-c)^2 + (d-y)^2 \right].$$

This completes the proof of Theorem 2.

4 Tow open problems

1. Can these inequalities be established for higher order of derivatives?
2. Can the analog of (1.5), (2.3), (3.1) and (3.6) be established for the preinvex functions as well as the harmonic functions?

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