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Some Results on Fuzzy Metric Spaces

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Abstract

In this paper, first some characterization of various fuzzy bounded sets in FM-spaces are given. Some important fixed point theorems for multi-valued mappings in FM-spaces will be given.

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1 Introduction

The theory of fuzzy sets was introduced by L.Zadeh in 1965 [22]. After the pioneering work of Zadeh, there has been a great effort to obtain fuzzy analogues of classical theories. Among other fields, a progressive developments is made in the field of fuzzy topology. One of the most important problems in fuzzy topology is to obtain an appropriate concept of fuzzy metric space. This problem has been investigated by many authors from different points of view. In particular, George and Veeramani [4] have introduced and studied a notion of fuzzy metric space. Furthermore, the class of topological spaces that are fuzzy metrizable agrees with the class of metrizable- topological spaces (see [4, 5]). This result permits Gregori and Romaguera to restate some classical theorems on metric completeness and metric (pre) compactness in the realm of fuzzy metric spaces [5–7]. Nevertheless, the theory of fuzzy metric completion is very different from the classical theories of metric completion. In fact, there are fuzzy metric spaces which are non-completable [7]. A significant difference between a classical metric and a fuzzy metric is that this last one includes in its definition a parameter t. This fact has been successfully used in engineering applications such as colour image filtering [1, 2, 12–16] and perceptual colour differences [17]. The fuzzy fixed point theory has become an area of interest for specialists in fixed point theory, or fuzzy mathematics has offered new possibilities for fixed point theorists, see [19–21]

The structure of the paper is as follows. After preliminaries, in section 3, we construct some new metric sets and we study the topological properties of these sets. Section 4 is devoted to study the fixed point theorem for multi-valued mappings in fuzzy metric spaces and finally, in section 5, we prove some common fixed point theorems for multi-valued mappings in fuzzy metric spaces.

2 Preliminaries

In this section we present some definitions and terminology will used in the sequel.

Definition 2.1. *Let X be an arbitrary set. A fuzzy set A in X is a function with domain X and values in $[0,1]$. That is, $A : X \to [0,1]$.*

Let $(\Lambda, \leq)$ be a partially ordered non-empty set. A triangle function on Λ (or a t_Λ-norm) is a map $\tau : \Lambda \times \Lambda \to \Lambda$ that is associative, commutative, nondecreasing in both arguments (that is, $\tau(\lambda_1, \lambda_2) \leq \tau(\lambda_3, \lambda_4)$, whenever $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_4$) and has an element $\lambda_0 \in \Lambda$ as identity (i.e., $\tau(\lambda, \lambda_0) = \lambda$ for all $\lambda \in \Lambda$). A t-norm is a triangle function $\Delta : [0,1]^2 \to [0,1]$ that has 1 as identity, see [18]. Some typical examples of t-norm are the following:

$$\begin{aligned}
&\Delta(a,b) = ab, \quad \text{(product)}\\
&\Delta(a,b) = \min\{a,b\}, \quad \text{(minimum)}\\
&\Delta(a,b) = \max\{a+b-1, 0\}, \quad \text{(Lukasiewicz)}\\
&\Delta(a,b) = \frac{ab}{a+b-ab}, \quad \text{(Hamacher)}
\end{aligned}$$

Definition 2.2. *[10] A triple (X, M, Δ) is called a fuzzy metric space (briefly, a FM-space) if X is an arbitrary (non-empty) set, Δ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0, \infty)$ such that the following axioms hold:*

(FM-1) $M(x, y, 0) = 0$ *for all* $x, y \in X$,

(FM-2) $M(x, y, t) = 1$ *for every* $t > 0$ *if and only if* $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$ *for all* $x, y \in X$ *and* $t > 0$,

(FM-4) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous for all* $x, y \in X$,

(FM-5) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$ *for all* $x, y, z \in X$ *and for all* $t, s \in [0, \infty)$.

We will refer to the fuzzy metric spaces in the sense of Kramosil and Michalek as KM-fuzzy metric spaces. If, in the above definition, the condition (FM-5) is replaced by the condition

(FM-5A) $M(x, z, \max\{t, s\}) \geq \Delta(M(x, y, t), M(y, z, s))$ for all $x, y, z \in X$ and for all $t, s \in [0, \infty)$,

then (X, M, Δ) is called a strong metric space. It is easy to check that (FM-5A) implies (FM-5), that is, every strong fuzzy metric space is it self a fuzzy metric space.

Definition 2.3. *[4] A triple* (X, M, Δ) *is called a fuzzy metric space (briefly, a FM-space) if* X *is an arbitrary (non-empty) set,* Δ *is a continuous* t*-norm and* M *is a fuzzy set on* $X \times X \times [0, \infty)$ *such that the following axioms hold:*

(FM-1) $M(x, y, t) > 0$ *for all* $x, y, z \in X$,

(FM-2) $M(x, y, t) = 1$ *for every* $t > 0$ *if and only if* $x = y$,

(FM-3) $M(x, y, t) = M(y, x, t)$ *for all* $x, y \in X$ *and* $t > 0$,

(FM-4) $M(x, y, .) : [0, \infty) \to [0, 1]$ *is left continuous for all* $x, y \in X$,

(FM-5) $M(x, z, t + s) \geq \Delta(M(x, y, t), M(y, z, s))$ *for all* $x, y, z \in X$ *and for all* $t, s \in [0, \infty)$.

We will refer to the fuzzy metric spaces in the sense of George and Veeramani as GV-fuzzy metric spaces.

Example 2.4. *([4])(1) Let* (X, d) *be a metric space. Define a* t*-norm by* $\Delta(a, b) = ab$*, and set*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)}, \quad \text{for all } x, y \in x \text{ and } t > 0.$$

Then (X, M_d, Δ) *is a strong fuzzy metric space;*M_d *is called the standard fuzzy metric induced by* d*. It is interesting to note that the topology induced by the*

M_d and the corresponding metric d coincide.
(2) Let (X, d) be a metric space. Define a t-norm by $\Delta(a, b) = ab$, and set

$$M(x, y, t) = exp\left[-\frac{d(x, y)}{t}\right], \quad , \textit{for all } x, y \in x \textit{ and } t > 0.$$

Then (X, M, Δ) is a strong fuzzy metric space.

Remark 2.5. *[4] Let (X, M, Δ) be a FM-space. For all $T \in (0, \infty)$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $r \in (0, 1)$ is defined by*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}. \tag{1}$$

Let (X, M, Δ) be a FM-space and let τ_M be the set of all $A \subset X$ with $x \in A$ if and only if there exists $t > 0$ and $r \in (0, 1)$ such that $B(x, r, t) \subset A$. Then τ_M is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable.

Lemma 2.6. *[8] Let (X, M, Δ) be a FM-spaces. Then $M(x, y, t)$ is non-decreasing with respect to t, for all $x, y \in X$.*

Definition 2.7. *[8] Let (X, M, Δ) be a FM-spaces. (i) A sequence $\{x_n\}$ in X is said to be convergent to $x \in X$ if*

$$\lim_{n\to\infty} M(x_n, x, t) = 1 \textit{ for all } t > 0.$$

(ii) The sequence $\{x_n\}$ in X is said to be Cauchy if

$$\lim_{n,m\to\infty} M(x_n, x_m, t) = 1 \textit{ for all } t > 0.$$

or equivalently, if for any $\epsilon \in (0, 1)$ and $t > 0$, there exists an $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n.m \geq n_0$.
(iii) The FM-space (X, M, Δ) is said to be complete if every Cauchy sequence in X is convergent.

Lemma 2.8. *[11] Let (X, M, Δ) be a FM-spaces. If there exists $k \in (0, 1)$ such that*

$$M(x, y, kt) \geq M(x, y, t)$$

for all $x, y \in X$ and $t > 0$, then $x = y$.

Definition 2.9. *Let (X, M, Δ) be a FM-spaces with continuous t-norm Δ.*

1. *A sequence $\{x_n\}$ in X is said to be τ-convergent to $x \in X$, denoted by $x_n \xrightarrow{\tau} x$, if for any $\epsilon > 0$ and $\lambda \in (0, 1)$, there exists a positive integer $N_0(\epsilon, \lambda)$, such that $M(x_n, x, \epsilon) > 1 - \lambda$ whenever $n \geq N_0$.*

2. *A sequence* $\{x_n\}$ *in* X *is said to be* τ*-Cauchy if if for any* $\epsilon > 0$ *and* $\lambda \in (0,1)$*, there exists a positive integer* $N_0(\epsilon, \lambda)$*, such that* $M(x_n, x_m, \epsilon) > 1 - \lambda$ *whenever* $n, m \geq N_0$*.*

3. (X, M, Δ) *is said to be* τ*-complete if every* τ*-Cauchy sequence is* τ*-convergent to some point in* X*.*

4. *A mapping* $T : X \to X$ *is said to be* τ*-continuous if, for any sequence* $\{x_n\}$ *in* X *such that* $x_n \xrightarrow{\tau} x$*,* $Tx_n \xrightarrow{\tau} Tx$*.*

3 Diameter and Distance Between Sets

Throughout this section, (X, M, Δ) will denote a Fuzzy metric space with a continuous t-norm Δ.

Definition 3.1. *Let* A *be a non-empty subset of* X*. Then the function defined by*

$$D_A(t) = \sup_{s<t}[\inf_{p,q \in A} M(p, q, s)],$$

will be called the fuzzy diameter of A*. If* $\sup_{t>0} D_A(t) = 1$*, then* A *is called a fuzzy bounded set.*

Definition 3.2. *A non-empty subset* A *of* X *is said to be*

1. *bounded if* $\sup\limits_t D_A(t) = 1$*,*

2. *semi-bounded if* $0 < \sup\limits_t D_A(t) < 1$*,*

3. *unbounded if* $D_A(t) = 0$*.*

We now establish the properties of the fuzzy diameter. The proofs requiring only routine calculation will be omitted.

Theorem 3.3. *For any non-empty subset* A *of* X*, the function* $D_A(.)$ *is non-decreasing in* t*,* $D_A(0) = 0$*,* $\sup_{t>0} D_A(t) = 1$ *and* $D_A(t)$ *is left-continuous.*

Theorem 3.4. *If* A *is a non-empty subset of* X*, then* $D_A(t) = 1$ *for all* $t > 0$ *if and only if* A *consists of a single point.*

Theorem 3.5. *If* A *and* B *are subsets of* X *and* $A \subset B$*, then* $D_A \geq D_B$*.*

Theorem 3.6. *If* A *and* B *are subsets of* X *such that* $A \cap B = \emptyset$*, then*

$$D_{A\cup B}(t + s) \geq \Delta(D_A(t), D_B(s)). \tag{2}$$

for all $t, s > 0$*.*

Proof. Let t and s be given. To establish (2), we first show that

$$\inf_{p,q\in A\cup B} M(p,q,s+t) \geq \Delta\left[\inf_{p,q\in A} M(p,q,s), \inf_{p,q\in B} M(p,q,t)\right]. \tag{3}$$

There are two distinct cases to consider:

Case(1):

$$\inf_{p,q\in A\cup B} M(p,q,s+t) = \inf_{p\in A,q\in B} M(p,q,s+t). \tag{4}$$

Now for any triple of points p, q and r in X, we have

$$M(p,q,s+t) \geq \Delta(M(p,r,s), M(r,q,t)).$$

Taking the infimum of both sides of this inequality as p ranges over A, q ranges over B and r ranges over $A\cap B$, and using (4), we have

$$\inf_{p,q\in A\cup B} M(p,q,s+t) \geq \inf_{p\in A,q\in B,r\in A\cap B} \Delta(M(p,r,s), M(r,q,t)).$$

However, since Δ is continuous and non-decreasing, we obtain

$$\inf_{p,q\in A\cup B} M(p,q,s+t) \geq \Delta\left[\inf_{p,r\in A} M(p,r,s), \inf_{r,q\in B} M(r,q,t)\right].$$

Case (2):

$$\inf_{p,q\in A\cup B} M(p,q,s+t) < \inf_{p\in A,q\in B} M(p,q,s+t).$$

In this case, one of the following equalities

$$\inf_{p,q\in A\cup B} M(p,q,s+t) < \inf_{p,q\in A} M(p,q,s+t)$$

or

$$\inf_{p,q\in A\cup B} M(p,q,s+t) < \inf_{p,q\in B} M(p,q,s+t)$$

must hold. If the first equality holds, we have

$$\inf_{p,q\in A\cup B} M(p,q,s+t) \geq \Delta\left[\inf_{p,q\in A} M(p,q,s+t), \inf_{p,q\in B} M(p,q,s+t)\right].$$

The same argument works for the second equality. This establish (3). Finally, using the fact that the rectangle

$$\{(s',t') : 0 \leq s' \leq s, 0 \leq t' \leq t\}$$

is contained in the triangle $(s', t') : s', t' \geq 0, s' + t' < s + t$, the inequality (3) and the continuity of Δ, we have

$$\begin{aligned} D_{A\cup B}(s'+t') &= \sup_{s+t<s'+t'} [\inf_{p,q\in A\cup B} M(p,q,s+t)] \\ &\geq \sup_{s<s',t<t'} [\inf_{p,q\in A\cup B} M(p,q,s+t)] \\ &\geq \Delta\left(\sup_{s<s'}[\inf_{p,q\in A} M(p,q,s)], \sup_{t<t'}[\inf_{p,q\in B} M(p,q,t)]\right) \\ &= \Delta(D_A(s'), D_B(t')). \end{aligned}$$

This achieves the proof. $\square$

Theorem 3.7. *If A is a non-empty subset of X, then $D_A = D_{\overline{A}}$, where $\overline{A}$ denotes the closure of A in the (ϵ, λ)-topology on X.*

Proof. Since $A \subset \overline{A}$, it follows from Theorem 3.5 that $D_A \geq D_{\overline{A}}$. Let $\eta > 0$ be given. In view of the uniform continuity of M with respect to the Levy metric space L on $\mathcal{D}$, there exists an $\epsilon > 0$ and a $\lambda > 0$ such that for any four points p_1, p_2, p_3 and p_4 in X,

$$L(M(p_1,p_2,t), M(p_3,p_4,t)) < \eta,\ \ t > 0.$$

whenever $M(p_1,p_3,\epsilon) > 1-\lambda$ and $M(p_2,p_4,\epsilon) > 1-\lambda$.

Next, with each point $\bar{p}$ in $\overline{A}$, associate a point $p(\bar{p})$ in $\overline{A}$ such that $M(p(\bar{p}), \bar{p}, t) > 1-\lambda$ for every $t > 0$. Then, in view of the above argument, for any pair of the points $\bar{p}$ and $\bar{q}$ in A,

$$L(M(p(\bar{p}), q(\bar{q}), t), M(p,q,t)) < \eta\ t > 0.$$

In particular, for all $t > 0$, we have

$$M(p(\bar{p}), q(\bar{q}), t-\eta) - \eta \leq M(p,q,t).$$

Let $A_\eta = \{p(\bar{p}) : \bar{p} \in \overline{A}\}$. Then since $A_\eta \subseteq A$, we have

$$\begin{aligned} \inf_{\bar{p},\bar{q}\in\overline{A}} M(\bar{p},\bar{q},t) &\geq \inf_{\bar{p},\bar{q}\in\overline{A}} M(p(\bar{p}), q(\bar{q}), t-\eta) - \eta \\ &= \inf_{p,q\in A_n} M(p,q,t-\eta) - \eta \\ &\geq \inf_{p,q\in A} M(p,q,t-\eta) - \eta. \end{aligned}$$

Now, taking the supremum for $t < s$, the above inequality yields

$$
\begin{aligned}
D_{\overline{A}}(s) &= \sup_{t<s}\left[\inf_{\bar p,\bar q\in\overline{A}} M(\bar p,\bar q,t)\right]\\
&\geq \sup_{t>s}\left[\inf_{\bar p,\bar q\in\overline{A}} M(p(\bar p),q(\bar q),t-\eta)\right]-\eta\\
&= \sup_{t<s-\eta}\left[\inf_{p,q\in A_n} M(p,q,t-\eta)\right]-\eta\\
&= D_A(s-\eta)-\eta.
\end{aligned}
$$

Since the above inequality is valid for all η and since D_A is left-continuous, it follows that

$$D_{\overline{A}}(s) \geq D_A(s)$$

for all $s > 0$. Whence $D_{\overline{A}}(s) = D_A(s)$ for all $s > 0$ and so the proof is complete. □

Definition 3.8. *Let A and B be non-empty subsets of X. The fuzzy distance between A and B is the function $M(A,B,.)$ defined by*

$$M(A,B,t) = \sup_{s<t}\Delta\left(\inf_{p\in A}\left[\sup_{q\in B} M(p,q,s)\right], \inf_{q\in B}\left[\sup_{p\in A} M(p,q,s)\right]\right) \tag{5}$$

for all $t \geq 0$.

In establishing the properties of $M(A,B,.)$, we again omit the routine proof.

Theorem 3.9. *The fuzzy distance $M(A,B,.)$ is non-decreasing in t, $M(A,B,0) = 0$, $\sup_{t>0} M(A,B,t) = 1$ and $M(A,B,t)$ is left-continuous in t.*

Theorem 3.10. *If A and B are non-empty subsets of X, then $M(A,B,.) = M(B,A,.)$.*

Theorem 3.11. *If A is non=empty subset of X, then $M(A,A,t) = 1$ for all $t > 0$.*

Theorem 3.12. *If A and B are non-empty subsets of X, then $M(A,B,.) = M(\overline{A},\overline{B},.)$.*

Proof. It is sufficient to show that $M(A,B,.) = M(A,\overline{B},.)$ since this result together with Theorem 3.10 yields

$$M(A,B,.) = M(A,\overline{B},.) = M(\overline{B},\overline{A},.) = M(\overline{A},\overline{B},.).$$

With this in mind, we first show that $M(A, \overline{B}, t) \leq M(A, B, t)$ for all $t > 0$. Since $B \subseteq \overline{B}$, for all $t > 0$,

$$\inf_{q \in B} \left[\sup_{p \in A} M(p, q, t) \right] \geq \inf_{q \in \overline{B}} \left[\sup_{p \in A} M(p, q, t) \right]. \tag{6}$$

Let $\eta > 0$ be given. The argument in the proof of Theorem 3.7 establishes that for each point $\bar{q} \in \overline{B}$, there exists a point $q(\bar{q})$ in B such that for all $t > 0$,

$$M(p, q, t - \eta) - \eta \leq M(p, q(\bar{q}), t).$$

Let $B_\eta = \{q(\bar{q}) : \bar{q} \in B$. Since $B_\eta \subseteq B$, we have

$$\sup_{\bar{q} \in \overline{B}} M(p, \bar{q}, t - \eta) \leq \sup_{\bar{q} \in \overline{B}} M(p, q(\bar{q}), t) = \sup_{q \in B_\eta} M(p, q, t) = \sup_{q \in B} M(p, q, t).$$

Thus, we have

$$\inf_{p \in A} \left[\sup_{\bar{q} \in \overline{B}} M(p, \bar{q}, t - \eta) \right] - \eta \leq \inf_{p \in A} \left[\sup_{q \in B} M(p, q, t) \right].$$

Moreover, taking the supremum on $t < s$, the above inequality yields, for any η,

$$\begin{aligned} f(s) = \sup_{t<s} \left(\inf_{p \in A} \left[\sup_{q \in B} M(p, q, t) \right] \right) &\geq \sup_{t<s} \left(\inf_{p \in A} \left[\sup_{\bar{q} \in \overline{B}} M(p, \bar{q}, t - \eta) \right] \right) - \eta \\ &= \sup_{t<s-\eta} \left(\inf_{p \in A} \left[\sup_{\bar{q} \in \overline{B}} M(p, \bar{q}, t) \right] \right) - \eta \\ &= g(s - \eta) - \eta. \end{aligned}$$

Now since both f and g are left-continuous and η is arbitrary, it follows that $f(s) \geq g(s)$. This together with (6) and the continuity of Δ yields

$$\begin{aligned} M(A, B, s) &= \Delta \left\{ \sup_{t<s} \left(\inf_{p \in A} \left[\sup_{q \in B} M(p, q, t) \right] \right), \sup_{t<s} \left(\inf_{q \in B} \left[\sup_{p \in A} M(p, q, t) \right] \right) \right\} \\ &\geq \Delta \left\{ \sup_{t<s} \left(\inf_{p \in A} \left[\sup_{\bar{q} \in \overline{B}} M(p, \bar{q}, t) \right] \right), \sup_{t<s} \left(\inf_{\bar{q} \in \overline{B}} \left[\sup_{p \in A} M(p, q, t) \right] \right) \right\} \\ &= \sup_{t<s} \Delta \left\{ \left(\inf_{p \in A} \left[\sup_{\bar{q} \in \overline{B}} M(p, \bar{q}, t) \right] \right), \left(\inf_{\bar{q} \in \overline{B}} \left[\sup_{p \in A} M(p, q, t) \right] \right) \right\} \\ &= M(A, \overline{B}, s). \end{aligned}$$

A similar argument shows that $M(A, \overline{B}, t) \geq M(A, B, t)$ for all $t > 0$. Combining these inequalities yields the desired result. This completes the proof. $\square$

Theorem 3.13. *If A and B are subsets of X, Then $M(A,B,t)=1$ for all $t>0$ if and only if $\overline{A}=\overline{B}$.*

Proof. Suppose that $M(A,B,t)=1$ for all $t>0$ and let $\epsilon>0$ be given. Then we have

$$\begin{aligned} M(A,B,\epsilon) &= \Delta\left\{\sup_{t<\epsilon}\left(\inf_{p\in A}\left[\sup_{q\in B}M(p,q,t)\right]\right),\sup_{t<\epsilon}\left(\inf_{q\in B}\left[\sup_{p\in A}M(p,q,t)\right]\right)\right\} \\ &= \sup_{t<\epsilon}\left(\inf_{q\in B}\left[\sup_{p\in A}M(p,q,t)\right]\right)=\inf_{q\in B}\left[\sup_{p\in A}M(p,q,\epsilon)\right]. \end{aligned}$$

So that for any $q\in B$ and every $\lambda>0$ there exists a point $p\in A$ for which $M(p,q,\epsilon)>1-\lambda$. Therefore, q is accumulation point of A and we have $B\subseteq\overline{A}$. A similar argument shows that $A\subseteq\overline{B}$.

Conversely, suppose that $\overline{A}=\overline{B}$. In view of Theorems (3.11) and (3.12), $M(A,B,t)=M(\overline{A},\overline{B},t)=M(\overline{A},\overline{A},t)=1$ for all $t>0$. This completes the proof. □

Theorem 3.14. *If A,B and C are subsets of X, then for all $t,s>0$, we have*

$$M(A,B,t+s)\geq\Delta(M(A,C,t),M(B,C,s)).$$

Proof. Let $u,v>0$ be given. Then, for any triple of points p,q and r in X, we have

$$M(p,q,u+v)\geq\Delta(M(p,r,u),M(r,q,v)).$$

Making use of the continuity and monotonicity of Δ, we have the inequality:

$$\sup_{q\in B}M(p,q,u+v)\geq\Delta\left(\sup_{r\in C}M(p,r,u),\inf_{r\in C}\left[\sup_{q\in B}M(r,q,v)\right]\right).$$

Thus, we have

$$\inf_{p\in A}\left[\sup_{q\in B}M(p,q,u+v)\right]\geq\Delta\left(\inf_{p\in A}\left[\sup_{r\in C}M(p,r,u)\right],\inf_{r\in C}\left[\sup_{q\in B}M(r,q,v)\right]\right).$$

Similarly, we have

$$\inf_{q\in B}\left[\sup_{p\in A}M(p,q,u+v)\right]\geq\Delta\left(\inf_{r\in C}\left[\sup_{p\in A}M(p,r,u)\right],\inf_{q\in B}\left[\sup_{r\in C}M(r,q,v)\right]\right).$$

Therefore, since Δ is associative, we have

$$\begin{aligned} &\Delta\left(\inf_{p\in A}\left[\sup_{q\in B}M(p,q,u+v)\right],\inf_{q\in B}\left[\sup_{p\in A}M(p,q,u+v)\right]\right) \\ &\geq\Delta\left\{\Delta\left(\inf_{r\in C}\left[\sup_{p\in A}M(p,r,u)\right],\inf_{p\in A}\left[\sup_{r\in C}M(r,p,u)\right]\right),\right. \\ &\left.\Delta\left(\inf_{q\in B}\left[\sup_{r\in C}M(r,q,v)\right],\inf_{r\in C}\left[\sup_{q\in B}M(r,q,v)\right]\right)\right\}. \end{aligned}$$

Now arguing as in the last step of the proof of Theorem (3.6), we have

$$\begin{aligned}
M(A,B,t+s) &= \sup_{u+v<t+s} \Delta\left(\inf_{p\in A}\left[\sup_{q\in B} M(p,q,u+v)\right], \inf_{q\in B}\left[\sup_{p\in A} M(p,q,u+v)\right]\right)\\
&\geq \sup_{u<t,v<s} \Delta\left(\inf_{p\in A}\left[\sup_{q\in B} M(p,q,u+v)\right], \inf_{q\in B}\left[\sup_{p\in A} M(p,q,u+v)\right]\right)\\
&= \Delta\left\{\sup_{u<t}\Delta\left(\inf_{p\in A}\left[\sup_{r\in C} M(p,r,u)\right], \inf_{r\in C}\left[\sup_{p\in A} M(p,r,u)\right]\right),\right.\\
&\left.\sup_{v<s}\Delta\left(\inf_{q\in B}\left[\sup_{r\in C} M(q,r,v)\right], \inf_{r\in C}\left[\sup_{q\in B} M(q,r,v)\right]\right)\right\}\\
&= \Delta\left(M(A,C,t), M(C,B,s)\right).
\end{aligned}$$

This achieves the proof. □

Remark 3.15. *Let (X,M,Δ) be a FM-space with a continuous t-norm Δ and let $\mathcal{C}$ be the non-empty collection of non-empty subsets of X. Then the function $M_{\mathcal{C}}$ defined for any A and B in $\mathcal{C}$ by $M_{\mathcal{C}}(A,B,t) = M(A,B,t)$, where $M(A,B,t)$ defined by (5) is a fuzzy set on $\mathcal{C}\times\mathcal{C}\times[0,\infty)$.*

Furthermore, as a direct consequence of Theorems (3.9) $\sim$ (3.14), we have the following:

Theorem 3.16. *If each set in $\mathcal{C}$ is closed, then $(\mathcal{C}, M_{\mathcal{C}}, \Delta)$ is a fuzzy metric space.*

Definition 3.17. *Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ and A be a non-empty subset of X. We define the function $D_A(t)$ by*

$$D_A(t) = \sup_{s<t}\inf_{p,q\in A} D_A(t) \text{ for all } t>0. \tag{7}$$

If $\sup_{t>0} D_A(t) = 1$, then A is called a fuzzy bounded set and $D_A(t)$ the fuzzy diameter of A.

Proposition 3.18. *Let (X,M,Δ) be a fuzzy metric space with a continuous t-norm Δ.*

1. *If A is a fuzzy bounded set, then $D_A(0) = 0$, $\sup_{t>0} D_A(t) = 1$, $D_A(t)$ is non-decreasing in t and left-continuous in t.*

2. *If $A, B \subset X$ are two fuzzy bounded sets, then $A\cup B$ is also fuzzy bounded set of X.*

Proof. (1) This follows directly by using the properties of sup(.) and inf(.).
(2) Since A and B are fuzzy bounded, then so is $B\setminus A$. From [3, Theorem 10], we have

$$D_{A\cup B}(t) = D_{A\cup B\setminus A}(t) \geq \Delta(D_A(t/2), D_{B\setminus A}(t/2)),$$

for all $t > 0$ and so, by the continuity of Δ, we have

$$\begin{aligned}\sup_{t>0} D_{A\cup B}(t) &\geq \sup_{t>0} \Delta(D_A(t/2), D_{B\setminus A}(t/2))\\ &= \Delta(\sup_{t>0} D_A(t/2), \sup_{t>0} D_{B\setminus A}(t/2))\\ &= \Delta(1,1) = 1.\end{aligned}$$

This completes the proof. $\square$

In the rest of this section, we always assume that (X, M, Δ) is a fuzzy metric space with a continuous t-norm Δ and Ω is the the family of all non-empty τ-closed fuzzy bounded sets. We define a mapping $\widetilde{M}(A, B, .)$ at $t \in \mathbb{R}^+$ by

$$\widetilde{M}(A,B,t) = \sup_{s<t} \Delta(\inf_{a\in A} \sup_{b\in B} M(a,b,s), \inf_{b\in B} \sup_{a\in A} M(a,b,s)),\ \ A, B \in \Omega. \tag{8}$$

Then $\widetilde{M}$ is called a fuzzy Hausdorff metric induced by M.

Proposition 3.19. *Let (X, M, Δ) be a fuzzy metric space. Then $\widetilde{M}$ is a fuzzy set on $\Omega \times \Omega \times [0, \infty)$ satisfying the following conditions:*

1. *$\widetilde{M}(A, B, t)$ is non-increasing and left-continuous in t and $\widetilde{M}(A, B, 0) = 0$ and $\sup_{t>0} \widetilde{M}(A, B, t) = 1$.*
2. *$\widetilde{M}(A, B, t) = 1$ for all $t > 0$ if and only if $A = B$ for all $A, B \in \Omega$ and $t > 0$,*
3. *$\widetilde{M}(A, B, 0) = 0$, for all $A, B \in \Omega$ and $t > 0$,*
4. *$\widetilde{M}(A, B, t) = M(B, A, t)$ for all $A, B \in \Omega$ and $t > 0$,*
5. *$\widetilde{M}(A, B, t_1 + t_2) \geq \Delta(\widetilde{M}(A, C, t_1), \widetilde{M}(C, B, t_2))$ for all $A, B, C \in \Omega$ and $t_1, t_2 > 0$.*

and hence $(\Omega, \widetilde{M}, \Delta)$ is a fuzzy metric space.

Proof. By the definition of $\widetilde{M}$, it is easy that $\widetilde{M}(A, B, t)$ is non-increasing and left-continuous in t and $\widetilde{M}(A, B, 0) = 0$. Now, we prove that

$$\sup_{t>0} \widetilde{M}(A, B, t) = 1.$$

In fact, since $A, B \in \Omega$, we have $A \cup B \in \Omega$. By the continuity of Δ, we have

$$\begin{aligned}\sup_{t>0} \widetilde{M}(A,B,t) &= \sup_{t>0}\sup_{s<t} \Delta(\inf_{a\in A}\sup_{b\in B} M(a,b,s), \inf_{b\in B}\sup_{a\in A} M(a,b,s)) \\ &\geq \Delta(\sup_{t>0}\sup_{s<t}\inf_{a\in A}\sup_{b\in B} M(a,b,s), \sup_{t>0}\sup_{s<t}\inf_{b\in B}\sup_{a\in A} M(a,b,s)) \\ &\geq \Delta(\sup_{t>0} D_{A\cup B}(t), \sup_{t>0} D_{A\cup B}(t)) \\ &= \Delta(1,1) = 1.\end{aligned}$$

To show that $\widetilde{M}$ satisfies condition (2), we first suppose that, for all $t > 0$, $\widetilde{M}(A,B,t) = 1$. By the continuity of Δ, for any $s > 0$, we have

$$\begin{aligned}1 = \widetilde{M}(A,B,\epsilon) &= \sup_{s<\epsilon} \Delta(\inf_{a\in A}\sup_{b\in B} M(a,b,s), \inf_{b\in B}\sup_{a\in A} M(a,b,s)) \\ &= \Delta(\sup_{s<\epsilon}\inf_{a\in A}\sup_{b\in B} M(a,b,s), \sup_{s<\epsilon}\inf_{b\in B}\sup_{a\in A} M(a,b,s))\end{aligned}$$

which implies that

$$\sup_{s<\epsilon}\inf_{a\in A}\sup_{b\in B} M(a,b,s) = 1, \tag{9}$$

$$\sup_{s<\epsilon}\inf_{b\in B}\sup_{a\in A} M(a,b,s) = 1. \tag{10}$$

From (9), it follows that $\sup_{b\in B} M(a,b,\epsilon) = 1$ for all $a \in A$. Therefore, for any $a \in A$ and any $\lambda > 0$, there exists $b_* \in B$ such that

$$M(a, b_*, \epsilon) > 1 - \lambda.$$

This shows that the point a is a τ-accumulation point of B and hence $a \in B$, i.e., $A \subseteq B$. Similarly, we can prove that $B \subseteq A$. Therefore, we have $A = B$. Conversely, if $A = B$, then for any $t > 0$ and any $s \in (0,1)$, we have

$$\widetilde{M}(A,B,t) \geq \Delta(\inf_{a\in A}\sup_{b\in B} M(a,b,s), \inf_{b\in B}\sup_{a\in A} M(a,b,s)) \geq \Delta(1,1) = 1.$$

The rest of the proof follows by Theorem 3.14. This ends the proof. □

Definition 3.20. *Let (X, M, Δ) be a fuzzy metric space with a continuous t-norm Δ and let $A \in \Omega$ and $x \in X$. The fuzzy distance between a point x and a set A is the function defined by*

$$M(x,A,t) = \sup_{s<t}\sup_{y\in A} M(x,y,s, \ \textit{for all } t \geq 0. \tag{11}$$

Proposition 3.21. *Let (X, M, Δ) be a fuzzy metric space with a continuous t-norm Δ and let $A \in \Omega$ and x, y be arbitrary points of X. Then*

1. *$M(x,A,t) = 1$ for all $t > 0$ if and only if $x \in A$,*

2. $M(x,A,t1+t_2) \geq \Delta(M(x,y,t_1), M(y,A,t_2))$ *for all* $t_1,t_2 \geq 0$,

3. *For any* $A,B \in \Omega$ *and* $x \in A$,

$$M(x,B,t) \geq \widetilde{M}(A,B,t), \text{ for all } t \geq 0.$$

Proof. (1) If $x \in A$, for any $t > 0$ and $s \in (0,1)$, we have

$$M(x,A,t) \geq \sup_{y\in A} M(x,y,s) \geq M(x,x,s) = 1,$$

which shows that

$$M(x,A,t) = 1,\ t > 0.$$

Conversely, if $M(x,A,t) = 1$ for all $t > 0$, then for any $\epsilon > 0$, we have

$$1 = M(x,A,\epsilon) = \sup_{t<\epsilon}\sup_{y\in A} M(x,y,\epsilon) = \sup_{y\in A} M(x,y,\epsilon).$$

This implies that, for any $\lambda > 0$, there exists $y_* \in A$ such that

$$M(x,y_*,\epsilon) > 1-\lambda.$$

It follows that the point x is a τ-accumulation point of A and so $x \in A$.
(2) By the triangle inequality of M and continuity of Δ, we have

$$\begin{aligned} M(x,A,t_1+t_2) &= \sup_{s_1+s_2<t_1+t_2}\ \sup_{z\in A} M(x,z,s_1+s_2) \\ &\geq \sup_{s_1+s_2<t_1+t_2} \Delta(M(x,y,s_1), \sup_{z\in A} M(y,z,s_2) \\ &\geq (\sup_{s_1<t_1} M(x,y,s_1), \sup_{s_2<t_2}\sup_{z\in A} M(y,z,s_2)) \\ &= \Delta(M(x,y,t_1), M(y,A,t_2)). \end{aligned}$$

(3) If $x \in A$, then we have

$$\begin{aligned} M(A,B,t) &= \sup_{s<t}\sup_{b\in B} M(x,b,s) \\ &\geq \sup_{s<t}\inf_{a\in A}\sup_{b\in B} M(a,b,s) \\ &= \sup_{s<t}\Delta(\inf_{a\in A}\sup_{b\in B} M(a,b,s),1) \\ &\geq \sup_{s<t}\Delta(\inf_{a\in A}\sup_{b\in B} M(a,b,s), \inf_{b\in B}\sup_{a\in A} M(a,b,s)) \\ &= \widetilde{M}(A,B,t),\ t \geq 0. \end{aligned}$$

This ends the proof. $\square$

4 Fixed Point Theorems for Multi-valued Mappings in Fuzzy Metric Spaces

Let (X, M, Δ_m) be a τ-complete fuzzy metric space with $\Delta_m = \min$. Let Ω be the family of all non-empty τ-closed and fuzzy bounded sets, $\widetilde{M}$ be fuzzy Hausdorff metric defined by (8) and $M(x, A, t)$ be the fuzzy distance defined by (11) for all $t > 0$.

Theorem 4.1. *Let T_i be a sequence of multi-valued mappings $T_i : X \to \Omega$, $i = 1, 2, \cdots$. Suppose that there exists a constant $k > 1$ such that for any $i, j \in \mathbb{N}$, and any $x, y \in X$,*

$$\widetilde{M}(T_i x, T_j y, t) \geq \min\{M(x, y, kt), M(x, T_i x, kt), M(y, T_j y, kt)\}, \ t \geq 0. \quad (12)$$

Suppose further that for any $x \in X$ and $a \in T_n x$, $n \in \mathbb{N}$, there exists $b \in T_{n+1} a$ such that

$$M(a, b, t) \geq \widetilde{M}(T_n x, T_{n+1} a, t), \quad t \geq 0. \quad (13)$$

Then there exists a point x_* such that $x_* \in \bigcap_{i=1}^{\infty} T_i x_*$.

Proof. For any $x_0 \in X$, take $x_1 \in T x_0 \in \Omega$. By the assumptions, there exists a point $x_2 \in T x_1 \in \Omega$ such that

$$M(x_1, x_2, t) \geq \widetilde{M}(T_1 x_0, T_2 x_1, t), \quad t \geq 0.$$

Similarly, there exists a point $x_3 \in T_3 x_2$ such that

$$M(x_2, x_3, t) \geq \widetilde{M}(T_2 x_1, T_3 x_2, t), \quad t \geq 0.$$

Continuing this procedure, we can obtain a sequence $\{x_n\}$ in X satisfying the following conditions:

(i) $x_n \in T_n x_{n-1}, n = 1, 2, \cdots,$

(ii) $M(x_n, x_{n+1}, t) \geq \widetilde{M}(T_n x_{n-1}, T_{n+1} x_n, t), \quad t \geq 0.$

It is easy to prove that a sequence $\{x_n\}$ in X is a Cauchy sequence. By the τ-completeness of (X, M, Δ), we can suppose $x_n \xrightarrow{\tau} x_* \in X$.

Now we prove that the point x_* is a common fixed point of $\{T_i\}_{i=1}^{\infty}$. In fact, it follows from Proposition 3.21 that

$$
\begin{aligned}
M(x_*, T_i x_*, t) &\geq \Delta\left(M\left(x_*, x_{n+1}, \left(\left(1-\frac{1}{\beta}\right)t\right)\right), M\left(x_{n+1}, x_*, \left(\frac{t}{\beta}\right)\right)\right) \\
&\geq \Delta\left(M\left(x_*, x_{n+1}, \left(\left(1-\frac{1}{\beta}\right)t\right)\right), M\left(T_{n+1}x_n, T_i x_*, \left(\frac{t}{\beta}\right)\right)\right) \qquad (14) \\
&\geq \Delta\left(M\left(x_*, x_{n+1}, \left(\left(1-\frac{1}{\beta}\right)t\right)\right),\right. \\
&\left.\min\left\{M\left(x_n, x_*, \left(\frac{kt}{\beta}\right)\right), M\left(x_n, T_{n+1}x_n, \left(\frac{kt}{\beta}\right)\right), M\left(x_*, T_i x_*, \left(\frac{kt}{\beta}\right)\right)\right\}\right),
\end{aligned}
$$

where $\beta > k$ is a constant. In addition, by Proposition 3.21, we have

$$
\begin{aligned}
M\left(x_n, T_{n+1}x_n, \left(\frac{kt}{\beta}\right)\right) &\geq \Delta\left(M\left(x_n, x_{n+1}, \left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right), M\left(x_{n+1}, T_{n+1}x_n, \left(\frac{t}{\beta^2}\right)\right)\right) \\
&= M\left(x_n, x_{n+1}, \left(\frac{k}{\beta}-\frac{1}{\beta^2}\right)t\right).
\end{aligned}
$$

Substituting the above inequality into (14) and letting $n \to \infty$, we have, by continuity of Δ,

$$
\begin{aligned}
M(x_*, T_i x_*, t) &\geq M\left(x_*, T_i x_*, \left(\frac{k}{\beta}\right)t\right) \geq M\left(x_*, T_i x_*, \left(\frac{k}{\beta}\right)^2 t\right) \\
&\geq \cdots \geq M\left(x_*, T_i x_*, \left(\frac{k}{\beta}\right)^m t\right), \ m = 1, 2, \cdots.
\end{aligned}
$$

Letting $m \to \infty$ on the right, we have, for all $t > 0$ and $i = 1, 2, \cdots$,

$$M(x_*, T_i x_*, t) = 1$$

and so we have $x_* \in T_i x_*$, $i = 1, 2, \cdots$. Therefore, by Proposition 3.21, we have

$$x_* \in \bigcap_{i=1}^{\infty} T_i x_*.$$

This achieves the proof. □

5 Common Fixed Points Theorems for Multi-valued Mappings in FM-Spaces

Throughout this section, we assume that (X, M, Δ) is a fuzzy metric space with the (ϵ, λ)-topology τ and continuous t-norm Δ. Let $CB(X)$ be the family of all non-empty τ-closed subsets of X and $C(X)$ be the family of all non-empty τ-compact subsets of X.

Definition 5.1. *Let (X, M, Δ) be a fuzzy FM-space, $A, B \in CB(X)$ and $x \in A$. We define $M(x, A, .)$ and $M(A, B, .)$ by*

$$M(x, A, t) = \sup_{y\in A} M(x, y, t),$$
$$M(A, B, t) = \sup_{s<t} \Delta(\inf_{x\in A} \sup_{y\in B} M(x, y, s), \inf_{y\in B} \sup_{x\in A} M(x, y, s))$$

for all $t \in \mathbb{R}^+$, respectively. We say that $M(x, A, .)$ is the fuzzy distance from the point x to the set A and $M(A, B, .)$ is the fuzzy distance from A to B induced by M.

Lemma 5.2. *Let (X, M, Δ) be a fuzzy FM-space with a continuous t-norm Δ, $A \in CB(X)$ and $x, y \in X$. Then we have the following:*

1. *For any $B \in CB(X)$, $x \in A$ and $t \in \mathbb{R}^+$,*
$$\inf_{x\in A} \sup_{y\in B} M(x, y, t) \leq M(x, B, t),$$
2. *$M(x, A, t) = 1$ for all $t > 0$ if and only if $x \in A$,*
3. *$M(x, A, t_1 + t_2) \geq \Delta(M(x, y, t_1), M(y, A, t_2))$ for all $t_1, t_2 > 0$,*
4. *$M(x, A, t)$ is a left-continuous function at t.*

Proof. (1)∼ (3) follows from Proposition 3.21. Noting that

$$\sup_{s<t} \sup_{y\in A} M(x, y, t) = \sup_{y\in A} M(x, y, t)$$

for all $t > 0$, the conclusion is obvious. □

Definition 5.3. *Let $\phi : [0, \infty) \to [0, \infty)$ be a strictly increasing function such that $\phi(0) = 0$ and $\lim_{t\to+\infty} \phi(t) = +\infty$. Define a function $\psi : [0, \infty) \to [0, \infty)$ by*

$$\psi(t) = \begin{cases} 0, & \text{if } t = 0; \\ \inf\{s > 0 : \phi(s) > t\}, & \text{if } t > 0. \end{cases} \tag{15}$$

Definition 5.4. *We say that a function $\phi : [0, \infty) \to [0, \infty)$ satisfies the condition (Φ) if it is strictly increasing and left-continuous function such that $\phi(0) = 0$, $\lim_{t\to+\infty} \phi(t) = +\infty$ and $\sum_{n=0}^{\infty} \phi^n(t) < \infty$ for all $t > 0$.*

Lemma 5.5. *[18] Let $\phi : [0, \infty) \to [0, \infty)$ satisfy the condition (Φ) and let ψ be defined by (15). Then we have the following:*

1. $\phi(t) < t$ *for all* $t > 0$,
2. $\phi(\psi(T)) \leq t$ *and* $\psi(\phi(t)) = t$ *for all* $t \geq 0$,
3. $\psi(t) \geq t$ *for all* $t \geq 0$,
4. $\lim_{n\to\infty} \psi^n(t) = +\infty$ *for all* $t > 0$.

Definition 5.6. *Let* Δ *be a t-norm satisfying the condition*

$$\sup_{0<t<1} \Delta(t,t) = 1.$$

Δ *is said to be of h-type if the family of functions* $\{\Delta^m\}_{m\in\mathbb{N}}$ *is equi-continuouys at* $t = 1$, *where*

$$\Delta^1(t) = \Delta(t,t), \Delta^m(t) = \Delta(t, \Delta^{m-1}(t)),\ t \in [0,1],\ m = 2, 3, \cdots .$$

Lemma 5.7. *Let* (X, M, Δ) *be a fuzzy FM-space, where* Δ *is a t-norm of h-type. If the sequence* $\{x_n\}$ *in* X *satisfies the following condition: for any* $n \in \mathbb{N}$ *and* $t > 0$,

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, \phi^n(\psi(t))), \tag{16}$$

where ϕ *is a function satisfying the condition* (Φ) *and* ψ *is defined by (15). Then* $\{x_n\}$ *is a* τ*-Cauchy sequence in* X.

Proof. Since Δ is a t-norm of h-type, for any $\lambda \in (0,1)$, there exists a $\delta(\lambda) > 0$ such that for any $t > \delta(\lambda)$, $\Delta^m(t) > 1 - \lambda$ for all $m \in \mathbb{N}$. Since ϕ satisfies the condition (Φ), $\sum_{n=0}^{\infty} \phi^n(\psi(t)) < \infty$. Hence for any $t > 0$, there exists $N \in \mathbb{N}$ such that $\sum_{i=n}^{\infty} \phi^i(\psi(t)) < t$ for any $n \geq N$. By (16) and Lemma 5.5, for any $n \geq N$, we have

$$\begin{aligned}
M(x_n, x_{n+m}, t) &\geq M\left(x_n, x_{n+m}, \sum_{i=n}^{\infty} \phi^i(\psi(t))\right) \\
&\geq \Delta\big(M(x_n, x_{n+1}, \phi^n(\psi(t))), \Delta(M(x_{n+1}, x_{n+2}, \phi^{n+1}(\psi(t)))), \cdots, \\
&\quad \Delta(M(x_{n+m-2}, x_{n+m-1}, \phi^{n+m-2}(\psi(t)))), \Delta(M(x_{n+m-1}, x_{n+m}, \phi^{n+m-1}(\psi(t))), \cdots)\big) \\
&\geq \Delta^m(x_0, x_1, \psi(t)) > 1 - \lambda
\end{aligned}$$

for all $m \in \mathbb{N}$, which implies that $\{x_n\}$ is a τ-Cauchy sequence in X. This achieves the proof. $\square$

Now we give our main result in this section:

Theorem 5.8. *Let (X, M, Δ) be a τ-complete fuzzy FM-space, where Δ is a continuous t-norm of h-type. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of multi-valued mappings $T_i : X \to CB(X)$, $i = 1, 2, \cdots$, satisfying the following condition:*

> *for any $i, j \in \mathbb{N}$, $x, y \in X$ and $u \in T_i x$, there exists a point $v \in T_j y$ such that*

$$M(u, v, \phi(t)) \geq \min\{M(x, y, t), M(x, T_i x, t), M(y, T_j y, t)\} \tag{17}$$

for all $t > 0$, where the function ϕ satisfies the condition (Φ). Then the family $\{T_i : i \in \mathbb{N}\}$ of multi-valued mappings has a common fixed point in X, i.e., there exists a point $x_* \in X$ such that $\bigcap_{i=1}^{\infty} T_i x_*$.

Proof. Take $x_0 \in X$ and $x_1 \in Tx_0$. By Lemma 5.5 and the condition (17), there exists $x_2 \in T_2 x_1$ such that

$$\begin{aligned} M(x_1, x_2, t) &\geq M(x_1, x_2, \phi(\psi(t))) \\ &\geq \min\{M(x_0, x_1, \psi(t)), M(x_0, T_1 x_0, \psi(t)), M(x_1, T_2 x_1, \psi(t))\} \\ &\geq \min\{M(x_0, x_1, \psi(t)), M(x_1, x_2, \psi(t))\} \end{aligned} \tag{18}$$

for all $t > 0$, where $\psi(t)$ is defined by (15). Using (18) repeatedly, we have

$$\begin{aligned} M(x_1, x_2, t) &\geq \min\{M(x_0, x_1, \psi(t)), M(x_0, x_1, \psi^2(t)), M(x_1, x_2, \psi^2(t))\} \\ &= \min\{M(x_0, x_1, \psi(t)), M(x_1, x_2, \psi^2(t))\} \\ &\geq \cdots \\ &\geq \min\{M(x_0, x_1, \psi(t)), M(x_1, x_2, \psi^n(t))\}. \end{aligned}$$

Letting $n \to \infty$, we have

$$M(x_1, x_2, t) \geq M(x_0, x_1, \psi(t))$$

for all $t > 0$. Taking this procedure repeatedly, we can define a sequence $\{x_n\}$ in X satisfying

$$x_{n+1} \in T_{n+1} x_n \text{ and } M(x_n, x_{n+1}, t) \geq M(x_n, x_{n-1}, \psi(t))$$

for all $t > 0$. Thus, for any $n \in \mathbb{N}$ and $t > 0$, we have

$$M(x_n, x_{n+1}, t) \geq M(x_n, x_{n-1}, \psi(t)) \geq \cdots \geq M(x_0, x_1, \psi^n(t)). \tag{19}$$

Therefore, by Lemma 5.7, $\{x_n\}$ is a τ-Cauchy sequence in X. Since (X, M, Δ) is τ-complete, we assume that $x_n \xrightarrow{\tau} x_* \in X$.

Next, we prove that x_* is a common fixed point of the family $\{T_i : i \in \mathbb{N}\}$. In fact, for any $t > 0$ and $i \in \mathbb{N}$ and $\epsilon \in (0, 1)$,from (19), we have

$$\begin{aligned} M(x_{n+1}, T_i x_*, t-\epsilon) &\geq M(x_{n+1}, T_i x_*, \phi(\psi(t-\epsilon))) \\ &= \sup_{y \in T_i x} M(x_{n+1}, y, \phi(\psi(t-\epsilon))) \qquad (20) \\ &\geq \min\{M(x_n, x_*, \psi(t-\epsilon)), M(x_n, x_{n+1}, \psi(t-\epsilon)), M(x_*, T_i x_*, \psi(t-\epsilon))\} \\ &\geq \min\{M(x_n, x_*, \psi(t-\epsilon)), M(x_0, x_1, \psi^{n+1}(t-\epsilon)), M(x_*, T_i x_*, \psi(t-\epsilon))\} \end{aligned}$$

If $M(x_*, T_i x_*, \psi(t-\epsilon)) = 1$, then we have

$$M(x_{n+1}, T_i x_*, t-\epsilon) \geq \min\{M(x_n, x_*, \psi(t-\epsilon)), M(x_0, x_1, \psi^{n+1}(t-\epsilon))\}.$$

Letting $n \to \infty$, we have $M(x_*, T_i x_*, t-\epsilon) \geq 1$. by the arbitrariness of $\epsilon \in (0, 1)$, we have $M(x_*, T_i x_*, t) = 1$ for all $t > 0$, i.e., $x_* \in T_i x_*$. thus the conclusion is proved.

If $M(x_*, T_i x_*, \psi(t-\epsilon)) < 1$, then letting $n \to \infty$ in (20), we have

$$M(x_*, T_i x_*, t-\epsilon) \geq M(x_*, T_i x_*, \psi(t)). \qquad (21)$$

By Lemma 5.2, we have

$$M(x_*, T_i x_*, t) \geq \Delta(M(x_*, x_{n+1}, \epsilon), M(x_{n+1}, T_i x_*, t-\epsilon)). \qquad (22)$$

Letting $n \to \infty$ in (22), from (21) and the continuity of Δ, we have

$$M(x_*, T_i x_*, t) \geq M(x_*, T_i x_*, t-\epsilon).$$

Thus, as $\epsilon \to t$, by the continuity of ψ and the left continuity of M, it follows that

$$M(x_*, T_i x_*, t) \geq M(x_*, T_i x_*, \psi(t)).$$

Taking this procedure repeatedly, we obtain

$$M(x_*, T_i x_*, t) \geq M(x_*, T_i x_*, \psi(t)) \geq \cdots \geq M(x_*, T_i x_*, \psi^n(t)).$$

Therefore, as $n \to \infty$, $M(x_*, T_i x_*, t) = 1$ for all $t > 0$, i.e., $x_* \in T_i x_*$. This completes the proof. □

Taking $\phi(t) = kt$, $0 < k < 1$, in Theorem 5.8, we obtain the following result.

Corollary 5.9. *Let (X, M, Δ) be a τ-complete fuzzy FM-space, where Δ is a continuous t-norm of h-type. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of multi-valued mappings $T_i : X \to CB(X)$, $i = 1, 2, \cdots$, satisfying the following condition:*

for any $i, j \in \mathbb{N}$, $x, y \in X$ and $u \in T_i x$, there exists a point $v \in T_j y$ such that

$$M(u, v, kt) \geq \min\{M(x, y, t), M(x, T_i x, t), M(y, T_j y, t)\}$$

for all $t > 0$, where $k \in (0, 1)$ is a constant. Then the family $\{T_i : i \in \mathbb{N}\}$ of multi-valued mappings has a common fixed point in X, i.e., there exists a point $x_* \in X$ such that $\bigcap_{i=1}^{\infty} T_i x_*$.

Corollary 5.10. *Let (X, M, Δ) be a τ-complete fuzzy FM-space, where Δ is a continuous t-norm, K be a non-empty fuzzy bounded subset of X and $T : K \to C(K)$ be a mapping satisfying the following condition: for any $x, y \in K$ and $u \in Tx$, there exists a point $v \in Ty$ such that*

$$M(u, v, kt) \geq M(x, y, t)$$

for all $t > 0$, where $k \in (0, 1)$ is a constant. Then T has a fixed point in K.

6 Open Problem

The open problem here is to construct some new metric sets and study the topological properties of these sets and study some fixed point theorem for multi-valued mappings in fuzzy metric spaces.

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