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Remark On The Paper "The Monotonicity Results And Sharp Inequalities For Some Power-Type Means Of Two Arguments"

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Abstract

In this paper, a solution of the following conjecture is given. For $a, b > 0$ ***with*** $a \neq b$ ***the inequality*** $N < T_p$ ***holds if and only if*** $p \geq 4/5$, ***where*** N, T_p ***are Neuman-Sándor mean, power-type second Seiffert mean, respectivelly.***

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1 Introduction

In the paper [4], the power-type mean $M_p(a, b)$ is defined as:

$$M_p = M^{1/p}(a^p, b^p) \text{ if } p \neq 0 \text{ and } M_0 = \sqrt{ab},$$

where $a, b > 0$, $a \neq b$, $p \in \mathcal{R}$, $M = A$, He, L, I, P, T, N, Z and Y stand for the arithmetic mean, Heronian mean, logarithmic mean, identric (exponential) mean, the first Seiffert mean, the second Seiffert mean, Neuman-Sandor mean, power-exponential mean and exponential-geometric mean, respectivelly. Author proved the power type means P_p, T_p, N_p, Z_p are increasing in p on $\mathcal{R}$ and established sharp inequalities among power-type means $A_p, He_p, L_p, I_p, N_p, Z_p, P_p, Y_p$. Lastly, the following conjecture was proposed.

Conjecture 1.1 *For $a, b > 0$, with $a \neq b$, the inequality $N < T_p$ holds if and only if $p \geq 4/5$.*

2 Preliminaries

The purpose of the paper is to prove the conjecture. First, we recall the definitions of N and T_p.

$T(a, b)$ is the second Seiffert mean defined in (see [4]) as

$$T(a,b) = \frac{a-b}{2\arctan\left(\frac{a-b}{a+b}\right)} \text{ for } a \neq b \text{ and } T(a,a) = a;$$

$N(a, b)$ is the Neuman-Sándor mean defined in (see [4]) as

$$N(a,b) = \frac{a-b}{2\text{arcsinh}\left(\frac{a-b}{a+b}\right)} \text{ for } a \neq b \text{ and } N(a,a) = a;$$

The mean $T_p(a, b)$ is defined as

$$T_p(a,b) = \left(T(a^p, b^p)\right)^{\frac{1}{p}}.$$

For some other details about means, (see [2], [4]) and related references cited there in.

3 Main results

The result of the paper is the following theorem.

Theorem 3.1 *Let $a, b > 0$, $a \neq b$. Then the inequality $N < T_p$ holds if and only if $p \geq 4/5$.*

First, we prove necessity. Without loss of generality, we assume that $0 < b < a$. Denote $x = b/a$. Let $N < T_p$. Using Taylor's theorem for $N(1, x)$ we have

$$N(1,x) = \frac{1-x}{2\text{arcsinh}\left(\frac{1-x}{1+x}\right)} = 1 + \frac{1}{2}(x-1) + \frac{1}{24}(x-1)^2 + u(x)(x-1)^3,$$

where $u(x)$ is a suitable function such that $\lim\limits_{x \to 1^-} u(x) = u(1) \neq \infty$.

Similarly,

$$T_p(1,x) = \left(\frac{1-x^p}{2\arctan\left(\frac{1-x^p}{1+x^p}\right)}\right)^{\frac{1}{p}} = 1+\frac{1}{2}(x-1)+\left(-\frac{1}{8}+\frac{10p}{48}\right)(x-1)^2 + v(p,x)(x-1)^3,$$

where $v(p,x)$ is a suitable function such that $\lim_{x\to 1^-} v(p,x) = v(p,1) \neq \infty$. (Without loss of generality we assume $p>0$.) It implies

$$\lim_{x\to 1^-}\frac{N(1,x)-T_p(1,x)}{(1-x)^2} = \frac{1}{6}-\frac{5p}{24} \le 0.$$

From this $p \ge 4/5$.
Sufficiency. We prove that $N(a,b) < T_{4/5}(a,b)$ for all positive $a \neq b$. From this and from T_p is increasing in p we obtain $N(a,b) < T_p(a,b)$ for $p \ge 4/5$. The inequality $N(a,b) < T_{4/5}(a,b)$ is equivalent to

$$\left(\frac{1-x}{2\mathrm{arcsinh}\left(\frac{1-x}{1+x}\right)}\right)^{4/5} < \frac{1-x^{4/5}}{2\arctan\left(\frac{1-x^{4/5}}{1+x^{4/5}}\right)} \quad \text{for } 0<x<1, \tag{1}$$

which can be rewriting as:

$$2^{1/5}\frac{(1-x)^{4/5}}{1-x^{4/5}}\arctan\left(\frac{1-x^{4/5}}{1+x^{4/5}}\right) < \left(\mathrm{arcsinh}\left(\frac{1-x}{1+x}\right)\right)^{4/5}. \tag{2}$$

We use two following inequalities and one formula. (The first one is evident, the second one see [3], formula follows from [1] 1625,9, NV 59,(6))

$$\begin{aligned} \arctan x &> x-\frac{x^3}{2} \quad \text{for } 0<x, \\ \mathrm{arcsinh} x &\ge \frac{3x}{2+\sqrt{1+x^2}} \quad \text{for } 0<x, \\ \arctan\left(\frac{1-x}{1+x}\right) &= \frac{\pi}{4}-\arctan x \quad \text{for } x>-1. \end{aligned}$$

We divide the proof of the inequality (2) into two cases. The first case is a proof of the inequality

$$2^{1/5}\frac{(1-x)^{4/5}}{1-x^{4/5}}\left(\frac{\pi}{4}-x^{4/5}+\frac{x^{12/5}}{2}\right) < \left(\mathrm{arcsinh}\left(\frac{1-x}{1+x}\right)\right)^{4/5}, \tag{3}$$

for $x \in (0, 0.1>$,
and the second case is a proof of the inequality

$$2^{1/5}\frac{(1-x)^{4/5}}{1-x^{4/5}}\left(\frac{\pi}{4}-\arctan\left(x^{4/5}\right)\right) < \left(\frac{3\left(\frac{1-x}{1+x}\right)}{2+\sqrt{1+\left(\frac{1-x}{1+x}\right)^2}}\right)^{4/5}, \tag{4}$$

for $x \in < 0.1, 1)$. If we show this, the proof will be done.

Proof of (3).

Denote

$$F(x) = \operatorname{arcsinh}\left(\frac{1-x}{1+x}\right) - 2^{1/4}\frac{(1-x)}{(1-x^{4/5})^{5/4}}\left(\frac{\pi}{4} - x^{4/5} + \frac{x^{12/5}}{2}\right)^{5/4}, \qquad (5)$$

for $x \in (0, 0.1 >$.

Because of $F(0) = 0.0021$, $F(0.1) = 0.0031$, $F'(0) = +\infty$, to prove (3) it suffices to show that $F'(x)$ has only one root in $(0, 0.1)$, where

$$\begin{aligned} F'(x) &= \frac{-2}{(1+x)^2\sqrt{1+\left(\frac{1-x}{1+x}\right)^2}} - 2^{1/4}\left[\frac{\left(\frac{\pi}{4} - x^{4/5} + \frac{x^{12/5}}{2}\right)^{1/4}}{(1-x^{4/5})^{10/4}}\times\right. \\ &\times \left(-\frac{\pi}{4} + x^{4/5} - \frac{x^{12/5}}{2} + \frac{5(1-x)}{4}\left(\frac{-4}{5x^{1/5}} + \frac{6x^{7/5}}{5}\right)\right)\times \\ &\times \left.(1-x^{4/5})^{5/4} + \left(\frac{\pi}{4} - x^{4/5} + \frac{x^{12/5}}{2}\right)^{5/4}(1-x)\frac{(1-x^{4/5})^{1/4}}{x^{1/5}}\right]. \end{aligned}$$

$F'(x) = 0$ is equivalent to

$$\begin{aligned} 2x^{4/5}\left(1-x^{4/5}\right)^9 &= (1+x)^4(1+x^2)^2\left(\frac{\pi}{4} - x^{4/5} + \frac{x^{12/5}}{2}\right)\times \\ \left(1 - \frac{\pi}{4} + \frac{\pi}{4}x^{1/5}\right. &- \left. 2x - \frac{3}{2}x^{8/5} + x^{9/5} + x^{12/5} + 2x^{13/5} - \frac{3}{2}x^{17/5}\right)^4, \qquad (6) \end{aligned}$$

for $x \in (0, 0.1 >$. Put $x = t^5$, then $t \in (0, 0.6310 >$. It is evident, that $F'(x) = 0$ has only one root $\in (0, 0.1 >$ if and only if

$$\begin{aligned} L(t) &= \frac{2t^4\left(1-t^4\right)^9}{(1+t^5)^4\left(1+t^{10}\right)^2\left(\frac{\pi}{4} - t^4 + \frac{t^{12}}{2}\right)}\times \\ &\times \frac{1}{\left(1 - \frac{\pi}{4} + \frac{\pi}{4}t - 2t^5 - 1.5t^8 + t^9 + t^{12} + 2t^{13} - 1.5t^{17}\right)^4} = 1 \qquad (7) \end{aligned}$$

has only one solution in $(0, 0.6310 >$. Because of $L(0) = 0$, $L(0.6310) = 1.5614$ it will be done, if we show that $L(t)$ is an increasing function on $(0, 0.6310 >$. Denote $L(t) = 2f_1(t)f_2(t)$, where

$$f_1(t) = \frac{1-t^4}{\frac{\pi}{4} - t^4 + \frac{t^{12}}{2}},$$

$$f_2(t) = \frac{t^4\left(1-t^4\right)^8}{(1+t^5)^4\left(1+t^{10}\right)^2} \times$$
$$\times \frac{1}{\left(1-\frac{\pi}{4}+\frac{\pi}{4}t-2t^5-1.5t^8+t^9+t^{12}+2t^{13}-1.5t^{17}\right)^4}.$$

If we show that f_1 and $\sqrt{f_2}$ are increasing functions, then L will be an increasing function.

Denote $w(u) = (1-u)/\left(\frac{\pi}{4} - u + \frac{u^3}{2}\right)$ for $u \in (0, 0.6310^4 >= (0, 0.1585 >$. Then $w'(u) > 0$ if and only if $s(u) = 1 - \frac{\pi}{4} - \frac{3}{2}u^2 + u^3 > 0$. Because of $s'(u) = 3u(u-1) < 0$, $s(0.1585) = 0.1809$ we have $f_1(t)$ is an increasing function on $(0, 0.6310 >$.

Denote $\sqrt{f_2} = f_3 f_4$ where

$$f_3(t) = \frac{tb^2(t)}{c(t)d(t)f(t)}, \quad f_4(t) = \frac{tb^2(t)}{c(t)f(t)}, \quad b(t) = 1 - t^4, \quad c(t) = 1 + t^5,$$

$$d(t) = 1 + t^{10}, \quad f(t) = 1 - \frac{\pi}{4} + \frac{\pi}{4}t - 2t^5 - 1.5t^8 + t^9 + t^{12} + 2t^{13} - 1.5t^{17}.$$

If we show that f_3 and f_4 are increasing functions, then f_2 will be an increasing function, so L will be an increasing function on $(0, 0.6310 >$. It will be done if we show $f_3' > 0$ and $f_4' > 0$. It is equivalent to prove

$$f_5(t) = (b(t)c(t)d(t) + 2tb'(t)c(t)d(t) - tb(t)c'(t)d(t) - tb(t)c(t)d'(t))f(t) - tb(t)c(t)d(t)f'(t) > 0, \tag{8}$$

$$f_6(t) = (b(t)c(t) + 2tb'(t)c(t) - tb(t)c'(t))f(t) - tb(t)c(t)f'(t) > 0. \tag{9}$$

We prove (8). Some calculations give

$$(b(t)c(t)d(t) + 2tb'(t)c(t)d(t) - tb(t)c'(t)d(t) - tb(t)c(t)d'(t)) = 1 - 9t^4 - 4t^5 - 14t^9 + t^{10} + t^{14} - 14t^{15} + 6t^{19},$$

$$tb(t)c(t)d(t) = t - t^5 + t^6 - t^{10} + t^{11} - t^{15} + t^{16} - t^{20}.$$

It implies

$$f_5(t) = 1 - \frac{\pi}{4} + \left(\frac{9\pi}{4} - 9\right)t^4 + (4-\pi)t^5 - \frac{5\pi}{4}t^6 + \left(\frac{7\pi}{2} - 14\right)t^9 + \frac{21}{2}t^8 + \left(-\frac{7\pi}{2} + 19\right)t^{10} - \frac{19}{2}t^{12} - 6t^{13} + \left(6 - \frac{\pi}{4}\right)t^{14} + (4\pi - 6)\,t^{15} + \left(3 - \frac{15\pi}{4}\right)t^{16} + 25t^{17} - \frac{57}{2}t^{18} - \left(14 + \frac{3\pi}{2}\right)t^{19} + \left(38 + \frac{7\pi}{4}\right)t^{20} - 14t^{21} + 5t^{22} + 19t^{23} - 45t^{24} + \frac{17}{2}t^{26} + 5t^{27} - 39t^{28} - 9t^{31} + \frac{169}{2}t^{32} - \frac{69}{2}t^{36}.$$

Denote $f_5(t) = \alpha_1(t) + t^{20}\alpha_2(t)$ where

$$\begin{aligned}\alpha_1(t) &= 1 - \frac{\pi}{4} + \left(\frac{9\pi}{4} - 9\right)t^4 + (4-\pi)t^5 - \frac{5\pi}{4}t^6 + \frac{21}{2}t^8 + \\ &\left(\frac{7\pi}{2} - 14\right)t^9 + \left(-\frac{7\pi}{2} + 19\right)t^{10} - \frac{19}{2}t^{12} - 6t^{13} + \left(6 - \frac{\pi}{4}\right)t^{14} + \\ &(4\pi - 6)\,t^{15} + \left(3 - \frac{15\pi}{4}\right)t^{16} + 25t^{17} - \frac{57}{2}t^{18} - \left(14 + \frac{3\pi}{2}\right)t^{19} + \\ &34t^{20},\end{aligned}$$

$$\begin{aligned}\alpha_2(t) &= \left(4 + \frac{7\pi}{4}\right) - 14t + 5t^2 + 19t^3 - 45t^4 + \frac{17}{2}t^6 + 5t^7 - 39t^8 - \\ &9t^{11} + \frac{169}{2}t^{12} - \frac{69}{2}t^{16}.\end{aligned}$$

We show $\alpha_1(t) > 0, \alpha_2(t) > O$. Put $\alpha_2(t) = \beta_1(t) + t^6\beta_2(t)$ where

$$\beta_1(t) = \left(4 + \frac{7\pi}{4}\right) - 14t + 5t^2 + 19t^3 - 45t^4$$

and

$$\beta_2(t) = \frac{17}{2} + 5t - 39t^2 - 9t^5 + \frac{169}{2}t^6 - \frac{69}{2}t^{10}.$$

Because of

$\beta_1(0.6310) = 0.2942$, $\beta_1'(0) = -14$, $\beta_1''(t) = 0 \Leftrightarrow t = 0.2778$, $t = -0.0667$, $\beta_1'(0.2778) = -10.6821$ we have $\beta_1(t) > 0$. (We used Cardano's formula and Matlab.)

We have $\beta_2 > 0$ if

$$s(t) = 17 + 10t - 82.6t^2 + 158t^6 > 0.$$

We used $t < 0.6310$. We prove $s(t) > 0$ if we show for example

$$\frac{17 + 10t}{t^2} > 158(1 - t) > 82.6 - 158t^4.$$

This is equivalent to

$$s_1(t) = 17 + 10t - 158t^2 + 158t^3 > 0$$

and

$$s_2(t) = 158 - 82.6 - 158t + 158t^4 > 0.$$

Using Cardano's formula we have $s_1(t) = 0$ only for $t = -0.2675$, which implies $s_1(t) > 0$. Next $s_2(t) = 0$ only for complex number t, which implies $s_2(t) > 0$. So we proved $\alpha_2(t) > 0$.

Put $\alpha_1(t) = \gamma_1(t) + t^{10}\gamma_2(t)$ where

$$\begin{aligned}\gamma_1(t) &= 1 - \frac{\pi}{4} + \left(\frac{9\pi}{4} - 9\right)t^4 + (4-\pi)t^5 - \frac{5\pi}{4}t^6 + \frac{21}{2}t^8 + \\ &\left(\frac{7\pi}{2} - 14\right)t^9 + \left(-\frac{7\pi}{2} + 14.8\right)t^{10},\end{aligned}$$

and

$$\begin{aligned}\gamma_2(t) &= 4.2 - \frac{19}{2}t^2 - 6t^3 + \left(6 - \frac{\pi}{4}\right)t^4 + (4\pi - 6)\,t^5 + \left(3 - \frac{15\pi}{4}\right)t^6 + \\ &25t^7 - \frac{57}{2}t^8 - \left(14 + \frac{3\pi}{2}\right)t^9 + 34t^{10}.\end{aligned}$$

Because of $\gamma_1(0.6310) = 0.00071784$ it suffices to show that $\gamma_1'(t) < 0$. It implies $\gamma_1(t) > 0$. $\gamma_1'(t) < 0$ is equivalent to $\delta_1(t) + t^4\delta_2(t) < 0$ where

$$\delta_1(t) = -36\left(1 - \frac{\pi}{4}\right) + 5(4-\pi)t - \frac{15\pi}{2}t^2 + 90t^4,$$

and

$$\delta_2(t) = -6 + 9\left(\frac{7\pi}{2} - 14\right)t + 10\left(-\frac{7\pi}{2} + 14.8\right)t^2.$$

Using Cardano's formula we have $\delta_1(t) = 0$ only for $t = 0.6330$ and $t = -0.2675$ which implies $\delta_1(t) < 0$. Next $\delta_2(t) = 0$ only if $t = 0.8883$, which implies $\delta_2(t) < 0$. So we proved $\gamma_1(t) > 0$.

Now we prove $\gamma_2(t) > 0$. We show that $\gamma_2'(t) < 0$. It implies the proof because of $\gamma_2(0.6310) = 0.1622$. We have

$$\begin{aligned}&\gamma_2'(t) = th(t) = t\,\Big[-19 - 18t + (24-\pi)\,t^2 + 5\,(4\pi - 6)\,t^3 + \\ &6\left(3 - \frac{15\pi}{4}\right)t^4 + 7 * 25t^5 - 57 * 4t^6 - 9\left(14 + \frac{3\pi}{2}\right)t^7 + 340t^8\Big].\end{aligned}$$

Because of $h(0) = -19$ and $h(0.6310) = -17.2052$ it suffices to show that h is a convex function. Simple computation gives

$$\begin{aligned}&h''(t) = 48 - 2\pi + 30\,(4\pi - 6)\,t + 54\,(4 - 5\pi)\,t^2 + 3500t^3 - \\ &6840t^4 - 378\left(14 + \frac{3\pi}{2}\right)t^5 + 24480t^6.\end{aligned}$$

We see that $h'''(t) = h_1(t) + h_2(t)$ where

$$h_1(t) = 30\,(4\pi - 6) + 108\,(4 - 5\pi)\,t + 2100t^2,$$

$$h_2(t) = 8400t^2 - 27360t^3 - 1890\left(14 + \frac{3\pi}{2}\right)t^4 + 6 * 24480t^5.$$

Using Cardano's formula we have $h_1(t) = 0$ only for complex number t which implies $h_1(t) > 0$. Next $h_2(t) = 0$ only if $t = 0$ or $t = -0.4518$ which implies $h_2(t) > 0$. So we proved $\alpha_1(t) > 0$ which implies $f_5(t) > 0$.

Now we show $f_6(t) > 0$. Simple calculation gives $f_6(t) = g_1(t) + t^{10}g_2(t)$, where

$$g_1(t) = 1 - \frac{\pi}{4} + \left(\frac{9\pi}{4} - 9\right)t^4 + (4-\pi)t^5 - \frac{5\pi}{4}t^6 + \frac{21}{2}t^8 - (4-\pi)\,t^9 + \left(8 - \frac{3\pi}{4}\right)t^{10},$$

$$g_2(t) = 10 - \frac{19}{2}t^2 - 6t^3 - 15t^4 + 3t^6 + 10t^7 - 29t^8 - 4t^{11} + \frac{99}{2}t^{12} - \frac{39}{2}t^{16}.$$

Using $t < 0.6310$ we have

$$g_2(t) > g_4(t) = 10 - \frac{19}{2}t^2 - 16.2t^3.$$

Because of $g_4(0.6310) = 2.1474$ and $g_4'(t) < 0$ we have that $g_2(t) > 0$. Next $g_1(t) = \varepsilon_1(t) + t^8\varepsilon_2(t)$ where

$$\varepsilon_1(t) = 1 - \frac{\pi}{4} + \left(\frac{9\pi}{4} - 9\right)t^4 + (4-\pi)t^5 - \frac{5\pi}{4}t^6 + 10.1t^8,$$

$$\varepsilon_2(t) = 0.4 - (4-\pi)t + \left(8 - \frac{3\pi}{4}\right)t^2.$$

We show $\varepsilon_1(t) > 0$, and $\varepsilon_2(t) > 0$. Because of $\varepsilon_1(0.6310) = 0.0002408$ and $\varepsilon_2(t) = 0$ only for t complex number, it suffices to show $\varepsilon_1'(t) < 0$ which is equivalent to

$$\varepsilon_3(t) = -36\left(1 - \frac{\pi}{4}\right) + 5\,(4-\pi)\,t - \frac{15\pi}{2}t^2 + 80.8t^4 < 0.$$

Using Cardano's formula and Matlab we have $\varepsilon_3(t) = 0$ only for $t = -0.7353$ and $t = 0.6573$ which implies $\varepsilon_1(t) > 0$. So we proved $f_6(t) > 0$.

Now we show the second case which is the inequality (4). (4) is equivalent to

$$\arctan\left(x^{4/5}\right) > \frac{\pi}{4} - \frac{3^{4/5}}{2^{1/5}}\frac{(1 - x^{4/5})}{\left(2(1+x) + \sqrt{2+2x^2}\right)^{4/5}} \tag{10}$$

for $x \in< 0.1, 1)$.

Denote

$$G_1(x) = \arctan\left(x^{4/5}\right) - \frac{\pi}{4} + \frac{3^{4/5}}{2^{1/5}} \frac{(1 - x^{4/5})}{\left(2(1+x) + \sqrt{2+2x^2}\right)^{4/5}}.$$

for $x \in< 0.1, 1)$. Because of $G_1(1) = 0$ it suffices to show that $G_1'(x) < 0$. Put $t = x^5$. We show $G'(t) = \frac{dG_1(x)}{dt} < 0$ for $t \in< 0.6310, 1)$. Simple computation gives that $G'(t) < 0$ is equivalent to

$$\frac{1}{1+t^8} - \left(\frac{81}{2}\right)^{1/5} \frac{\left(2(1+t^5) + \sqrt{2+2t^{10}}\right)^{4/5} + \frac{2t(1-t^4)(t^5+\sqrt{2+2t^{10}})}{\left(2(1+t^5)+\sqrt{2+2t^{10}}\right)^{1/5}\sqrt{2+2t^{10}}}}{\left(2(1+t^5) + \sqrt{2+2t^{10}}\right)^{8/5}} < 0$$

which can be rewriting as

$$\left(2(1+t^5) + \sqrt{2+2t^{10}}\right)^9 \left(\sqrt{2+2t^{10}}\right)^5 < 81 \star 16 \left(1+t^8\right)^5 \times \left[(1+t)\sqrt{2+2t^{10}} + 1 + t^6\right]^5 \tag{11}$$

for $t \in< 0.6310, 1)$.

We distinguish two cases. The first is (11) for $t \in< 0.6310, 0.67 >$, the second is (11) for $t \in< 0.67, 1 >$.

Proof of (11) for $t \in< 0.6310, 0.67 >$.

We use two following inequalities, which are evident.

$$\sqrt{1+t^{10}} < 1 + 0.5t^{10}, \quad \sqrt{1+t^{10}} > 1 + 0.4976t^{10} > 1 + 0.4362t^{10}$$

for $t \in< 0.6310, 0.67 >$. (11) will be proved if we show for example

$$8\left(\sqrt{2}(1+t^5) + 1 + \frac{t^{10}}{2}\right)^9 \left(1 + \frac{t^{10}}{2}\right)^5 < 81\left(1+t^8\right)^5 \times \left[(1+t)\sqrt{2}(1+0.4362t^{10}) + 1 + t^6\right]^5. \tag{12}$$

Denote

$$\begin{aligned} a(t) &= \sqrt{2}(1+t^5) + 1 + \frac{t^{10}}{2}, \quad b(t) = 1 + \frac{t^{10}}{2}, \\ c(t) &= 1 + t^8, \quad d(t) = (1+t)\sqrt{2}(1+0.4362t^{10}) + 1 + t^6, \\ H(t) &= \frac{8a^9b^5}{81c^5d^5}. \end{aligned}$$

Now we prove $H(t) < 1$ for $t \in < 0.6310, 0.67 >$. Using $2.5607 < a(t) < 2.6143$ for $t \in < 0.6310, 0.67 >$ it suffices to show that

$$8a(t)^{10}b(t)^5 < 81 \star 2.5607c(t)^5 d(t)^5,$$

which is equivalent to

$$O(t) = \frac{a(t)^2 b(t)}{c(t)d(t)} < 1.917. \qquad (13)$$

Using $0.6310 < t < 0.67$ we have that (13) will be done if we prove $S(t) = a(t)^2 - 1.8996c(t)d(t) < 0$. Simple calculations give

$$\begin{aligned} S(t) &< -0.1717 - 2.6863t + 6.8286t^5 - 1.8996t^6 - 4.585t^8 - 2.6863t^9 + \\ &\quad 3.2425t^{10} - 1.8996t^{14} - 1.1717t^{11} + \sqrt{2}t^{15} - 1.17171t^{18} - \\ &\quad 1.17171t^{19} + 0.25t^{20}. \end{aligned}$$

Using $6.8286t^5 < 1.376t$, $3.2425t^{10} < 2.1725t^9$ we have $S(t) < 0$.

Proof of (11) for $t \in < 0.67, 1 >$.

Denote

$$\begin{aligned} f^*(t) &= 2(1+t^5) + \sqrt{2+2t^{10}}, \\ g^*(t) &= \frac{(1+t^8)\left[(1+t)\sqrt{2+2t^{10}} + 1 + t^6\right]}{\sqrt{2+2t^{10}}}, \end{aligned}$$

then (11) is equivalent to $\lambda(t) = (f^*(t))^9/(g^*(t))^5 < 81 \star 16$.

Because of $\lambda(1) = 81 * 16$ it suffices to show that $\lambda'(t) > 0$ for $t \in < 0.67, 1)$. This is equivalent to $\phi(t) = 9f^{*'}(t)g^*(t) - 5f^*(t)g^{*'}(t) > 0$.

Simple calculations give

$$\begin{aligned} f^{*'}(t) &= 10t^4 + \frac{10t^9}{\sqrt{2+2t^{10}}}, \\ g^{*'}(t) &= 1 + 8t^7 + 9t^8 + \frac{8t^7(1+t^6)}{\sqrt{2+2t^{10}}} + \frac{(1+t^8)(12t^5 - 10t^9 + 2t^{15})}{(2+2t^{10})\sqrt{2+2t^{10}}}. \end{aligned}$$

$\phi(t) > 0$ is equivalent to $\sqrt{2+2t^{10}}\alpha^*(t) + \beta^*(t) > 0$ where

$$\begin{aligned} \alpha^*(t) &= -2 + 18t^4 + 10t^5 - 24t^7 - 18t^8 + 14t^9 - 2t^{10} + 2t^{12} - 14t^{13} + \\ &\quad 18t^{14} + 24t^{15} - 10t^{17} - 18t^{18} + 2t^{22}, \\ \beta^*(t) &= -2 + 18t^4 - 12t^5 - 32t^7 - 18t^8 + 28t^9 + 20t^{10} + 12t^{12} - 28t^{13} + \\ &\quad 28t^{14} - 2t^{15} - 20t^{17} - 28t^{18} + 18t^{19} + 32t^{20} + 12t^{22} - 2t^{23} + 2t^{27}. \end{aligned}$$

Using $\sqrt{2+2t^{10}} \geq \sqrt{2+2\star 0.67^{10}} = 1.4270$ we show that

$$\chi(t) = 1.4270\alpha^*(t) + \beta^*(t) > 0.$$

Simple calculations give

$$\begin{aligned}\chi(t) &= -4.854 + 43.686t^4 + 2.27t^5 - 66.248t^7 - 43.686t^8 + 47.978t^9 + \\ &\quad 17.146t^{10} + 14.854t^{12} - 47.978t^{13} + 53.686t^{14} + 32.248t^{15} - 34.27t^{17} \\ &\quad -53.686t^{18} + 18t^{19} + 32t^{20} + 14.854t^{22} - 2t^{23} + 2t^{27}.\end{aligned}$$

It is evident that $\chi(t) > m_1(t) + t^{10}m_2(t) > 0$, where

$$\begin{aligned}m_1(t) &= -4.854 + 43.686t^4 + 2.27t^5 - 66.248t^7 - 43.686t^8 + 47.978t^9 + \\ &\quad 21.146t^{10}, \\ m_2(t) &= -4 + 14.854t^2 - 47.978t^3 + 53.686t^4 + 32.248t^5 - 34.27t^7 - \\ &\quad 53.686t^8 + 18t^9 + 30t^{10}.\end{aligned}$$

We prove $m_1(t) > 0$, and $m_2(t) > 0$.

Because of $m_2(0.67) = 0.1897$ it suffices to show $m_2'(t) > 0$. We have

$$\begin{aligned}m_2'(t) &= t(29.708 - 143.934t + 214.744t^2 + 161.24t^3 - 239.89t^5 - \\ &\quad 429.488t^6 + 162t^7 + 300t^8) = t[\mu_1(t) + t^4\mu_2(t)]\end{aligned}$$

where

$$\begin{aligned}\mu_1(t) &= 29.708 - 143.934t + 214.744t^2 + 161.24t^3 - 261t^4, \\ \mu_2(t) &= 261 - 239.89t - 429.488t^2 + 162t^3 + 300t^4.\end{aligned}$$

Using Cardano's formula and Matlab we have $\mu_1(t) = 0$ only for $t = -0.9618$ and $t = 1.0027$. Next $\mu_2(t) = 0$ only for complex numbers, which implies $m_2(t) > 0$. Now we show $m_1(t) > 0$. Simple calculation gives

$$\begin{aligned}m_1'(t) &= t^3\varrho_1(t) = t^3(174.740 + 11.35t - 463.736t^3 - 349.488t^4 + \\ &\quad 431.802t^5 + 211.46t^6), \\ m_1''(t) &= t^2\varrho_2(t) = t^2(524.232 + 45.4t - 2782.4t^3 - 2446.4t^4 + 3454.4t^5 + \\ &\quad 1903.1t^6), \\ \varrho_1'(t) &= 11.35 - 3 * 463.736t^2 - 4 * 349.488t^3 + 5 * 431.802t^4 + \\ &= +6 * 211.46t^5, \\ \varrho_1''(t) &= t(-6 * 463.736 - 12 * 349.488t + 20 * 431.802t^2 + 30 * 211.46t^3), \\ \varrho_2'(t) &= 45.4 - 3 * 2782.4t^2 - 4 * 2446.4t^3 + 5 * 3454.4t^4 + 6 * 1903.1t^5, \\ \varrho_2''(t) &= t(-6 * 2782.4 - 12 * 2446.4t + 20 * 3454.4t^2 + 30 * 1903.1t^3).\end{aligned}$$

We show that

$$m_1(t) \begin{cases} I. & \text{is a concave function on } < 0.67, 0.9 > \\ II. & \text{is a decreasing function on } < 0.9, 0.9698 > \\ III. & \text{is a convex function on } < 0.91, 1 > \\ IV. & \text{is an increasing function on } < 0.97, 1 > . \end{cases}$$

Because of

$$\begin{aligned} & m_1(0.67) = 0.1574,\ m_1(0.9) = 0.618,\ m_1(0.968) = 0.0723, \\ & m_1(0.97) = 0.0723,\ m_1(1) = 0.292,\ m_1(0.91) = 0.5082, \\ & m_1'(0.9698) = -0.0247,\ m_1'(0.97) = 0.0532,\ \varrho_2'(0.91) = 4728.8, \\ & \varrho_2(0.67) = -136.6314,\ \varrho_2(0.9) = -17.1866,\ \varrho_1'(0.9) = 31.0811, \\ & \varrho_1'(0.97) = 427.3728, \\ & \varrho_2'' = 0 \text{ if and only if } t = -1.4268, t = -0.3571, t = 0.5739 \\ & \text{(we used Cardano's formula)}, \\ & \varrho_1'' = 0 \text{ if and only if } t = -1.6031, t = -0.4160, t = 0.6577 \\ & \text{(we used Cardano's formula)}, \end{aligned}$$

and the tangent line $y = 0.0723 - 0.0247(t - 0.9698)$ to $m_1(t)$ at the point $[0.9698, 0.0723]$ is equal to 0 for $t = 3.8969$ we get $m_1(t) > 0$. The proof is complete.

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