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Fractional reduced differential transform method for numerical computation of a system of linear and nonlinear fractional partial differential equations

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Abstract

This paper presents an alternative numerical computation of a system of linear and nonlinear fractional partial differential equations obtained by employing fractional reduced differential transform method (FRDTM), where Caputo type fractional derivative is taken. The effectiveness and convergence of FRDTM is tested by means of four problems, which indicate the validity and great potential of the FRDTM for solving system of fractional partial differential equations.

Keywords: *System of nonlinear fractional partial differential equations, Caputo time-fractional derivatives, Mittag-Leffler function, fractional reduced differential transform method, coupled viscous Burgers equation*

1 Introduction

Fractional differential equation have achieved great attention among researchers due to its wide range of applications in various meaningful phenomena in fluid mechanics, electrical networks, signal processing, diffusion, reaction processes and other fields of science and engineering [1]–[6], among them, the non-linear oscillation of earthquake can be modeled with fractional derivatives [7], the fluid-dynamic traffic model with fractional derivatives [8] can eliminate the deficiency arising from the assumption of continuum traffic flow, fractional non linear complex model for seepage flow in porous media in [9].

Keeping all this in mind, a lot of vigorous techniques has been introduced for getting an approximate solution of such type of fractional differential equations, among others, generalized differential transform method [10], variation iteration method (VIM) [11],[32],[34], local fractional variation iteration method [12], modified Laplace decomposition method [13], reproducing kernel Hilbert space method [14], homotopy analysis method (HAM) [15]-[17], [31], Adomian decomposition method [18], homotopy perturbation method (HPM) [30], [33] and homotopy perturbation Sumudu transform method [19].

Keskin and Oturanc proposed reduced differential transform method (RDTM) for finding approximate analytic solutions of partial differential equations [20]. After, seminal work of Keskin, FRDTM has been adopted to solve vigorous type differential equations arising in mathematics, physics and engineering [21]–[28]. The initial valued system of time-fractional partial differential equation has been solved by many research articles, see [29], [35]-[41].

The main aim of this paper is to present an implementation of fractional reduced differential transform (FRDT) method to compute an alternative approximate solution of initial valued autonomous system of linear and nonlinear fractional partial differential equations.

2 Background

The basic preliminaries on fractional calculus as appeared in [1], [2] is revisited to complete this work.

Definition 1 Let $\mu \in \mathbb{R}$, $m \in \mathbb{N}$. A function $f:\mathbb{R}^{+} \to \mathbb{R}$ belongs to the space C_{μ} if there exists a real number $k \in \mathbb{R}$ with $k > \mu$ such that $f(t) = t^{p} g(t)$, where $g \in C[0,\infty]$. Moreover, $C_{\alpha} \subset C_{\beta}$ whenever $\beta \le \alpha$ and $f \in \mathbb{C}_{\mu}^{m}$ if $f^{(m)} \in \mathbb{C}_{\mu}$.

Definition 2 Let J_{t}^{α} be Riemann – Liouville fractional integral operator and let $f \in \mathbb{C}_{\mu}$, then

(*): $J_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} f(\tau)\, d\tau,\ \alpha > 0$

(**): $J_t^0 f(t) = f(t)$, where $\Gamma(z) := \int_0^{\infty} e^{-t} t^{z-1} dt, z \in \mathbb{C}.$

Moreover, if $\lceil \alpha \rceil = m$, $m \in \mathbb{N}$, $f \in C_{\mu}^{m}$ $(\mu \geq -1), \alpha, \beta \geq 0\, and\, \gamma > -1$, then the operator J_x^{α} satisfy the following properties:

i) $J_x^{\alpha} J_x^{\beta} f(x) = J_x^{\alpha+\beta} f(x) = J_x^{\beta} J_x^{\alpha} f(x),$

ii) $J_x^{\alpha} x^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma+\alpha)} x^{\gamma+\alpha}$, x>0.

Caputo and Mainardi [4] developed a modified fractional differentiation operator D_x^{α} to overcome the discrepancy of Riemann-Liouville derivative.

Definition 3 If $m-1 < \alpha \leq m,\ m \in \mathbb{N}, t > 0$, then Caputo fractional derivative of $f \in C_{\mu}$ [4] is read as:

(1) $D_x^{\alpha} f(x) = J_x^{m-\alpha} D_x^m f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^{\mathrm{x}} (\mathrm{x\text{-}t})^{m-\alpha-1} f^{(m)}(t)\, \mathrm{dt}.$

The basic properties of D_t^{α} are as follows:

Lemma 1 If $m-1 < \alpha \leq m,\ m \in N$ and $f \in C_{\mu}^{m},\ \mu \geq -1$, then

a) $D_t^{\alpha} D_t^{\beta} f(t) = D_t^{\alpha+\beta} f(t) = D_t^{\beta} D_t^{\alpha} f(t),$

b) $D_t^{\alpha} t^{\gamma} = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)} t^{\gamma-\alpha}$, t>0

c) $D_t^{\alpha} J_t^{\alpha} f(t) = f(t)$, t>0,

d) $J_t^{\alpha} D_t^{\alpha} f(t) = f(t) - \sum_{k=0}^{m} f^{(k)}(0^+) \frac{t^k}{k!}$, t>0,

For details study of fractional derivatives we refer the readers to [1-6].

3 FRDT method

This section concerned with the discussion of some basic results as in [20]-[28], on fractional reduced differential transform to complete the paper. Throughout the paper, we denote the original function by $\phi(x,t)$ (lowercase) while it's fractional reduced differential transform (FRDT) by $\Phi_k(x,t)$ *(uppercase).*

Definition 4 *FRDT (spectrum) of an analytic and continuously differentiable function* $w(x,t)$ *is defined by*

(2) $W_k(x) = \frac{1}{\Gamma(k\alpha+1)}\left\{D_t^{\alpha k} w(x,t)\right\}_{t=t_0}$

where α *is order of fractional derivative. The inverse FRDT of* $W_k(x)$ *is defined as follows*

(3) $w(x,t) = \sum_{k=0}^{\infty} W_k(x)(t-t_0)^{k\alpha}$.

From Eq. (2) and (3), one get

(4) $w(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left\{D_t^{\alpha k} w(x,t)\right\}_{t=t_0} (t-t_0)^{k\alpha}$.

In particular for $t_0 = 0$ *, we get*

(5) $w(x,t) = \sum_{k=0}^{\infty} W_k(x) t^{k\alpha} = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}\left\{D_t^{\alpha k} w(x,t)\right\}_{t=0} t^{k\alpha}$.

Definition 5 *The Mittag-Leffler function* $E_\alpha(z)$ *with* $\alpha > 0$ *is defined by*

(6) $\mathrm{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}$

It is *valid in the whole complex plane, and is an advanced form of* $\exp(z)$ **a***nd* $\exp(z) = \lim_{\alpha\to 1} \mathrm{E}_\alpha(z)$.

Theorem 1 Let $U_k(x)$ and $V_k(x)$ be spectrum of analytic and continuously differentiable function $u(x,t)$ and $v(x,t)$ respectively, then

a) If $w(x,t) = u(x,t)v(x,t)$, *then*

$$W_k(x) = U_k(x) \otimes V_k(x) = \sum_{r=0}^{k} U_r(x) V_{k-r}(x).$$

b) If $w(x,t) = \ell_1 u(x,t) \pm \ell_2 v(x,t)$, *then*

$$W_k(x) = \ell_1 U_k(x) \pm \ell_2 V_k(x).$$

c) If $\psi(x,t) = u(x,t)v(x,t)w(\mathrm{x,t})$, *then*

$$\Psi_k(x) = U_k(x) \otimes V_k(x) \otimes W_k(x) = \sum_{r=0}^{k}\sum_{i=0}^{r} U_i(x)\, V_{r-i}(x) W_{k-r}(x).$$

d) If $\psi(x,t) = D_t^{N\alpha} u(x,t)$, *then* $\Psi_k(x) = \dfrac{\Gamma(1+(k+N)\alpha)}{\Gamma(1+k\alpha)} U_{k+N}(x)$.

e) If $\theta(x,t) = x^m t^n \psi(x,t)$, *then*

$$\Theta_k(x) = \begin{cases} x^m \Psi_{k\alpha-n}(x), & \text{if } k\alpha \geq n \\ 0, & \text{else.} \end{cases}$$

f) If $\theta(x,t) = x^m t^n$, *then*

$$\Theta_k(x) = x^m \delta(k\alpha - n),$$

where δ *is defined by* $\delta(k) = \begin{cases} 1 & \text{if } \ k=0 \\ 0 & \text{otherwise} \end{cases}.$

4 Numerical study

This section deals with the main goal of the paper, is to obtain approximate analytical solution of initial valued autonomous systems of linear and nonlinear FPDEs, by adopting FRDTM.

Problem 1 Consider the following initial valued autonomous system of the linear fractional partial differential equations with $(0 < \alpha, \beta < 1)$ as:

$$\text{(7)} \qquad \begin{cases} D_{*t}^{\alpha} u(x,t) - v_x(x,t) + v(x,t) + u(x,t) = 0 \\ D_{*t}^{\beta} v(x,t) - u_x(x,t) + v(x,t) + u(x,t) = 0 \\ u(x,0) = \sinh(x)\ , \ \ v(x,0) = \cosh(x) \end{cases}$$

FRDTM on Eq. (4.1) leads the following recurrence relation

$$\text{(8)} \qquad \begin{cases} \dfrac{\Gamma(1+(1+k)\alpha)}{\Gamma(1+k\alpha)} U_{k+1}(x,t) = \dfrac{d}{dx} V_k(x,t) - V_k(x,t) - U_k(x,t) \\ \dfrac{\Gamma(1+(1+k)\beta)}{\Gamma(1+k\beta)} V_{k+1}(x,t) = \dfrac{d}{dx} U_k(x,t) - V_k(x,t) - U_k(x,t) \\ U_0(x) = \sinh(x), V_0(x) = \cosh(x) \end{cases}$$

On solving the recurrence relation (8), we

$$U_1(x)=\frac{(-1)}{\Gamma(1+\alpha)}\cosh(x); \qquad V_1(x)=\frac{(-1)}{\Gamma(1+\beta)}\sinh(x)$$

$$U_2(x)=\frac{(-1)^2}{\Gamma(1+2\alpha)}\sinh(x); \qquad V_2(x)=\frac{(-1)^2}{\Gamma(1+2\beta)}\cosh(x)$$

get $U_3(x)=\dfrac{(-1)^3}{\Gamma(1+3\alpha)}\cosh(x)$; $\qquad V_3(x)=\dfrac{(-1)}{\Gamma(1+3\beta)}\sinh(x)$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

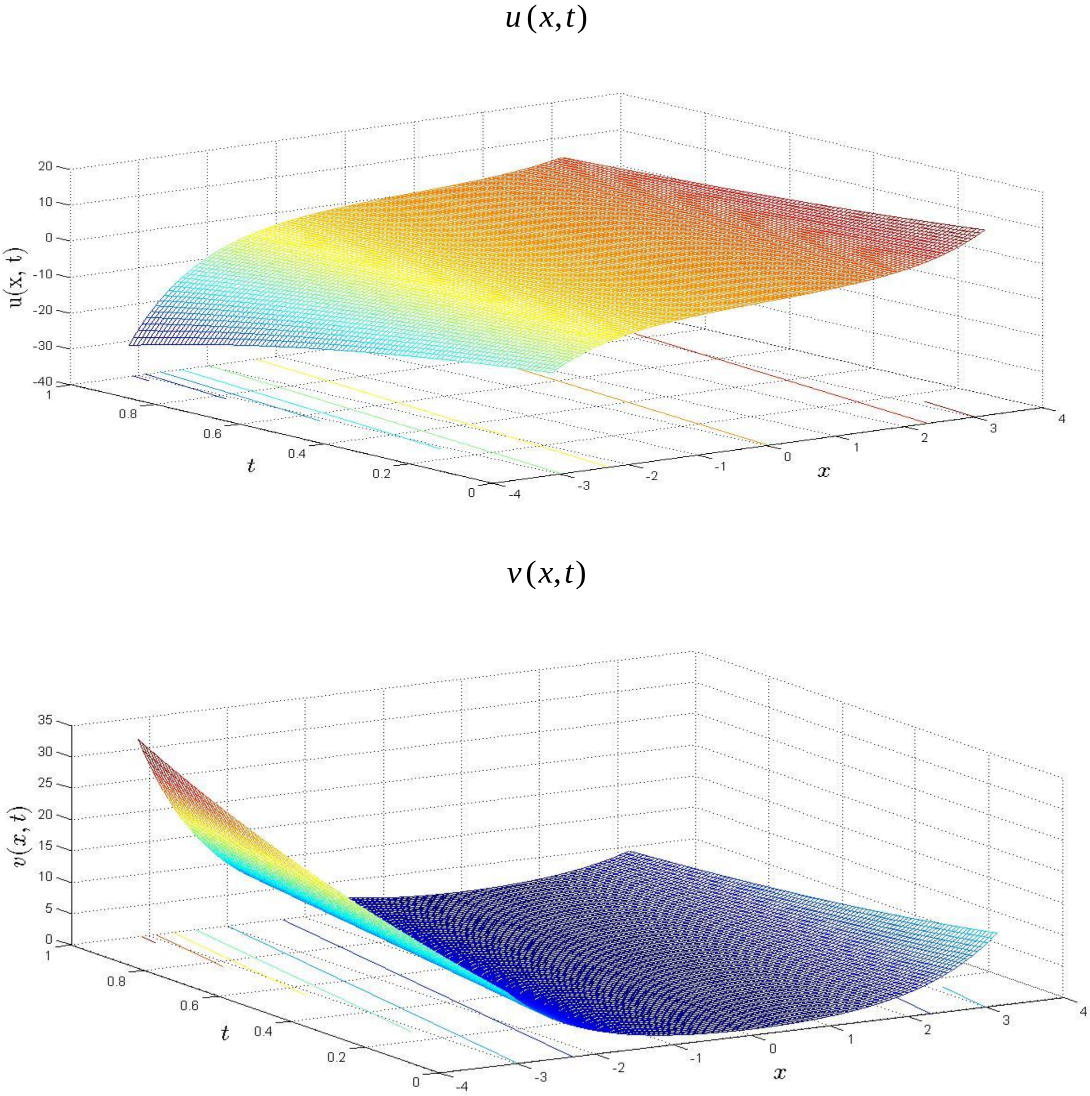


Fig. 1: The solution behavior of u,v of the IVS (7) in the computational domain $(-\pi,\pi)$.

By using inverse FRDTM, we have

(9) $$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{\alpha k}=\sinh(x)+\frac{(-1)}{\Gamma(1+\alpha)}\cosh(x)t^{\alpha}+\frac{(-1)^2}{\Gamma(1+2\alpha)}\sinh(x)t^{2\alpha}+\cdots$$
$$=\sinh(x)\left(1+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots\right)-\cosh(x)\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots\right).$$

(10) $$v(x,t)=\sum_{k=0}^{\infty}V_k(x)t^{\beta k}=\cosh(x)+\frac{(-1)}{\Gamma(1+\beta)}\sinh(x)t^{\beta}+\frac{(-1)^2}{\Gamma(1+2\beta)}\cosh(x)t^{2\beta}+\cdots$$
$$=\cosh(x)\left(1+\frac{t^{2\beta}}{\Gamma(1+2\beta)}+\cdots\right)-\sinh(x)\left(\frac{t^{\beta}}{\Gamma(1+\beta)}+\frac{t^{3\beta}}{\Gamma(1+3\beta)}+\cdots\right).$$

This is the required exact solution of system of linear fractional partial differential equations (7).

Moreover, for $\alpha=\beta=1$, Eq. (11) & (12) reduces to

(11) $$u(x,t)=\sinh(x-t); \qquad v(x,t)=\cosh(x-t).$$

The same exact solution is obtained by employing HAM [31], VIM [32] and HPM [33]. The solution behavior of u, v of the IVS (7) with $\alpha=\beta=1$ is depicted in Fig. 1.

Example 2 Consider the following initial valued nonlinear autonomous system of FPDEs:

(12) $$\begin{cases} D_{*t}^{\alpha}u+v_x w_y-v_y w_x=-u \\ D_{*t}^{\beta}v+u_x w_y+u_y w_x=v \\ D_{*t}^{\gamma}w+u_x v_y+u_y v_x=w \\ u(x,y,0)=e^{x+y}\ ,v(x,y,0)=e^{x-y}\ ,w(x,y,0)=e^{-x+y},\ 0<\alpha,\beta,\gamma<1 \end{cases}$$

FRDTM on Eq. (12) leads the following recurrence relation

(13) $$\begin{cases} \dfrac{\Gamma\{1+(1+k)\alpha\}}{\Gamma(1+k\alpha)}U_{k+1}(x,y)=-U_k-\sum\limits_{i=0}^{k}\left\{\left(\dfrac{\partial}{\partial x}V_i\right)\left(\dfrac{\partial}{\partial y}W_{k-i}\right)-\left(\dfrac{\partial}{\partial y}V_i\right)\left(\dfrac{\partial}{\partial x}W_{k-i}\right)\right\} \\ \dfrac{\Gamma\{1+(1+k)\beta\}}{\Gamma(1+k\beta)}V_{k+1}(x,y)=V_k-\sum\limits_{i=0}^{k}\left\{\left(\dfrac{\partial}{\partial x}U_i\right)\left(\dfrac{\partial}{\partial y}W_{k-i}\right)-\left(\dfrac{\partial}{\partial y}U_i\right)\left(\dfrac{\partial}{\partial x}W_{k-i}\right)\right\} \\ \dfrac{\Gamma\{1+(1+k)\gamma\}}{\Gamma(1+k\gamma)}W_{k+1}(x,y)=W_k-\sum\limits_{i=0}^{k}\left\{\left(\dfrac{\partial}{\partial x}U_i\right)\left(\dfrac{\partial}{\partial y}V_{k-i}\right)-\left(\dfrac{\partial}{\partial y}U_i\right)\left(\dfrac{\partial}{\partial x}V_{k-i}\right)\right\} \\ U_0(x,y)=e^{x+y}, \qquad V_0(x,y)=e^{x-y}, \qquad W_0(x,y)=e^{-x+y} \end{cases}$$

On solving the system (13), we get

$$
\begin{aligned}
&U_1(x,y)=\frac{(-1)}{\Gamma(1+\alpha)}e^{x+y}\ ,\quad V_1(x,y)=\frac{1}{\Gamma(1+\beta)}e^{x-y}\ ,W_1(x,y)=\frac{1}{\Gamma(1+\gamma)}e^{-x+y}\\
&U_2(x,y)=\frac{(-1)^2}{\Gamma(1+2\alpha)}e^{x+y}, V_2(x,y)=\frac{1}{\Gamma(1+2\beta)}e^{x-y}, W_2(x,y)=\frac{1}{\Gamma(1+2\gamma)}e^{-x+y}\\
&U_3(x,y)=\frac{(-1)^3}{\Gamma(1+3\alpha)}e^{x+y}, V_3(x,y)=\frac{1}{\Gamma(1+3\beta)}e^{x-y}, W_3(x,y)=\frac{1}{\Gamma(1+3\gamma)}e^{-x+y}\\
&\vdots \qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots\\
&U_k(x,y)=\frac{(-1)^k}{\Gamma(1+k\alpha)}e^{x+y}, V_k(x,y)=\frac{1}{\Gamma(1+k\beta)}e^{x-y}, W_k(x,y)=\frac{1}{\Gamma(1+k\gamma)}e^{-x+y} \qquad \forall\ k\geq 1.
\end{aligned}
$$

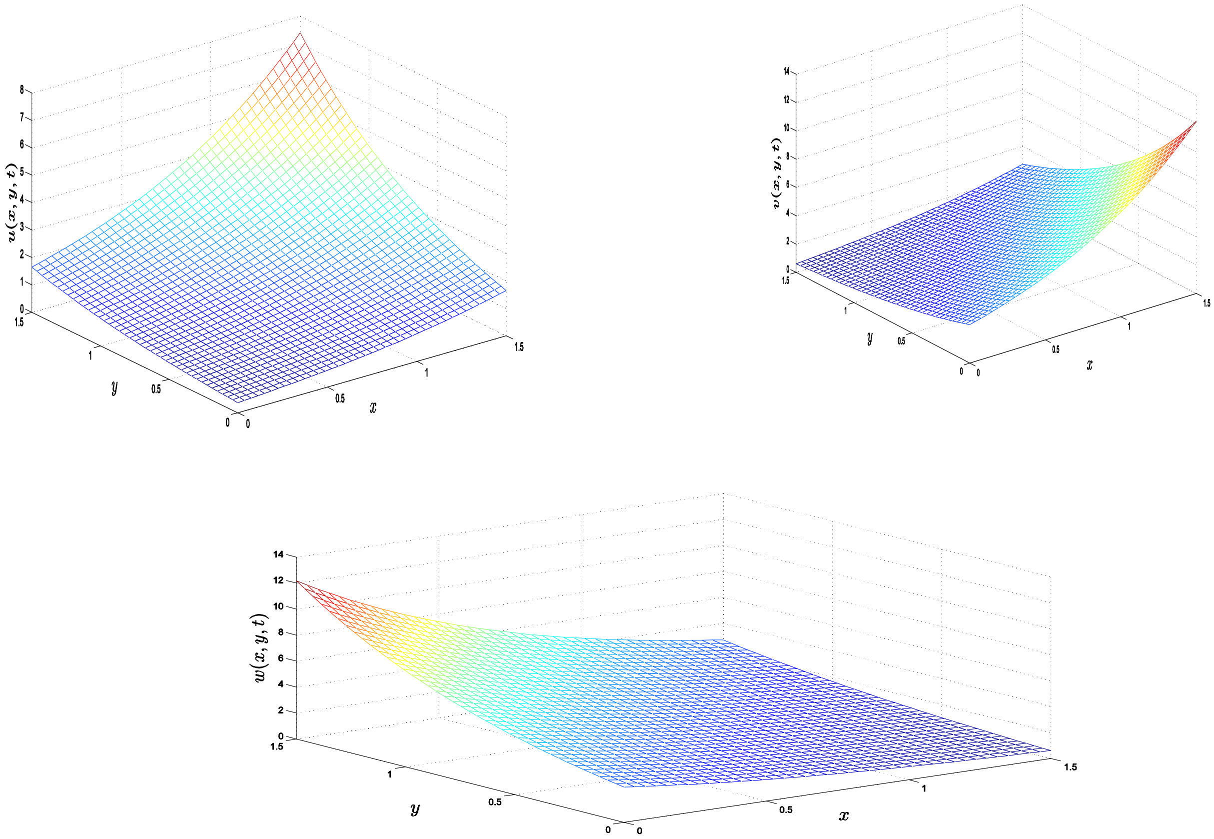


Fig. 2: The behavior of u, v, w of the IVS (12) with $\alpha=\beta=\gamma=1$ in computational domain $(0,\ 1.5)$.

The inverse FRDTM leads to

$$(14)\quad u(x,y,t)=\sum_{k=0}^{\infty}U_k(x,y)t^{\alpha k}=e^{x+y}\sum_{k=0}^{\infty}\frac{(-t^{\alpha})^k}{\Gamma(1+k\alpha)}=e^{x+y}E_{\alpha}(-t^{\alpha}),$$

$$(15)\quad v(x,y,t)=\sum_{k=0}^{\infty}V_k(x,y)t^{\beta k}=e^{x-y}\sum_{k=0}^{\infty}\frac{(t^{\beta})^k}{\Gamma(1+k\beta)}=e^{x-y}E_{\beta}(t^{\beta}),$$

$$(16)\quad w(x,y,t)=\sum_{k=0}^{\infty}W_k(x,y)t^{\gamma k}e^{-x+y}\sum_{k=0}^{\infty}\frac{(t^{\gamma})^k}{\Gamma(1+k\gamma)}=e^{-x+y}E_{\gamma}(t^{\gamma})$$

Moreover, for $\alpha=\beta=\gamma=1,$ Eq. (4.8)-(4.10) reduces to

$$(17)\quad u(x,y,t)=e^{x+y-t},\qquad v(x,y,t)=e^{x-y+t},\qquad w(x,y,t)=e^{-x+y+t}$$

This is the required exact solution of system (12) of non linear fractional partial differential equations (FPDEs), which is same as obtained in [31] using HAM. The solution behavior of $u,\ v,\ w$ at $t=1$ is depicted in Fig. 2.

Example 3: Consider the time-fractional coupled Burgers' equations [34]:

$$(18)\qquad \begin{cases} D_t^{\alpha}u=\dfrac{\partial^2 u}{\partial x^2}+2u\dfrac{\partial u}{\partial x}-\dfrac{\partial}{\partial x}\left(uv\right)\\ D_t^{\beta}v=\dfrac{\partial^2 v}{\partial x^2}+2v\dfrac{\partial v}{\partial x}-\dfrac{\partial}{\partial x}\left(uv\right)\\ u(x,0)=f(x)=\sin x,\quad v(x,0)=g(x)=\sin x,\quad 0<\alpha,\beta\le 1 \end{cases}$$

FRDTM on Eq. (18) leads the following recurrence relation

$$(19)\qquad \begin{cases} \dfrac{\Gamma\{1+(1+k)\alpha\}}{\Gamma(1+k\alpha)}U_{k+1}=\dfrac{d^2}{dx^2}U_k+\displaystyle\sum_{i=0}^{k}\left(2U_i\frac{d}{dx}U_{k-i}-\left(U_i\frac{d}{dx}V_{k-i}+V_i\frac{d}{dx}U_{k-i}\right)\right)\\ \dfrac{\Gamma\{1+(1+k)\beta\}}{\Gamma(1+k\beta)}V_{k+1}=\dfrac{d^2}{dx^2}V_k+\displaystyle\sum_{i=0}^{k}\left(2V_i\frac{d}{dx}V_{k-i}-\left(U_i\frac{d}{dx}V_{k-i}+V_i\frac{d}{dx}U_{k-i}\right)\right)\\ U_0=\sin x\\ V_0=\sin x \end{cases}$$

On solving the system (19), we have

$$U_1(x)=\frac{(-1)}{\Gamma(1+\alpha)}\sin x\ ;\qquad V_1(x)=\frac{(-1)}{\Gamma(1+\beta)}\sin x$$

$$U_2(x)=\left(\frac{\sin x}{\Gamma(1+2\alpha)}-\frac{2\sin x\cos x}{\Gamma(1+2\alpha)}+\frac{2\sin x\cos x\Gamma(1+\alpha)}{\Gamma(1+2\alpha)\Gamma(1+\beta)}\right),$$

$$V_2(x)=\left(\frac{1}{\Gamma(1+2\beta)}-\frac{2\cos x}{\Gamma(1+2\beta)}+\frac{2\cos x\Gamma(1+\beta)}{\Gamma(1+2\beta)\Gamma(1+\alpha)}\right)\sin x$$

$$U_3(x)=\left(\frac{8\cos x}{\Gamma(1+3\alpha)}-\frac{1}{\Gamma(1+3\alpha)}-\frac{8\cos x\Gamma(1+\alpha)\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\beta)}\right)\sin x$$
$$+\left(\frac{4\sin x\cos x}{\Gamma(1+3\alpha)}-\frac{4\sin x\cos 2x}{\Gamma(1+3\alpha)}+\frac{4\sin x\cos 2x\Gamma(1+\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\beta)}+\frac{2\sin x\cos x\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\{\Gamma(1+\alpha)\}^2}-\frac{4\sin x\cos^2 x}{\Gamma(1+3\alpha)}+\frac{2\sin x\cos^2 x\Gamma(1+\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\beta)}\right)$$
$$-\left(\frac{\sin x\cos x\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+2\beta)}-\frac{2\sin x\cos 2x\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+2\beta)}+\frac{2\sin x\cos 2x\Gamma(1+2\alpha)\Gamma(1+\beta)}{\Gamma(1+3\alpha)\Gamma(1+2\beta)\Gamma(1+\alpha)}\right.$$
$$\frac{2\sin x\cos x\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\alpha)\Gamma(1+\beta)}+\frac{\sin^2 x}{\Gamma(1+3\alpha)}-\frac{2\sin^2 x\cos x}{\Gamma(1+3\alpha)}+\frac{2\sin^2 x\cos x\Gamma(1+\alpha)}{\Gamma(1+3\alpha)\Gamma(1+\beta)}$$
$$\left.+\frac{\sin^2 x\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+2\beta)}-\frac{2\sin x\cos x\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)\Gamma(1+2\beta)}+\frac{2\sin^2 x\cos x\Gamma(1+2\alpha)\Gamma(1+\beta)}{\Gamma(1+3\alpha)\Gamma(1+2\beta)\Gamma(1+\alpha)}\right)$$

$$V_3(x)=\left(-\frac{\sin x}{\Gamma(1+3\beta)}+\frac{8\sin x\cos x}{\Gamma(1+3\beta)}-\frac{8\sin x\cos x\Gamma(1+\beta)\Gamma(1+2\beta)}{\Gamma(1+3\beta)\Gamma(1+\alpha)}\right)$$
$$+\left(\frac{4\sin x\cos x}{\Gamma(1+3\beta)}-\frac{4\sin x\cos 2x}{\Gamma(1+3\beta)}+\frac{4\sin x\cos 2x\Gamma(1+\beta)}{\Gamma(1+3\beta)\Gamma(1+\alpha)}+\frac{2\sin x\cos x\Gamma(1+2\beta)}{\Gamma(1+3\beta)\{\Gamma(1+\beta)\}^2}-\frac{4\sin x\cos^2 x}{\Gamma(1+3\beta)}+\frac{2\sin x\cos^2 x\Gamma(1+\beta)}{\Gamma(1+3\beta)\Gamma(1+\alpha)}\right)$$
$$-\left(\frac{\sin x\cos x\Gamma(1+2\beta)}{\Gamma(1+3\beta)\Gamma(1+2\alpha)}-\frac{2\sin x\cos 2x\Gamma(1+2\beta)}{\Gamma(1+3\beta)\Gamma(1+2\alpha)}+\frac{2\sin x\cos 2x\Gamma(1+2\beta)\Gamma(1+\alpha)}{\Gamma(1+3\beta)\Gamma(1+2\alpha)\Gamma(1+\beta)}\right.$$
$$\frac{2\sin x\cos x\Gamma(1+2\beta)}{\Gamma(1+3\beta)\Gamma(1+\beta)\Gamma(1+\alpha)}+\frac{\sin^2 x}{\Gamma(1+3\beta)}-\frac{2\sin^2 x\cos x}{\Gamma(1+3\beta)}+\frac{2\sin^2 x\cos x\Gamma(1+\beta)}{\Gamma(1+3\beta)\Gamma(1+\alpha)}$$
$$\left.+\frac{\sin^2 x\Gamma(1+2\beta)}{\Gamma(1+3\beta)\Gamma(1+2\alpha)}-\frac{2\sin x\cos x\Gamma(1+2\beta)}{\Gamma(1+3\beta)\Gamma(1+2\alpha)}+\frac{2\sin^2 x\cos x\Gamma(1+2\beta)\Gamma(1+\alpha)}{\Gamma(1+3\beta)\Gamma(1+2\alpha)\Gamma(1+\beta)}\right)$$

The inverse FRDTM, leads to

(20) $$u(x,t)=\sum_{k=0}^{\infty}U_k(x)t^{k\alpha}$$
$$=\sin x-\frac{\sin x}{\Gamma(1+\alpha)}t^{\alpha}+\left(\frac{\sin x}{\Gamma(1+2\alpha)}-\frac{2\sin x\cos x}{\Gamma(1+2\alpha)}+\frac{2\sin x\cos x\Gamma(1+\alpha)}{\Gamma(1+2\alpha)\Gamma(1+\beta)}\right)t^{2\alpha}+\cdots$$

(21) $$v(x,t)=\sum_{k=0}^{\infty}V_k(x)t^{k\beta}$$
$$=\sin x-\frac{\sin x}{\Gamma(1+\beta)}t^{\beta}+\left(\frac{\sin x}{\Gamma(1+2\beta)}-\frac{2\sin x\cos x}{\Gamma(1+2\beta)}+\frac{2\sin x\cos x\Gamma(1+\beta)}{\Gamma(1+2\beta)\Gamma(1+\alpha)}\right)t^{2\beta}+\cdots$$

The same solution is obtained by Yildirim and Kelleci [33] using HPM.

In particular, for $\alpha=\beta=1,$ the solutions (20)-(21) reduces to

(22) $u(x,t)=e^{-t}\sin x$, $v(x,t)=e^{-t}\sin x$

This is the required exact solution of the initial values system of classical coupled viscous Burgers equation (18). This is same as the solution obtained by HPM [33], VIM [34] for $\alpha = \beta = 1$. The physical behavior of u, v in domain $(-\pi, \pi)$ is depicted in Fig. 4, whereas Fig. 3 depicts the physical behavior of u, v in domain $(-10,10)$ for different values of α, β.

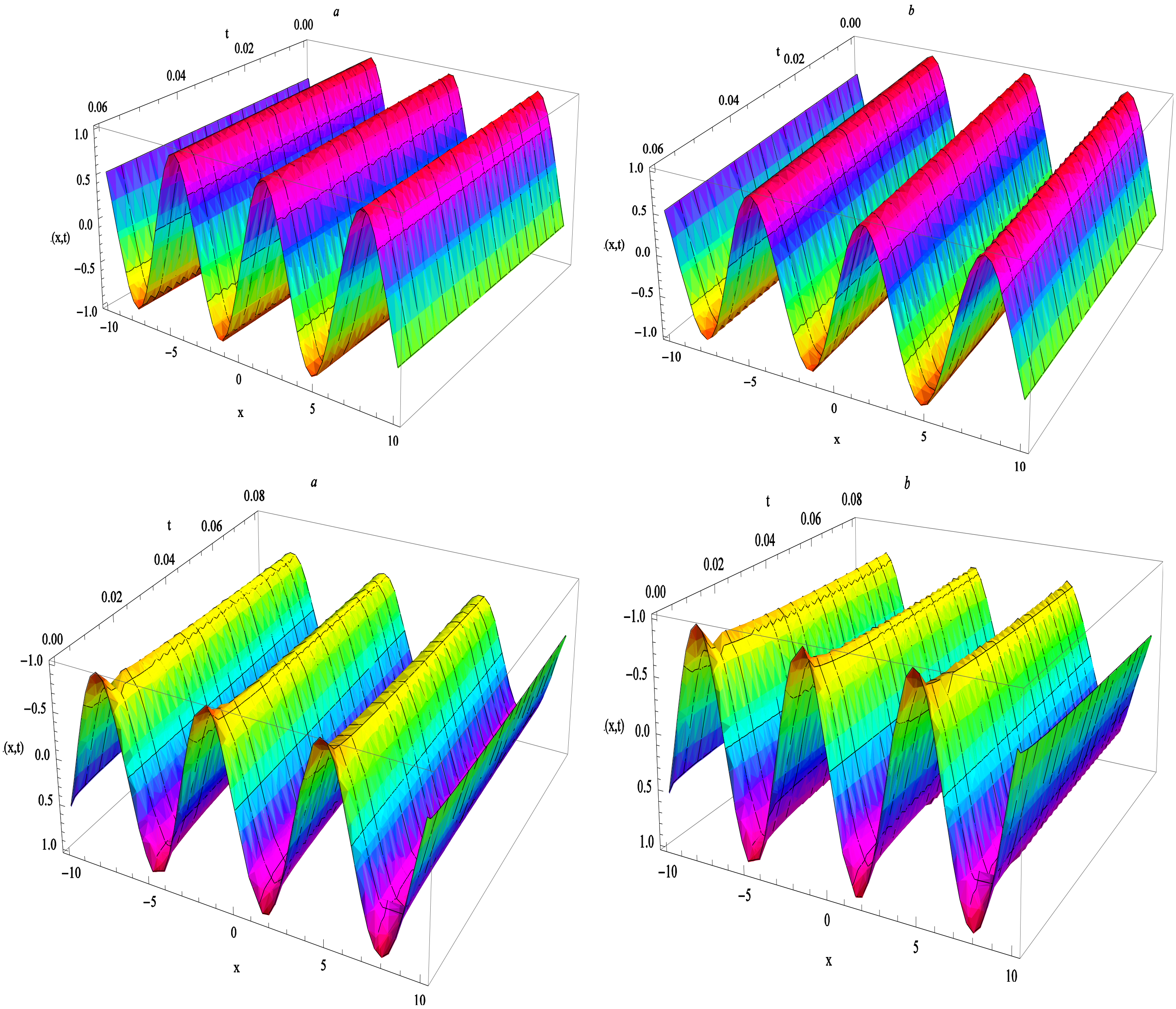


Fig. 3: The solution behavior of $a) u$, $b) v$ of (18) in domain $(-10,10)$ at different time levels for $\alpha = \beta = 1$ (upper) and $\alpha = 1/3$, $\beta = 0.2$ (lower).

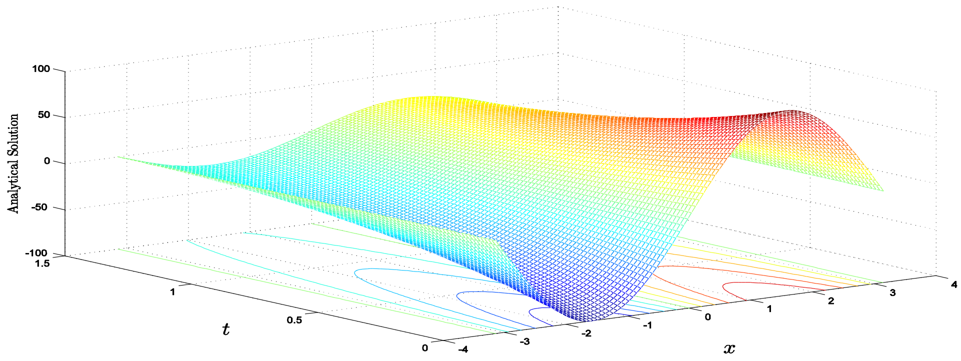


Fig. 4: Behavior of u, v of system (18) in domain $(-\pi,\pi)$ at different time levels.

Example 4 Consider the coupled system of nonlinear fractional reaction diffusion equation as in [30]:

$$(23)\quad \begin{cases} u_t^{\alpha} = u(1-u-v)+u_{xx}, & t>0, \\ v_t^{\alpha} = v_{xx} - uv, & \\ u(x,0) = \dfrac{e^{kx}}{\left[1+e^{0.5kx}\right]^2}, \quad v(x,0) = \dfrac{1}{\left[1+e^{0.5kx}\right]}, & \end{cases}$$

where k is constant.

The FRDT method on Eq. (23) reduces to a set of recurrence relation as follows:

$$(24)\quad \begin{cases} \dfrac{\Gamma(1+(1+k))\alpha}{\Gamma(1+k\alpha)} U_{k+1}(x) = \sum_{r=0}^{k} U_r(x)\left(1-U_{k-r}(x)-V_{k-r}(x)\right) + \dfrac{\partial^2 U_k(x)}{\partial x^2}, \\ \dfrac{\Gamma(1+(1+k))\alpha}{\Gamma(1+k\alpha)} V_{k+1}(x) = \dfrac{\partial^2 V_k(x)}{\partial x^2} - \sum_{r=0}^{k} U_r(x) V_{k-r}(x), \\ U_0 = \dfrac{e^{kx}}{\left[1+e^{0.5kx}\right]^2}, \quad V_0 = \dfrac{1}{\left[1+e^{0.5kx}\right]}, \end{cases}$$

On simplifying (24), we get

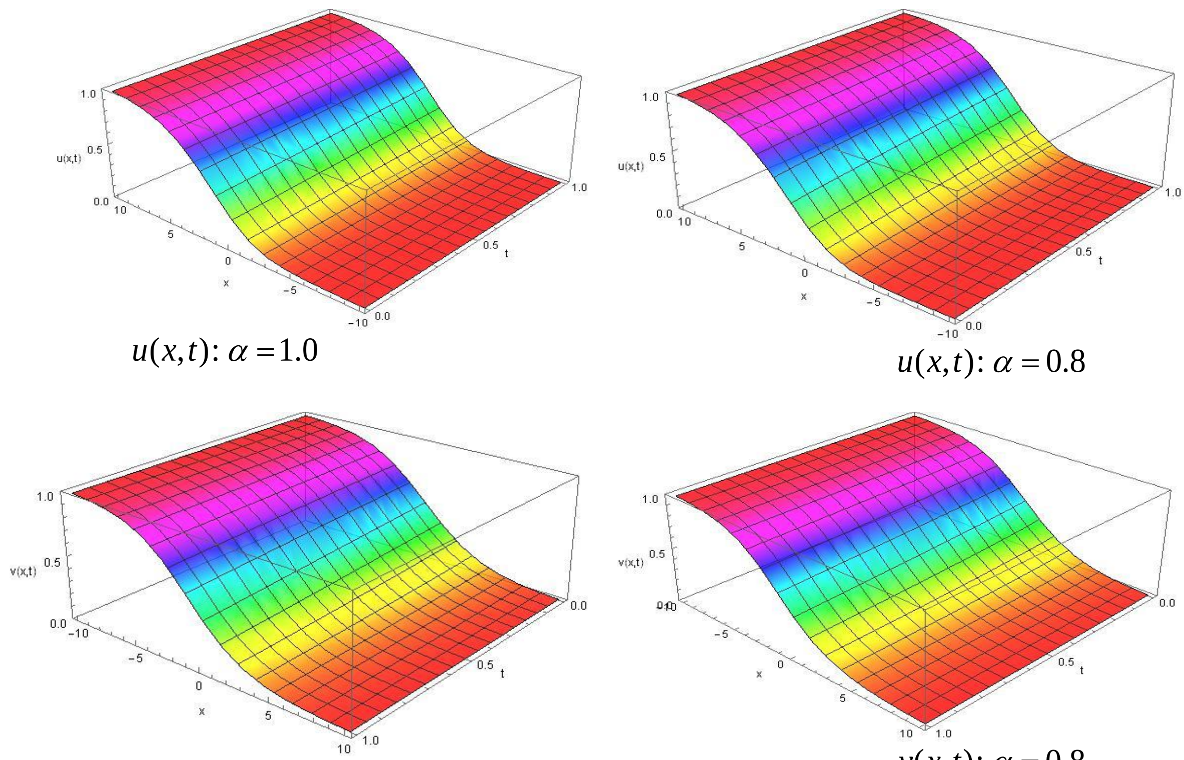
$$U_1 = \frac{1}{\Gamma[1+\alpha]}\frac{2e^{1.5kx} - c^2 e^{kx}\left(-2+e^{0.5kx}\right)}{2\left(1+e^{0.5kx}\right)^4},$$

$$V_1 = \frac{1}{\Gamma[1+\alpha]}\frac{-4e^{kx} + c^2\left(-e^{0.5kx}+e^{kx}\right),}{4\left(1+e^{0.5kx}\right)^3}$$

$$U_2 = \frac{1}{\Gamma[1+2\alpha]}\frac{e^{kx}\left(16e^{kx} + 4c^2 e^{0.5kx}\left(7-8e^{0.5kx}+e^{kx}\right) - c^4\left(-8+33e^{0.5kx}-18e^{kx}+e^{1.5kx}\right)\right)}{8\left(1+e^{0.5kx}\right)^6},$$

$$V_2 = \frac{1}{\Gamma[1+2\alpha]}\frac{\left(16e^{1.5kx}(e^{0.5kx}-1) - 8c^2 e^{kx}\left(4-5e^{0.5kx}+e^{kx}\right) + c^4\left(-e^{0.5kx}+11e^{kx}-11e^{1.5kx}+e^{2kx}\right)\right)}{16\left(1+e^{0.5kx}\right)^5},$$

$$\vdots$$



$u(x,t)$: $\alpha = 1.0$

$u(x,t)$: $\alpha = 0.8$

$v(x,t)$: $\alpha = 1.0$

$v(x,t)$: $\alpha = 0.8$

Fig. 5: The behaviour u and v of the Problem (23) for $\alpha = 0.8, 1$ at $t \in (0,1)$ with $k = 0.9, x \in (-10,10)$.

The inverse FRDT method leads to

$$u(x,t)=\frac{e^{kx}}{\left[1+e^{0.5kx}\right]^2}+\frac{1}{\Gamma[1+\alpha]}\frac{2e^{1.5kx}-c^2e^{kx}\left(-2+e^{0.5kx}\right)}{2\left(1+e^{0.5kx}\right)^4}t^{\alpha}+$$
$$\frac{1}{\Gamma[1+2\alpha]}\frac{e^{kx}\left(16e^{kx}+4c^2e^{0.5kx}\left(7-8e^{0.5kx}+e^{kx}\right)-c^4\left(-8+33e^{0.5kx}-18e^{kx}+e^{1.5kx}\right)\right)}{8\left(1+e^{0.5kx}\right)^6}t^{2\alpha}\cdots$$

$$v(x,t)=\frac{1}{\left[1+e^{0.5kx}\right]}+\frac{1}{\Gamma[1+\alpha]}\frac{-4e^{kx}+c^2\left(-e^{0.5kx}+e^{kx}\right)}{4\left(1+e^{0.5kx}\right)^3}t^{\alpha}+$$
$$\frac{1}{\Gamma[1+2\alpha]}\frac{\left(16e^{1.5kx}(e^{0.5kx}-1)-8c^2e^{kx}\left(4-5e^{0.5kx}+e^{kx}\right)+c^4\left(-e^{0.5kx}+11e^{kx}-11e^{1.5kx}+e^{2kx}\right)\right)}{16\left(1+e^{0.5kx}\right)^5}t^{2\alpha}\cdots$$

which is the required solution of the IVS of reaction-diffusion equation (23). The same solution is obtained by HPM [30]. The solution behavior of the system of reaction-diffusion equation (23) is depicted in Fig. 5.

5 Concluding remark

This paper is successfully implemented the FRDTM to solve the initial value autonomous system of time-fractional partial differential equations, including coupled viscous Burgers equations. The fractional derivative is taken into Caputo sense. The proposed solutions are obtained in the form of power series. The validity and efficiency of FRDTM has been confirmed by four test problems. It is found that the obtained solutions are agreed well with the solution obtained by HAM [31], HPM [30], [33] and VIM [32], [34].

The solutions are approximated without any discretization, perturbation, or restrictive conditions. The small size of computation of the scheme is the strength of the scheme.

6 Open Problem

The implementaion of finite difference method / collocation method for the numerical computation of the initial values nonlinear autonomous system of time-fractional partial differential equations is still a challenging problem.

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