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Fuzzy modules over a t-norm

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Abstract

In this paper we investigate some properties of submodules by using a t-norm T. We consider properties of intersection, sum and homomorphisms for fuzzy submodules.

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1 Introduction

The notion of a fuzzy subset was introduced by L.A Zadeh [9]. His seminal paper in 1965 has opened up new insights and applications in a wide range of scientific fields. Since then, some autors [2, 4, 8] applied this concept to group and ring theory. We characterize some basic properties of submodules with respect to a t-norm T. The organization of this paper is as follows: In section 2, some preliminary definitions and results are given. In section 3, union and intersection of fuzzy submodules are vestigated. In this section we consider the relationship between submodules and fuzzy submodules. Also we obtain some results for fuzzy submodules under the mapping between modules.

2 Preliminaries

Definition 2.1 *(See [6]) Let R be a ring. A commutative group $(M, +)$ is called a left R-module or a left module over R with respect to a mapping*

$$. : R \times M \to M$$

if for all $r, s \in R$ and $m, n \in M$,
(1) $r.(m+n) = r.m + r.n$,
(2) $r.(s.m) = (rs).m$,
(3) $(r+s).m = r.m + s.m$.
If R has an identity 1 and if $1.m = m$ for all $m \in M$, then M is called a unitary or unital left R-module.
A right R-module can be defined in a similar fashion.

Definition 2.2 *(See [6]) Let M be an R-module and N be a nonempty subset of M. Then N is called a submodule of M if N is a subgroup of M and for all $r \in R, a \in N$, we have $ra \in N$.*

Definition 2.3 *(See [5]) Let X a non-empty sets. A fuzzy subset μ of X is a function $\mu : X \to [0,1]$. Denote by $[0,1]^X$, the set of all fuzzy subset of X.*

Definition 2.4 *(See [1]) A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:*
(T1) $T(x,1) = x$ (neutral element),
(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),
(T3) $T(x,y) = T(y,x)$ (commutativity),
(T4) $T(x, T(y,z)) = T(T(x,y), z)$ (associativity),
for all $x, y, z \in [0,1]$.

Example 2.5 *The basic t-norms are $T_m(x,y) = \min\{x,y\}$,$T_b(x,y) = \max\{0, x+y-1\}$ and $T_p(x,y) = xy$, which are called standard intersection, bounded sum and algebraic product respectively.*

Lemma 2.6 *(See [1]) Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.7 *(See [7]) The intersection of fuzzy subsets μ_1 and μ_1 in a set X with respect to a t-norm T we mean the fuzzy subset $\mu = \mu_1 \cap \mu_2$ in the set X such that for any $x \in X$*

$$\mu(x) = (\mu_1 \cap \mu_2)(x) = T(\mu_1(x), \mu_2(x)).$$

Definition 2.8 *(See [7]) Let R be a ring and M be a (left) R-module. A function $\mu : M \to [0,1]$ is a fuzzy R-submodule of M with respect to a t-norm T if and only if for all $x, y \in M$ and for all $r \in R$ the following conditions are satisfied;*
(1) $\mu(x+y) \geq T(\mu(x), \mu(y))$,
(2) $\mu(rx) \geq \mu(x)$,
(3) $\mu(0) = 1$.

Denote by $TFM(R)$, the set of all fuzzy R-submodules of M with respect to a t-norm T.

Definition 2.9 *(See [3]) Let X and Λ be two non-empty sets such that $\{\mu_i \mid i \in \Lambda\} \subseteq [0,1]^X$. The union $\cup_{i\in\Lambda} \mu_i$ is defined by $\cup_{i\in\Lambda} \mu_i(x) = \sup\{\mu_i(x) \mid i \in \Lambda\}$.*

Definition 2.10 *(See [5]) Let f be a mapping from R-module M into R-module N. Let $\mu \in [0,1]^M$ and $\nu \in [0,1]^N$. Define $f(\mu) \in [0,1]^N$ and $f^{-1}(\nu) \in [0,1]^M$ as $\forall y \in N$, $f(\mu)(y) = \sup\{\mu(x) \mid x \in M, f(x) = y\}$ if $f^{-1}(y) \neq \emptyset$ and $f(\mu)(y) = 0$ if $f^{-1}(y) = \emptyset$.*
Also $\forall x \in M$, $f^{-1}(\nu)(x) = \nu(f(x))$.

3 Main results

Proposition 3.1 *Let $\mu_1, \mu_2 \in TFM(R)$. Then $(\mu_1 \cap \mu_2) \in TFM(R)$.*

Proof 3.2 *Let $x, y \in M$ and $r \in R$.*
(1)
$$(\mu_1 \cap \mu_2)(x+y) = T(\mu_1(x+y), \mu_2(x+y))$$
$$\geq T(T(\mu_1(x), \mu_1(y)), T(\mu_2(x), \mu_2(y)))$$

(from Lemma 2.6)

$$= T(T(\mu_1(x), \mu_2(x)), T(\mu_1(y), \mu_2(y)))$$
$$= T((\mu_1 \cap \mu_2)(x), (\mu_1 \cap \mu_2)(y))$$

(2)

$$(\mu_1 \cap \mu_2)(rx) = T(\mu_1(rx), \mu_2(rx)) \geq T(\mu_1(x), \mu_2(x)) = (\mu_1 \cap \mu_2)(x).$$

(3)
$$(\mu_1 \cap \mu_2)(0) = T(\mu_1(0), \mu_2(0)) = T(1,1) = 1.$$

Corollary 3.3 *Let $\{\mu_i \mid i \in I_n = 1, 2, ..., n\} \subseteq TFM(R)$. Then so is $\cap_{i\in I_n} \mu_i$.*

Proposition 3.4 *Let $\mu \in TFM(R)$. Then $A = \{x \mid x \in M, \mu(x) = 1\}$ is a submodule of the module M.*

Proof 3.5 *Let $x, y \in A$ and $r \in R$. Then*
(1) $\mu(x+y) \geq T(\mu(x), \mu(y)) = T(1,1) = 1$ *and so* $x + y \in A$.
(2) Since $\mu(rx) \geq \mu(x) = 1$ *so* $rx \in A$.
(3) Finally, $\mu(0) = 1$ *and* $0 \in A$.
Hence A is an R-submodule of A.

Proposition 3.6 *Let f be an epimorphism from R-module M into R-module N. If $\mu \in TFM(R)$, then $f(\mu) \in TFN(R)$.*

Proof 3.7 *Let $y_1, y_2 \in N$.*
(1)

$$\begin{aligned} f(\mu)(y_1+y_2) &= \sup\{\mu(x_1+x_2) \mid x_1, x_2 \in M, f(x_1) = y_1, f(x_2) = y_2\} \\ &\geq \sup\{T(\mu(x_1), \mu(x_2)) \mid x_1, x_2 \in M, f(x_1) = y_1, f(x_2) = y_2\} \\ &= T(\sup\{\mu(x_1) \mid f(x_1) = y_1\}, \sup\{\mu(x_2) \mid f(x_2) = y_2\}) \\ &= T(f(\mu)(y_1), f(\mu)(y_2)). \end{aligned}$$

(2) Let $x \in M$ and $r \in R$.

$$\begin{aligned} f(\mu)(ry) &= \sup\{\mu(rx) \mid rx \in M, f(rx) = ry\} \\ &\geq \sup\{\mu(x) \mid x \in M, f(x) = y\} \\ &= f(\mu)(y). \end{aligned}$$

(3) $f(\mu)(0) = \sup\{\mu(0) \mid 0 \in M, f(0) = 0\} = 1$.
Therefore $f(\mu) \in TFN(R)$.

Proposition 3.8 *Let f be an epimorphism from R-module M into R-module N. If $\nu \in TFN(R)$, then $f^{-1}(\nu) \in TFM(R)$.*

Proof 3.9 *Let $x_1, x_2 \in M$. Then*
(1)

$$\begin{aligned} f^{-1}(\nu)(x_1+x_2) &= \nu(f(x_1+x_2)) = \nu(f(x_1)+f(x_2)) \\ &\geq T(\nu(f(x_1), \nu(f(x_2)) \\ &= T(f^{-1}(\nu)(x_1), f^{-1}(\nu)(x_2)). \end{aligned}$$

(2) Let $x \in M$ and $r \in R$. $f^{-1}(\nu)(rx) = \nu(f(rx)) = \nu(rf(x)) \geq \nu(f(x)) = f^{-1}(\nu)(x)$.
(3) $f^{-1}(\nu)(0) = \nu(f(0)) = \nu(0) = 1$.
Hence $f^{-1}(\nu) \in TFM(R)$.

Definition 3.10 *The sum on fuzzy subsets μ_1 and μ_2 of an R-module M is defined as follows:*

$$(\mu_1 + \mu_2)(x) = \sup\{T(\mu_1(y), \mu_2(z)) \mid x = y + z \in M\}.$$

Proposition 3.11 *Let $\mu_1, \mu_2 \in TFM(R)$. Then $(\mu_1 + \mu_2) \in TFM(R)$.*

Proof 3.12 *(1) Let $x_1, x_2, y_1, y_2, z_1, z_2 \in M$. Then*

$$(\mu_1 + \mu_2)(x_1 + x_2)$$

$$= \sup\{T(\mu_1(y_1 + y_2), \mu_2(z_1 + z_2)) \mid x_1 + x_2 = y_1 + y_2 + z_1 + z_2\}$$

$$\geq \sup\{T(T(\mu_1(y_1), \mu_1(y_2)), T(\mu_2(z_1), \mu_2(z_2))) \mid x_1 + x_2 = y_1 + z_1 + y_2 + z_2\}$$

$$(\ from\ Lemma\ 2.6\ )$$

$$= \sup\{T(T(\mu_1(y_1), \mu_2(z_1)), T(\mu_1(y_2), \mu_2(z_2))) \mid x_1 + x_2 = y_1 + z_1 + y_2 + z_2\}$$

$$= T(\sup\{T(\mu_1(y_1), \mu_2(z_1)) \mid x_1 = y_1 + z_1)\}, \sup\{T(\mu_1(y_2), \mu_2(z_2)) \mid x_2 = y_2 + z_2\})$$

$$= T((\mu_1 + \mu_2)(x_1), (\mu_1 + \mu_2)(x_2)).$$

(2) Let $x, y, z \in M$ and $r \in R$.

$$(\mu_1 + \mu_2)(rx) = \sup\{T(\mu_1(ry), \mu_2(rz)) \mid rx = ry + rz\}$$

$$\geq \sup\{T(\mu_1(y), \mu_2(z)) \mid x = y + z\} = (\mu_1 + \mu_2)(x).$$

(3)

$$(\mu_1 + \mu_2)(0) = \sup\{T(\mu_1(0), \mu_2(0)) \mid 0 = 0 + 0\} = T(1, 1) = 1.$$

Proposition 3.13 *Let $\mu \in TFM(R)$ and T be idempotent. Then for any $t \in [0, 1]$, $A_t = \{x \mid x \in M, \mu(x) \geq t\}$ will be a submodule of the module M.*

Proof 3.14 *Let $x, y \in A_t$ and $r \in R$.*
(1) $\mu(x + y) \geq T(\mu(x),)\mu(y) \geq T(t, t) = t$. Then $x + y \in A_t$.
(2) $\mu(rx) \geq \mu(x) \geq t$, so $rx \in A_t$.
(3) $\mu(0) = 1 \geq t$ and it means that $0 \in A_t$.
Now we can say that A_t is a submodule of the module M.

Proposition 3.15 *Let M be an R-module and N be a subset of M. If $\mu : N \to \{0, 1\}$ be the characteristic function, then $\mu \in TFM(R)$ if and only if N is a submodule of M.*

Proof 3.16 *Let $\mu \in TFM(R)$ and we prove that N is a submodule of M. Let $x, y \in N \subseteq M$ and $r \in R$. Now*

$$\mu(x+y) \geq T(\mu(x), \mu(y)) = T(1,1) = 1$$

so $x + y \in N$.
Also $\mu(rx) \geq \mu(x) = 1$ and $rx \in N$.
Finally $\mu(0) = 1$ means that $0 \in N$.
Therefore N is a submodule of M.
Conversely, let N is a submodule of M and we prove that $\mu \in TFM(R)$. Suppose $x, y \in M$ and we investigate the following conditions
(1) If $x, y \in N$, then

$$\mu(x+y) = 1 \geq 1 = T(1,1) = T(\mu(x), \mu(y)).$$

(2) For any $x \in N$ and $y \notin N$,

$$\mu(x+y) \geq 0 = T(1,0) = T(\mu(x), \mu(y)).$$

(3) Let $x \notin N$ and $y \in N$. Then

$$\mu(x+y) \geq 0 = T(0,1) = T(\mu(x), \mu(y)).$$

(4) Finally, if $x, y \notin N$, then

$$\mu(x+y) \geq 0 = T(0,0) = T(\mu(x), \mu(y)).$$

Therefore from (1)-(4) we have that

$$\mu(x+y) \geq T(\mu(x), \mu(y)).$$

Now let $x \in M$ and $r \in R$. Then we have that
(1) If $x \in N$, then $\mu(rx) = 1 \geq \mu(x)$.
(2) If $x \notin N$, then $\mu(rx) = 0 \geq 0 = \mu(x)$.
Therefore from (1) and (2) we have that $\mu(rx) \geq \mu(x)$.
Finally, since $0 \in N$ we have $\mu(0) = 1$.
Hence $\mu \in TFM(R)$.

4 Open Problem

The open problem here is to investigate some properties of fuzzy ideals in BCI, BCK, BL and MV-algebras by using a t-norm T as we did.

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