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Decomposition of an Ortholattice

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Abstract

In this paper B-elements and C-elements are defined in an ortholattice. We obtain an equivalent condition for an ortholattice to become a distributive lattice and hence Boolean algebra in terms of B-elements. Using B-elements two congruences are studied. Finally for each C-element, we obtain a decomposition for an ortholattice.

Keywords: *Ortholattice, Congruence and Decomposition.*

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1 Introduction

In [3], every Boolean algebra is isomorphic to the direct product of two element Boolean algebras. The concept of Boolean algebra plays a key role in lattice theory and mathematical logic. Ortholattices are one of the generalization of Boolean algebras. Several authors discussed about the structure of an ortholattice. In [2], Ivan chazda characterized the ideals of an ortholattice. By an ortholattice we mean an algebra $(L, \vee, \wedge, ', 0, 1)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded lattice and $'$ is an unary operations satisfying the following identities

$$x'' = x$$
$$x \wedge x' = 0 \text{ and } x \vee x' = 1$$

$$(x \wedge y)' = x' \vee y' \text{ and } (x \vee y)' = x' \wedge y'$$
$$0' = 1 \text{ and } 1' = 0.$$

In this paper, we introduce B-elements in an ortholattice and obtain some properties on them which are useful in consequent sections. If a is a B-element in an ortholattice L, we obtain two congruences θ_a, ψ_a on L. In fact θ_a (and ψ_a) need not be a congruence in an ortholattice L, where $a \in L$. We obtain two ortholattices namely L_a and R_a, which are not subalgebras of L, for any $a \in L$. We define C-elements in an ortholattice. If a is a C-element and a & a' are $\wedge$-distributive, then we prove that L is isomorphic to $L_a \times L_{a'}$. Similarly, if a is a C-element and a & a' are $\vee$-distributive, then L is isomorphic to $R_a \times R_{a'}$.

2 B-elements in ortholattices

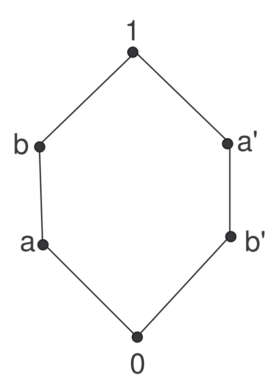
In this section, we define B-elements in an ortholattice and provide several examples for it. Mainly, we obtain a necessary and sufficient condition for an ortholattice to become a Boolean algebra.

Definition 2.1 *An element a of an ortholattice $(L, \vee, \wedge, ', 0, 1)$ is said to be a B-element with respect to $\wedge$, if for any $x, y \in L$, $a \wedge x = a \wedge y$ implies $a \wedge x' = a \wedge y'$. Correspondingly, we can define B-element with respect to $\vee$.*

From now onwards by L we mean an ortholattice $(L, \vee, \wedge, ', 0, 1)$ in this paper unless and otherwise stated by the authors.

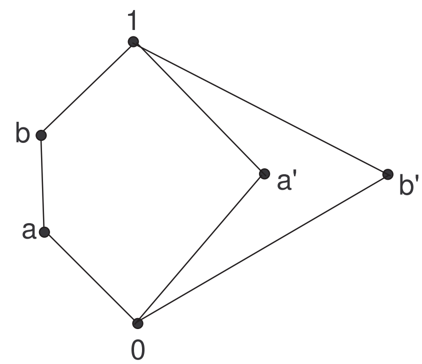
Note 2.2 *1. 0 and 1 are always B-elements with respect to $\wedge$ and $\vee$ in L.*
2. For any $a \in L$, a is a B-element with respect to $\wedge$ if and only if a' is a B-element with respect to $\vee$.

Example 2.3 *Let $L = \{0, a, b, a', b', 1\}$ be an ortholattice whose Hasse-diagram is*



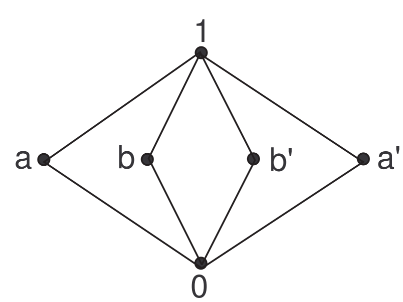
Then a and b' are B-elements with respect to $\wedge$.

Example 2.4 *Let $L = \{0, a, b, a', b', 1\}$ be an ortholattice whose Hasse-diagram is*



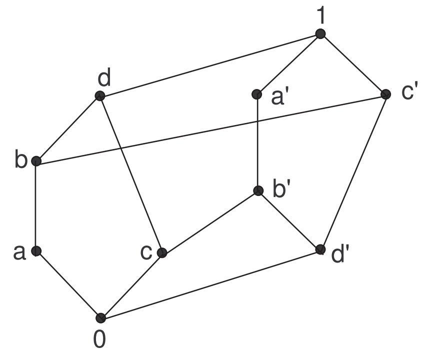
Then a is a B-element with respect to $\wedge$.

Example 2.5 *Let $L = \{0, a, b, a', b', 1\}$ be an ortholattice whose Hasse-diagram is*



Then there is no B-elements with respect to $\wedge$ or $\vee$ except 0 and 1.

Example 2.6 *Let $L = \{0, a, b, c, d, a', b', c', d', 1\}$ be an ortholattice whose Hasse-diagram is*



Then a, c, and d' are B-elements with respect to $\wedge$. b is neither B-element with respect to $\wedge$ nor $\vee$.

Theorem 2.7 *Let L be an ortholattice in which every element is B-element with respect to $\wedge$. Then L is distributive and hence a Boolean algebra.*

Proof: Suppose that L is an ortholattice in which every element is a B-element with respect to $\wedge$. We claim that L has no copy of N_5 and M_5.
Case 1. Let us assume that L has a copy of N_5. Then there exist $a, b, c \in L$ such that $a > b, a \vee c = b \vee c$ and $a \wedge c = b \wedge c$. Then $a' < b', a' \vee c' = b' \vee c'$ and $a' \wedge c' = b' \wedge c'$. Take $x = a \wedge b'$. If $x = 0$, then $a \wedge b' = x = 0 = a \wedge a'$. By our hypothesis, $a \wedge b = a \wedge a = a$, which is not true in N_5. So, $x \neq 0$. Now,

$$c \wedge x = c \wedge (a \wedge b') = (c \wedge a) \wedge b' = (c \wedge b) \wedge b' = 0 = 0 \wedge x$$

$$c' \wedge x = c' \wedge (b' \wedge a) = (c' \wedge b') \wedge a = (c' \wedge a') \wedge a = 0$$

By our hypothesis, $0 = x \wedge c' = x \wedge 0' = x \wedge 1 = x$. Which is a contradiction to $x \neq 0$. Therefore L has no copy of N_5.
Case 2. Let us assume that L has a copy of M_5. Then there exist $a, b, c \in L$ such that $a \vee b = a \vee c = b \vee c$ and $a \wedge b = a \wedge c = b \wedge c$. We have same conditions on a', b', c'. Take $x = a \wedge b'$. If $x = 0$, then $a \wedge b' = x = 0 = a \wedge a'$. By our hypothesis, $a \wedge b = a \wedge a = a$. But this implies $a \leq b$, which is not true in M_5. So, $x \neq 0$. Now,

$$c \wedge x = c \wedge (a \wedge b') = (c \wedge a) \wedge b' = (c \wedge b) \wedge b' = 0 = 0 \wedge x$$

$$c' \wedge x = c' \wedge (b' \wedge a) = (c' \wedge b') \wedge a = (c' \wedge a') \wedge a = 0.$$

By our hypothesis, we get $0 = x \wedge c' = x \wedge 0' = x \wedge 1 = x$. Which is a contradiction to $x \neq 0$. Therefore L has no copy of M_5. Thus L is distributive and hence a Boolean algebra. □

Corollary 2.8 *Let L be an ortholattice in which every element is B-element with respect to $\vee$. Then L is distributive and hence a Boolean algebra.*

3 Congruences on ortholattices

Let us denote B be the set of B-elements with respect to $\wedge$ and B' is the set of B-elements with respect to $\vee$. For each B-element, we study two congruences. In an ortholattice, we present two ortholattice structures which are not sub ortholattices.

Theorem 3.1 *For any $a \in B$, the set $\theta_a = \{(x,y) \mid a \wedge x = a \wedge y\}$ is a congruence on L.*

Proof: It is easy to verify that θ_a is an equivalence relation on L. Let $x_1, y_1, x_2, y_2 \in L$ such that $a \wedge x_1 = a \wedge y_1$ and $a \wedge x_2 = a \wedge y_2$. Then $a \wedge x_1' = a \wedge y_1'$ and $a \wedge x_2' = a \wedge y_2'$ (since $a \in B$). Now,

$$\begin{aligned} a \wedge x_1 \wedge x_2 &= a \wedge a \wedge x_1 \wedge x_2 \\ &= a \wedge x_1 \wedge a \wedge x_2 \\ &= a \wedge y_1 \wedge a \wedge y_2 \quad \text{(by our assumption)} \\ &= a \wedge a \wedge y_1 \wedge y_2 \\ &= a \wedge y_1 \wedge y_2 \end{aligned}$$

and

$$\begin{aligned} a \wedge (x_1 \vee x_2)' &= a \wedge x_1' \wedge x_2' \\ &= a \wedge x_1' \wedge a \wedge x_2' \\ &= a \wedge y_1' \wedge a \wedge y_2' \quad \text{(by our assumption)} \\ &= a \wedge y_1' \wedge y_2' \\ &= a \wedge (y_1 \vee y_2)'. \end{aligned}$$

So, $a \wedge (x_1 \vee x_2) = a \wedge (y_1 \vee y_2)$ (since $a \in B$). Therefore $(x_1 \wedge x_2, y_1 \wedge y_2), (x_1 \vee x_2, y_1 \vee y_2) \in \theta_a$. Hence θ_a is a congruence on L. □

Note 3.2 *If a is not a B-element with respect to $\wedge$, then θ_a need not be a congruence on L. For, see Example 2.3., θ_b is not a congruence (because $(a', b') \in \theta_b$ but $(a, b) \notin \theta_b$), where as b is not a B-element with respect to $\wedge$.*

Theorem 3.3 *For any $a \in B'$, the set $\psi_a = \{(x,y) \mid a \vee x = a \vee y\}$ is a congruence on L.*

Proof: It is easy to prove that ψ_a is an equivalence relation on L. Let $x_1, y_1, x_2, y_2 \in L$ such that $a \vee x_1 = a \vee y_1$ and $a \vee x_2 = a \vee y_2$. Then $a \vee x_1' = a \vee y_1'$ and $a \vee x_2' = a \vee y_2'$ (since $a \in B'$). Now,

$$\begin{aligned} a \vee x_1 \vee x_2 &= a \vee a \vee x_1 \vee x_2 \\ &= a \vee x_1 \vee a \vee x_2 \\ &= a \vee y_1 \vee a \vee y_2 \quad \text{(by our assumption)} \\ &= a \vee y_1 \vee y_2 \end{aligned}$$

and

$$\begin{aligned}
a \vee (x_1 \wedge x_2)' &= a \vee x_1' \vee x_2' \\
&= a \vee x_1' \vee x_2' \\
&= a \vee a \vee x_1' \vee x_2' \\
&= a \vee y_1' \vee a \vee y_2' \quad \text{(by our assumption)} \\
&= a \vee y_1' \vee y_2' \\
&= a \vee (y_1 \wedge y_2)'.
\end{aligned}$$

So, $a \vee (x_1 \wedge x_2) = a \vee (y_1 \wedge y_2)$ (since $a \in B'$). Therefore $(x_1 \wedge x_2, y_1 \wedge y_2), (x_1 \vee x_2, y_1 \vee y_2) \in \psi_a$. Hence ψ_a is a congruence on L. □

Note 3.4 *If a is not a B-element with respect to $\vee$, then ψ_a need not be a congruence on L. For, see Example 2.3., $\psi_{b'}$ is not a congruence (because $(a, b) \in \psi_{b'}$ but $(a', b') \notin \psi_{b'}$), where as b' is not a B-element with respect to $\vee$.*

Definition 3.5 *An element a of L is said to be $\wedge$-distributive, if for any $x, y \in L, a \wedge (x \vee y) = (a \wedge x) \vee (a \wedge y)$.*

It is easy to prove that if a, b are $\wedge$-distributive, then $a \wedge b$ is also $\wedge$-distributive.

Theorem 3.6 *Let a be a $\wedge$-distributive element in L. Then the set $L_a = \{a \wedge x \mid x \in L\}$ is itself an ortholattice with induced operations $\vee, \wedge$ and the unary operation $*$ defined by $x^* = (a \wedge x)^* = a \wedge x'$, for all $x \in L_a$.*

Proof: It is easy to verify that $(L_a, \vee, \wedge, 0, a)$ is a bounded lattice. Let $x, y \in L$ such that $x = a \wedge x, y = a \wedge y$. Then,

$$x^{**} = (a \wedge x)^{**} = a \wedge (a \wedge x')' = a \wedge (a' \vee x) = a \wedge x = x,$$

$$(x \vee y)^* = a \wedge (x \vee y)' = a \wedge x' \wedge y' = (a \wedge x') \wedge (a \wedge y') = x^* \wedge y^*,$$

and

$$\begin{aligned}
(x \wedge y)^* &= a \wedge (x \wedge y)' \\
&= a \wedge (x' \vee y') \\
&= (a \wedge x') \vee (a \wedge y') \quad \text{(since } a \text{ is } \wedge\text{-distributive)} \\
&= x^* \vee y^*
\end{aligned}$$

Therefore $(L_a, \vee, \wedge,^*, 0, a)$ is itself an ortholattice. □

Theorem 3.7 *Let a be a $\wedge$-distributive element in B. Then the mapping $f : L \to L_a$ defined by $f(x) = a \wedge x$, for all $x \in L$, is a homomorphism from L onto L_a.*

Proof: Let $x, y \in L$. Then

$$f(x \wedge y) = a \wedge x \wedge y = (a \wedge x) \wedge (a \wedge y) = f(x) \wedge f(y)$$

$$\begin{aligned} f(x \vee y) &= a \wedge (x \vee y) \\ &= (a \wedge x) \vee (a \wedge y) \quad \text{(since } a \text{ is } \wedge\text{-distributive)} \\ &= f(x) \vee f(y) \end{aligned}$$

and $f(x') = a \wedge x' = a \wedge (a' \vee x')$ (since a is $\wedge$-distributive) $= a \wedge (a \wedge x)' = (f(x))^*$. Therefore f is a homomorphism from L onto L_a. □

Definition 3.8 *An element a of L is said to be $\vee$-distributive, if for any $x, y \in L$, $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y)$.*

It is easy to prove that if a and b are $\vee$-distributive, then $a \vee b$ is also $\vee$-distributive.

Theorem 3.9 *Let a be a $\vee$-distributive element in L. Then the set $R_a = \{a \vee x \mid x \in L\}$ is itself an ortholattice with the induced operations $\vee$ & $\wedge$, and the unary operation $*$ is defined by $x^* = (a \vee x)^* = a \vee x'$, for all $x \in R_a$.*

Proof: Let $x, y \in L$ such that $x = a \vee x$ and $y = a \vee y$. Then,

$$x \vee y = (a \vee x) \vee (a \vee y) = a \vee (x \vee y) \in R_a,$$

$$x \wedge y = (a \vee x) \wedge (a \vee y) = a \vee (x \wedge y) \in R_a \text{ (since } a \text{ is } \vee-\text{distributive),}$$

and

$$x^* = a \vee x' \in R_a.$$

Therefore $(R_a, \vee, \wedge, ^*, a, 1)$ is a bounded lattice. For $x, y \in L$,

$$\begin{aligned} x^{**} &= (a \vee x')^* \\ &= a \vee (a' \wedge x'') \\ &= (a \vee a') \wedge (a \vee x) \quad \text{(since } a \text{ is } \vee-\text{distributive)} \\ &= 1 \wedge (a \vee x) \\ &= a \vee x \\ &= x. \end{aligned}$$

$$\begin{aligned} (x \wedge y)^* &= a \vee (x \wedge y)' \\ &= a \vee (x' \vee y') \\ &= (a \vee x') \vee (a \vee y') \\ &= x^* \vee y^*. \end{aligned}$$

$$\begin{aligned} (x \vee y)^* &= a \vee (x \vee y)' \\ &= a \vee (x' \wedge y') \\ &= (a \vee x') \wedge (a \vee y') \quad \text{(since } a \text{ is } \vee-\text{distributive)} \\ &= x^* \wedge y^*. \end{aligned}$$

Therefore R_a is an ortholattice. □

Theorem 3.10 *Let a be a $\vee$-distributive element in B. Then the mapping $g : L \to R_a$ defined by $g(x) = a \vee x$, for all $x \in L$, is a homomorphism from L onto R_a.*

Proof: Let $x, y \in L$. Then

$$\begin{aligned} g(x \wedge y) &= a \vee (x \wedge y) \\ &= (a \vee x) \wedge (a \vee y) \quad \text{(since } a \text{ is } \vee\text{-distributive)} \\ &= g(x) \wedge g(y) \end{aligned}$$

$$g(x \vee y) = a \vee (x \vee y) = (a \vee x) \vee (a \vee y) = g(x) \vee g(y).$$

and $g(x') = a \vee x' = 1 \wedge (a \vee x') = a \vee (a' \wedge x') = a \vee (a \vee x)' = a \vee g(x)' = g(x)^*$ (since a is $\vee$-distributive). Therefore g is a homomorphism from L onto R_a.
□

4 C-elements in ortholattices

In this section, we define C-elements in an ortholattice. For each C-element, we obtain a factor congruence and hence it leads to a decomposition for an ortholattice.

Definition 4.1 *A B-element a with respect to $\wedge$ (or with respect to $\vee$) of an ortholattice L is said to be a C-element, if it satisfies the following conditions; for any $x, y \in L$,*
(i) $(a \vee a') \wedge x = (a \wedge x) \vee (a' \wedge x) = x$
(ii) $a \wedge [(a \wedge x) \vee (a' \wedge y)] = a \wedge x$
(iii) $a' \wedge [(a \wedge x) \vee (a' \wedge y)] = a' \wedge y$

It can be easy to verify that if a is a C-element of L, then a' is also a C-element of L.

Note 4.2 *Every B-element need not be a C-element in L. For, see Example 2.3., a is a B-element but not a C-element in L (because $a = (a \wedge b) \vee (a' \wedge b) \neq b$)*

Lemma 4.3 *For any C-element a of L, $\theta_a \cap \theta_{a'} = \Delta$.*

Proof: Let $(x, y) \in \theta_a \cap \theta_{a'}$. Then $a \wedge x = a \wedge y$ and $a' \wedge x = a' \wedge y$. Now,

$$\begin{aligned} x = 1 \wedge x &= (a \vee a') \wedge x \\ &= (a \wedge x) \vee (a' \wedge x) \quad \text{(since } a \text{ is a C-element)} \\ &= (a \wedge y) \vee (a' \wedge y) \\ &= (a \vee a') \wedge y \quad \text{(since } a \text{ is a C-element)} \\ &= 1 \wedge y = y \end{aligned}$$

Therefore $\theta_a \cap \theta_{a'} = \Delta$. □

Lemma 4.4 *For any C-element a of L, $\theta_a o\ \theta_{a'} = L \times L$.*

Proof: Let $x, y \in L$. Take $t = (a \wedge x) \vee (a' \wedge y)$. Then $a \wedge t = a \wedge [(a \wedge x) \vee (a' \wedge y)] = a \wedge x$ (since a is a central element). Therefore $(t, x) \in \theta_a$. Similarly, $a' \wedge t = a' \wedge [(a \wedge x) \vee (a' \wedge y)] = a' \wedge y$ (since a is a central element). Therefore $(t, y) \in \theta_{a'}$. Hence $(x, y) \in \theta_a o\ \theta_{a'}$. Thus $\theta_a o \theta_{a'} = L \times L$. □

Now, we have the following from the above two lemmas

Theorem 4.5 *If a is a C-element of L, then θ_a is a factor congruence on L.*

Theorem 4.6 *If a is a C-element of L and a & a' are $\wedge$-distributive then $L \cong L_a \times L_{a'}$.*

Proof: Define $h : L \to L_a \times L_{a'}$ by $h(x) = (a \wedge x, a' \wedge x)$ for all $x \in L$. Then h is well-defined and onto. Let $x, y \in L$ such that $h(x) = h(y)$. Then $a \wedge x = a \wedge y$ and $a' \wedge x = a' \wedge y$. Now,

$$\begin{aligned} x &= (a \vee a') \wedge x \\ &= (a \wedge x) \vee (a' \wedge x) \quad \text{(since } a \text{ is a C-element)} \\ &= (a \wedge y) \vee (a' \wedge y) \\ &= (a \vee a') \wedge y \quad \text{(since } a \text{ is a C-element)} \\ &= y \end{aligned}$$

Therefore h is one-one. Hence h is bijective. It is easy to verify that h is a homomorphism and hence h is an isomorphism from L onto $L_a \times L_{a'}$. □

We conclude this paper with the following which are similar to 4.3, 4.4, 4.5 and 4.6.

Lemma 4.7 *If a is a C-element of L, then*
(i) $\psi_a \cap \psi_{a'} = \Delta$
(ii) $\psi_a\ o\ \psi_{a'} = L \times L$.

Lemma 4.8 *If a is a C-element of L, then ψ_a is a factor congruence on L*

Theorem 4.9 *If a is a C-element of L and a & a' are $\vee$-distributive, then $L \cong R_a \times R_{a'}$.*

5 Open problems

1. If one can exhibit algebraic operations on congruences (θ_a and ψ_a) in an ortholattice, then it may leads some fruitful results in terms of B-elements and vice versa.

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