On \((\beta, \alpha)\)-logarithmically convex functions in the first and second sense with their inequalities

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Abstract

In the paper, the authors introduce a new concept "\((\beta, \alpha)\)-logarithmically convex functions in the first and second sense" and establish Hermite-Hadamard type integral inequalities for these convexities.

Keywords: convexity, logarithmically convexity, Hermite-Hadamard inequality.

1 Preliminaries

A function \(f: I \rightarrow \mathbb{R}\) is said to be convex on \(I\) if the inequality

\[ f(\lambda u + (1 - \lambda) v) \leq \lambda f(u) + (1 - \lambda) f(v) \]  

holds for all \(u, v \in I\) and \(\lambda \in [0, 1]\). We say that \(f\) is concave if \(-f\) is convex.

In this section we give some necessary definitions which are used throughout this paper. Let \(f: I \subseteq \mathbb{R} \rightarrow \mathbb{R}\) be a convex mapping and \(u, v \in I\) with \(u < v\). The following double inequality

\[ f \left( \frac{u + v}{2} \right) \leq \frac{1}{v - u} \int_u^v f(z) \, dz \leq \frac{f(u) + f(v)}{2} \]  

is known in the literature as Hadamard’s inequality (or Hermite-Hadamard inequality) for convex mapping. Note that some of the classical inequalities for means can be derived from (2) for appropriate particular selections of the
mapping \( f \). If \( f \) is a positive concave function, then the inequalities are reversed. For some results which generalize, improve and extend the inequality (2) see [2, 3, 6, 8, 9, 10, 11, 12, 14] and the references therein.

**Definition 1.1** [7] The function \( f : I \to \mathbb{R} \) is said to be \((\alpha, \beta)\)-convex function if for all \((\alpha, \beta) \in [0, 1]^2\) and \( \lambda \in [0, 1] \) we have

\[
    f(\lambda u + (1 - \lambda) v) \leq \lambda^\alpha f(u) + (1 - \lambda)^\beta f(v).
\]

The function \( f : I \subset \mathbb{R} \to [0, \infty) \) is said to be log-convex or multiplicative convex if \( \log f \) is convex, or, equivalently, if for all \( u, v \in I \) and \( \lambda \in [0, 1] \), one has the inequality

\[
    f(\lambda u + (1 - \lambda) v) \leq [f(u)]^\lambda [f(v)]^{(1-\lambda)}.
\]

In [1], Akdemir and Tunc defined the class of \( s \)-logarithmically convex functions in the first sense as follows:

**Definition 1.2** [1] A function \( f : I \subset \mathbb{R}_0 \to \mathbb{R}_+ \) is said to be \( s \)-logarithmically convex in the first sense if

\[
    f(\alpha u + \beta v) \leq [f(u)]^\alpha [f(v)]^\beta
\]

for some \( s \in (0, 1] \), where \( u, v \in I \) and \( \alpha s + \beta s = 1 \).

\( s \)-logarithmically convex functions in the second sense was defined in [13] as follows:

**Definition 1.3** [13] A function \( f : I \subset \mathbb{R}_0 \to \mathbb{R}_+ \) is said to be \( s \)-logarithmically convex in the second sense if

\[
    f(\lambda u + (1 - \lambda) v) \leq [f(u)]^\lambda [f(v)]^{(1-\lambda)}
\]

for some \( s \in (0, 1] \), where \( u, v \in I \) and \( \lambda \in [0, 1] \).

It can be easily checked for \( s = 1 \), in Definition 1.2 or inequality (4), then \( f \) becomes the ordinary logarithmically convex function on \( I \).

**Lemma 1.4** [3, Lemma 2.1] Let \( f : I \subset \mathbb{R}_0 \to \mathbb{R}_+ \) be a differentiable mapping on \( I^0 \), \( a, b \in I \) with \( a < b \). If \( f' \in L[a, b] \), then the following equality holds:

\[
    \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx = \frac{b-a}{2} \int_0^1 (1 - 2\lambda) f'(\lambda a + (1 - \lambda) b) \, d\lambda.
\]

The main purpose of this paper is to define new classes of convex functions, which is called the \((\beta, \alpha)\)-logarithmically convex functions in the first sense and second sense. Some new Hermite-Hadamard inequalities to obtain for these two new extensions of logarithmically convex functions.
2 New Definitions

Motivated by Definitions 1.2 and 1.3, now we introduce concepts of \((\beta, \alpha)\)-logarithmically convex functions in the first sense and second sense.

**Definition 2.1** A function \(f : I \subset \mathbb{R}_0 \to \mathbb{R}_+\) is said to be \((\beta, \alpha)\)-logarithmically convex in the first sense if

\[
f (tx + (1 - t)y) \leq [f (x)]^t [f (y)]^{(1-t)^\alpha}\]

for some \((\beta, \alpha) \in (0,1]^2\), where \(x, y \in I\) and \(t \in [0,1]\). We denote by \(L^{\beta,\alpha}_1 (I)\) the set of all \((\beta, \alpha)\)-logarithmically convex in the first sense functions on \(I\).

**Remark 2.2** In Definition 2.1, if we take

\(\text{i) } \beta = \alpha = 1\), then \(f\) is the standard logarithmically convex function on \(I\).

\(\text{ii) } \beta = \alpha = s\), then \(f\) is the \(s\)-logarithmically convex function in the first sense on \(I\).

**Definition 2.3** A function \(f : I \subset \mathbb{R}_0 \to \mathbb{R}_+\) is said to be \((\beta, \alpha)\)-logarithmically convex in the second sense if

\[
f (tx + (1 - t)y) \leq [f (x)]^t [f (y)]^{(1-t)^\alpha}\]

for some \((\beta, \alpha) \in (0,1]^2\), where \(x, y \in I\) and \(t \in [0,1]\). We denote by \(L^{\beta,\alpha}_2 (I)\) the set of all \((\beta, \alpha)\)-logarithmically convex in the second sense functions on \(I\).

**Remark 2.4** In Definition 2.3, if we take

\(\text{i) } \beta = \alpha = 1\), then \(f\) is the standard logarithmically convex function on \(I\).

\(\text{ii) } \beta = \alpha = s\), then \(f\) is the \(s\)-logarithmically convex function in the second sense on \(I\).

**Definition 2.5** Let \(f : [0,b] = I \to \mathbb{R}\) be a function, \((\beta, \alpha) \in (0,1]^2\). Then \(f\) is said to be \((\beta, \alpha)\)-Godunova-Levin-log-convex functions in the first sense if

the inequality

\[
f (ta + (1 - t)b) \leq [f (a)]^\frac{1}{\beta} [f (b)]^\frac{1}{1-\beta}
\]

holds for all \(a, b \in I\) and \(t \in (0,1)\). It can be easily that for \((\beta, \alpha) \in \{(1,1), (s,s), (\alpha, \alpha)\}\) one obtains the following classes of functions: Godunova-Levin-log-convex function, \(s\)-Godunova-Levin-log-convex function in the first sense, \(\alpha\)-Godunova-Levin-log-convex function in the first sense.

**Definition 2.6** Let \(f : [0,b] = I \to \mathbb{R}\) be a function \((\beta, \alpha) \in (0,1]^2\). Then \(f\) is said to be \((\beta, \alpha)\)-Godunova-Levin-log-convex functions in the second sense if

the inequality

\[
f (ta + (1 - t)b) \leq [f (a)]^\frac{1}{\beta} [f (b)]^\frac{1}{1-\beta}
\]
holds for all \(a, b \in I\) and \(t \in (0, 1)\). It can be easily that for \((\beta, \alpha) \in \{(1, 1), (s, s), (\alpha, \alpha)\}\) one obtains the following classes of functions: Godunova-Levin-log-convex function, \(s\)-Godunova-Levin-log-convex function in the second sense, \(\alpha\)-Godunova-Levin-log-convex function in the second sense.

**Lemma 2.7** (See [13]) If \(0 < \mu \leq 1 \leq \eta\), \(0 < \alpha, s \leq 1\), then

\[
\mu^{\alpha s} \leq \mu^{s\alpha} \quad \text{and} \quad \eta^{\alpha s} \leq \eta^{s\alpha+1-s}. \tag{7}
\]

**Lemma 2.8** Let \(t \in [0, 1]\). Then

\[
\int_0^1 |1 - 2t| k^t dt = \left[ \frac{k - 1}{\ln k} - 2 \left( \frac{\sqrt{k} - 1}{\ln k} \right)^2 \right] = M(k; \beta, \alpha) \tag{8}
\]

where \(k = \frac{|f'(a)|^\beta}{|f'(b)|^\beta}\).

**Proof:** Proof is directly clear via integrating by parts.

### 3 Main Results

**Theorem 3.1** Let \(f : I \subset \mathbb{R} \to \mathbb{R}_+\) be a differentiable mapping on \(I^o\), \(a, b \in I\) with \(a < b\) and \(f \in L[a, b]\). If \(f\) is \((\beta, \alpha)\)-logarithmically convex in the second sense, \((\beta, \alpha) \in (0, 1]^2\), then the following inequality holds:

\[
\frac{1}{b-a} \int_a^b f(x)dx \leq \left\{ \begin{array}{c}
L \left( [f(a)]^\beta, [f(b)]^\alpha \right) \\
[f(b)]^{1-\alpha} L \left( [f(a)]^\beta, [f(b)]^\alpha \right) \\
[f(a)]^{1-\beta} L \left( [f(a)]^\beta, [f(b)]^\alpha \right)
\end{array} \right\}, \quad 0 < f(a), f(b) \leq 1
\]

where \(L\) is logarithmic mean.

**Proof:** Since \(f \in L_2^{\beta, \alpha}(I)\), we have

\[
f(ta + (1-t)b) \leq [f(a)]^\eta \cdot [f(b)]^{(1-t)\alpha} \tag{9}
\]

for all \(t \in [0, 1]\). Integrating this inequality on \([0, 1]\), we get

\[
\int_0^1 f(ta + (1-t)b) dt = \frac{1}{b-a} \int_a^b f(x)dx \leq \int_0^1 [f(a)]^\eta \cdot [f(b)]^{(1-t)\alpha} dt.
\]
From (7), if \(0 < f(a), f(b) \leq 1\), we get
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq [f(b)]^\alpha \int_0^1 \left( \frac{[f(a)]^\beta}{[f(b)]^\alpha} \right)^t dt
\]
\[
= [f(b)]^\alpha \left[ \frac{[f(a)]^\beta}{[f(b)]^\alpha} - \frac{1}{\ln \left( \frac{[f(a)]^\beta}{[f(b)]^\alpha} \right)} \right]
\]
\[
= \frac{[f(a)]^\beta - [f(b)]^\alpha}{\ln [f(a)]^\beta - \ln [f(b)]^\alpha} = L \left( [f(a)]^\beta, [f(b)]^\alpha \right).
\]
If \(0 < f(a) \leq 1 \leq f(b)\), it is easy to see that
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq [f(b)]^\alpha \int_0^1 \left( \frac{[f(a)]^\beta}{[f(b)]^\alpha} \right)^t dt
\]
\[
= [f(b)]^{1-\alpha} L \left( [f(a)]^\beta, [f(b)]^\alpha \right).
\]
If \(0 \leq f(b) \leq 1 \leq f(a)\), it is easy to see that
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq [f(a)]^{1-\beta} [f(b)]^\alpha \int_0^1 \left( \frac{[f(a)]^\beta}{[f(b)]^\alpha} \right)^t dt
\]
\[
= [f(a)]^{1-\beta} L \left( [f(a)]^\beta, [f(b)]^\alpha \right).
\]
If \(1 \leq f(a), f(b)\), so we reach
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq [f(a)]^{1-\beta} [f(b)]^\alpha \int_0^1 \left( \frac{[f(a)]^\beta}{[f(b)]^\alpha} \right)^t dt
\]
\[
= [f(a)]^{1-\beta} [f(b)]^{1-\alpha} L \left( [f(a)]^\beta, [f(b)]^\alpha \right).
\]
The proof is completed by combining the above four inequality.

**Remark 3.2** i) In Theorem 3.1, if we take \(\beta = \alpha = 1\), then we have (see [4])
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq L ([f(a)], [f(b)]).
\]

ii) In Theorem 3.1, if we choose \(\beta = \alpha = s\), then we obtain following inequality
\[
\frac{1}{b-a} \int_a^b f(x)dx \leq \left\{ \begin{array}{ll}
L ([f(a)]^s, [f(b)]^s), & 0 < f(a), f(b) \leq 1 \\
[f(b)]^{1-s} L ([f(a)]^s, [f(b)]^s), & 0 < f(a) \leq 1 \leq f(b) \\
[f(a)]^{1-s} L ([f(a)]^s, [f(b)]^s), & 0 \leq f(b) \leq 1 \leq f(a) \\
[f(a)]^{1-s} [f(b)]^{1-s} L ([f(a)]^s, [f(b)]^s), & 1 \leq f(a), f(b)
\end{array} \right.
\]
Theorem 3.3 Let $f : I \subset \mathbb{R} \rightarrow \mathbb{R}_+$ be a differentiable mapping on $I^o$, $a, b \in I^o$ with $a < b$ and $f \in L[a,b]$. If $|f'| \in L^2_{\beta, \alpha}(I)$, $(\beta, \alpha) \in (0, 1]^2$, then the following inequality holds:

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \begin{cases} |f'(b)|^\alpha M(k; \beta, \alpha), & 0 < f'(a), f'(b) \leq 1 \\ |f'(b)| M(k; \beta, \alpha), & 0 < f'(a) \leq 1 \leq f'(b) \\ |f'(a)|^{1-\beta} |f'(b)|^\alpha M(k; \beta, \alpha), & 0 \leq f'(b) \leq 1 \leq f'(a) \\ |f'(a)|^{1-\beta} |f'(b)| M(k; \beta, \alpha), & 1 \leq f'(a), f'(b) \end{cases}$$

where $M(k; \beta, \alpha)$ and $k$ are given by Lemma 2.8.

**Proof:** As $|f'| \in L^2_{\beta, \alpha}(I)$, using Lemma 2.7 and 2.8 respectively, additionally if we take $0 < f'(a), f'(b) \leq 1$, then

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \int_0^1 \left| 1 - 2t \right| f'(ta + (1-t)b) \, dt$$

$$\leq \int_0^1 \left| 1 - 2t \right| |f'(a)|^\beta |f'(b)|^{(1-t)^\alpha} \, dt$$

$$\leq \int_0^1 \left| 1 - 2t \right| |f'(a)|^{\beta t} |f'(b)|^\alpha (1-t)^\alpha \, dt$$

$$= |f'(b)|^\alpha \int_0^1 \left| 1 - 2t \right| t \left[ \frac{|f'(a)|^\beta}{|f'(b)|^\alpha} \right]^t \, dt$$

$$= |f'(b)|^\alpha \int_0^1 \left| 1 - 2t \right| k^t \, dt$$

$$= |f'(b)|^\alpha M(k; \beta, \alpha).$$
If we take \( 0 < f' (a) \leq 1 \leq f' (b) \), we obtain

\[
\left| \frac{f (a) + f (b)}{2} - \frac{1}{b - a} \int_a^b f (x) \, dx \right| \leq \int_0^1 |1 - 2t| |f' (ta + (1 - t)b)| \, dt
\]

\[
\leq \int_0^1 |1 - 2t| |f'(a)|^\alpha |f' (b)|^{(1-t)\alpha} \, dt
\]

\[
\leq \int_0^1 |1 - 2t| |f'(a)|^{\beta|f' (b)|^{(1-t) + 1 - \alpha} \, dt
\]

\[
= |f' (b)| \int_0^1 |1 - 2t| \left[ \frac{|f'(a)|^\beta}{|f' (b)|^\alpha} \right] t \, dt
\]

\[
= |f' (b)| M (k; \beta, \alpha).
\]

If we obtain \( 0 \leq f' (b) \leq 1 \leq f' (a) \), we get

\[
\left| \frac{f (a) + f (b)}{2} - \frac{1}{b - a} \int_a^b f (x) \, dx \right| \leq \int_0^1 |1 - 2t| |f' (ta + (1 - t)b)| \, dt
\]

\[
\leq \int_0^1 |1 - 2t| |f'(a)|^\alpha |f' (b)|^{(1-t)\alpha} \, dt
\]

\[
\leq \int_0^1 |1 - 2t| |f'(a)|^{\beta+1-\beta} |f' (b)|^{\alpha(1-t)} \, dt
\]

\[
= |f'(a)|^{1-\beta} |f' (b)|^\alpha \int_0^1 |1 - 2t| \left[ \frac{|f'(a)|^\beta}{|f' (b)|^\alpha} \right] t \, dt
\]

\[
= |f'(a)|^{1-\beta} |f' (b)|^\alpha M (k; \beta, \alpha).
\]
If we take $1 \leq f'(a), f'(b)$, we get
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \int_0^1 |1 - 2t| f'(ta + (1-t)b) dt \\
\leq \int_0^1 |1 - 2t| |f'(a)|^\beta |f'(b)|^{(1-t)\alpha} \, dt \\
\leq \int_0^1 |1 - 2t| |f'(a)|^{\beta+1-\beta} |f'(b)|^{\alpha(1-t)+1-\alpha} \, dt \\
= |f'(a)|^{1-\beta} |f'(b)| \int_0^1 |1 - 2t| \left[ \frac{|f'(a)|^{\beta}}{|f'(b)|^{\alpha}} \right]^{t} \, dt \\
= |f'(a)|^{1-\beta} |f'(b)| M(k; \beta, \alpha).
\]
We reach required result.

**Remark 3.4**

i) In Theorem 3.3, if we take $\beta = \alpha = 1$, then we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq |f'(b)| M(k; 1, 1),
\]
ii) In Theorem 3.3, if we take $\beta = \alpha = s$, then we have
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \begin{cases} |f'(b)|^s M(k; s, s) & , 0 < f'(a), f'(b) \leq 1 \\
|f'(b)| M(k; s, s) & , 0 < f'(a) \leq 1 \leq f'(b) \\
|f'(a)|^{1-s} |f'(b)|^s M(k; s, s) & , 0 \leq f'(b) \leq 1 \leq f'(a) \\
|f'(a)|^{1-s} |f'(b)| M(k; s, s) & , 1 \leq f'(a), f'(b) \end{cases}
\]
where $M(k; \beta, \alpha)$ is as in Lemma 2.8.

**Theorem 3.5** Under the conditions of Theorem 3.3, if $|f'|^q \in L_2^{\beta,\alpha}(I)$, $(\beta, \alpha) \in (0,1)^2$ for $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:
\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \left( \frac{1}{p+1} \right)^{1/p} \times \\
\begin{cases} \\
\left[ L \left( |f'(a)|^{3q}, |f'(b)|^{aq} \right) \right]^{1/q} & , 0 < f'(a), f'(b) \leq 1 \\
\left[ |f'(b)|^{1-\alpha} L \left( |f'(a)|^{3q}, |f'(b)|^{aq} \right) \right]^{1/q} & , 0 < f'(a) \leq 1 \leq f'(b) \\
\left[ |f'(a)|^{1-\beta} L \left( |f'(a)|^{3q}, |f'(b)|^{aq} \right) \right]^{1/q} & , 0 \leq f'(b) \leq 1 \leq f'(a) \\
\left[ |f'(a)|^{1-\beta} |f'(b)|^{1-\alpha} L \left( |f'(a)|^{3q}, |f'(b)|^{aq} \right) \right]^{1/q} & , 1 \leq f'(a), f'(b) \end{cases}
\]
where $M(k; \beta, \alpha)$ and $k$ are given by Lemma 2.8.
Proof: Under the assumptions, using Hölder inequality, we obtain

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| \leq \int_0^1 \left| 1 - 2t \right| \left| f'(ta + (1-t)b) \right| dt
\]

\[
\leq \left( \int_0^1 \left| 1 - 2t \right|^p dt \right)^{1/p} \left( \int_0^1 \left| f'(ta + (1-t)b) \right|^q dt \right)^{1/q}
\]

\[
\leq \left( \frac{1}{p+1} \right)^{1/p} \left( \int_0^1 \left| f'(a) \right|^q \left| f'(b) \right|^q (1-t) \alpha dt \right)^{1/q}
\]

For \( 0 < f'(a), f'(b) \leq 1 \), we get

\[
\int_0^1 \left| f'(a) \right|^{\beta q} \left| f'(b) \right|^{\alpha (1-t)} dt \leq \int_0^1 \left| f'(a) \right|^{\beta q} \left| f'(b) \right|^{\alpha (1-t)} dt
\]

\[
= \left| f'(b) \right|^{\alpha q} \int_0^1 \left[ \frac{\left| f'(a) \right|^{\beta q}}{\left| f'(b) \right|^{\alpha q}} \right]^{t} dt
\]

\[
= \left| f'(b) \right|^{\alpha q} \left[ \frac{\left| f'(a) \right|^{\beta q}}{\left| f'(b) \right|^{\alpha q}} - 1 \right]
\]

\[
= \frac{\left| f'(a) \right|^{\beta q} - \left| f'(b) \right|^{\alpha q}}{\ln \left[ \left| f'(a) \right|^{\beta q} / \left| f'(b) \right|^{\alpha q} \right]}
\]

\[
= L \left( \left| f'(a) \right|^{\beta q} , \left| f'(b) \right|^{\alpha q} \right).
\]

For \( 0 < f'(a) \leq 1 \leq f'(b) \), we get

\[
\int_0^1 \left| f'(a) \right|^{\beta q} \left| f'(b) \right|^{\alpha (1-t)} dt \leq \int_0^1 \left| f'(a) \right|^{\beta q} \left| f'(b) \right|^{\alpha (1-t) + 1 - \alpha} dt
\]

\[
= \left| f'(b) \right|^{\alpha q + 1 - \alpha} \int_0^1 \left[ \frac{\left| f'(a) \right|^{\beta q}}{\left| f'(b) \right|^{\alpha q}} \right]^{t} dt
\]

\[
= \left| f'(b) \right|^{1 - \alpha} \frac{\left| f'(a) \right|^{\beta q} - \left| f'(b) \right|^{\alpha q}}{\ln \left[ \left| f'(a) \right|^{\beta q} / \left| f'(b) \right|^{\alpha q} \right]}
\]

\[
= \left| f'(b) \right|^{1 - \alpha} L \left( \left| f'(a) \right|^{\beta q} , \left| f'(b) \right|^{\alpha q} \right).
\]
For $0 \leq f'(b) \leq 1 \leq f'(a)$, we get

$$
\int_0^1 |f'(a)|^{\alpha q} |f'(b)|^{q(1-t)\alpha} dt \leq \int_0^1 |f'(a)|^{\beta qt+1-\beta} |f'(b)|^{\alpha q(1-t)} dt
$$

$$
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q} \int_0^1 \left( \frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}} \right)^t dt
$$

$$
= |f'(a)|^{1-\beta} \frac{|f'(a)|^{\beta q} - |f'(b)|^{\alpha q}}{\ln |f'(a)|^\beta - \ln |f'(b)|^\alpha}
$$

$$
= |f'(a)|^{1-\beta} L \left( |f'(a)|^{\beta q}, |f'(b)|^{\alpha q} \right).
$$

For $1 \leq f'(a), f'(b)$, we get

$$
\int_0^1 |f'(a)|^{\alpha q} |f'(b)|^{q(1-t)\alpha} dt \leq \int_0^1 |f'(a)|^{\beta qt+1-\beta} |f'(b)|^{\alpha q(1-t)+1-\alpha} dt
$$

$$
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q+1-\alpha} \int_0^1 \left( \frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}} \right)^t dt
$$

$$
= |f'(a)|^{1-\beta} |f'(b)|^{1-\alpha} \frac{|f'(a)|^{\beta q} - |f'(b)|^{\alpha q}}{\ln |f'(a)|^\beta - \ln |f'(b)|^\alpha}
$$

$$
= |f'(a)|^{1-\beta} |f'(b)|^{1-\alpha} L \left( |f'(a)|^{\beta q}, |f'(b)|^{\alpha q} \right).
$$

The proof is completed.

**Theorem 3.6** Under the conditions of Theorem 3.3, if $|f'|^q \in L^2_{\alpha \beta} (I)$, for $q \geq 1$ and $(\beta, \alpha) \in (0,1)^2$, then the following inequality holds:

$$
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2(q-1)/q} \times
$$

$$
\begin{cases}
|f'(b)|^q M(k^q; \beta q, \alpha q)^{1/q}, & 0 < f'(a), f'(b) \leq 1 \\
|f'(b)|^{q+1-\alpha} M(k^q; \beta q, \alpha q)^{1/q}, & 0 < f'(a) \leq 1 \leq f'(b) \\
|f'(a)|^{1-\beta} |f'(b)|^{\alpha q} M(k^q; \beta q, \alpha q)^{1/q}, & 0 \leq f'(b) \leq 1 \leq f'(a) \\
|f'(a)|^{1-\beta} |f'(b)|^{q+1-\alpha} M(k^q; \beta q, \alpha q)^{1/q}, & 1 \leq f'(a), f'(b)
\end{cases}
$$

where $M(k; \beta, \alpha)$ and $k$ are defined by Lemma 2.8.
On \((\beta, \alpha)\)-logarithmically convex functions

**Proof:** By virtue of Definition 2.3, Lemma 2.7 and 2.8, and Hölder inequality, \(q \geq 1\), we have

\[
\left| \frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x) \, dx \right| 
\leq \int_0^1 |1 - 2t| |f'(ta + (1-t)b)| \, dt
\]

\[
\leq \left( \int_0^1 |1 - 2t| \, dt \right)^{(q-1)/q} \left( \int_0^1 |1 - 2t| |f'(ta + (1-t)b)|^q \, dt \right)^{1/q}
\]

\[
\leq \frac{1}{2(q-1)/q} \left( \int_0^1 |1 - 2t| |f'(a)|^{\frac{q\beta}{q}} |f'(b)|^{q(1-\alpha)} \, dt \right)^{1/q}
\]

If \(0 < f'(a), f'(b) \leq 1\), we obtain

\[
\int_0^1 |1 - 2t| |f'(a)|^{\frac{q\beta}{q}} |f'(b)|^{q(1-\alpha)} \, dt 
\leq \int_0^1 |1 - 2t| |f'(a)|^{\beta q} |f'(b)|^{q(1-\alpha)} |f'(a)|^{\beta q} |f'(b)|^{q(1-\alpha)} \, dt
\]

\[
= |f'(b)|^{\alpha q} \int_0^1 |1 - 2t| \frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}} \, dt
\]

\[
= |f'(b)|^{\alpha q} \int_0^1 |1 - 2t| k^q \, dt
\]

\[
= |f'(b)|^{\alpha q} M(k^q; \beta q, \alpha q).
\]

If \(0 < f'(a) \leq 1 \leq f'(b)\), we get

\[
\int_0^1 |1 - 2t| |f'(a)|^{\frac{q\beta}{q}} |f'(b)|^{q(1-\alpha)} \, dt 
\leq \int_0^1 |1 - 2t| |f'(a)|^{\beta q} |f'(b)|^{q(1-\alpha)+1-\alpha} \, dt
\]

\[
= |f'(b)|^{\alpha q+1-\alpha} \int_0^1 |1 - 2t| \frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}} \, dt
\]

\[
= |f'(b)|^{\alpha q+1-\alpha} \int_0^1 |1 - 2t| k^q \, dt
\]

\[
= |f'(b)|^{\alpha q+1-\alpha} M(k^q; \beta q, \alpha q).
\]
If $0 \leq f'(b) \leq 1 \leq f'(a)$, hence

$$
\int_0^1 |1 - 2t| |f'(a)|^{\beta q} |f'(b)|^{\alpha (1-t)} dt
\leq \int_0^1 |1 - 2t| |f'(a)|^{\beta q+1-\beta} |f'(b)|^{\alpha q(1-t)} dt
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q} \int_0^1 |1 - 2t| \left[ \frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}} \right] t dt
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q} \int_0^1 |1 - 2t| k^q dt
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q} M(k^q; \beta q, \alpha q).
$$

If $1 \leq f'(a) , f'(b)$, then

$$
\int_0^1 |1 - 2t| |f'(a)|^{\beta q} |f'(b)|^{\alpha (1-t)} dt
\leq \int_0^1 |1 - 2t| |f'(a)|^{\beta q+1-\beta} |f'(b)|^{\alpha q(1-t)+1-\alpha} dt
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q+1-\alpha} \int_0^1 |1 - 2t| \left[ \frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}} \right] t dt
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q+1-\alpha} \int_0^1 |1 - 2t| k^q dt
= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q+1-\alpha} M(k^q; \beta q, \alpha q).
$$

We reach desired result.

4 Open Problem

It is a well-known fact that if $f$ is a convex function on the interval $I \subset \mathbb{R}$, then the Hadamard’s inequality retains for the convex functions. As a matter of
fact, it has been demonstrated lots of this type inequalities for several convex functions. Therefore, there is one questions as follows:

Under what conditions, the composition \( f \circ g \) or \( fg \) are \((\beta, \alpha)\)–logarithmically convex function on \( I \)? Can we prove Hadamard type inequalities for \( f \circ g \) or \( fg \).

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**References**


