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On (β,α)-logarithmically convex functions in the first and second sense with their inequalities

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Abstract

In the paper, the authors introduce a new concept "(β,α)-logarithmically convex functions in the first and second sense" and establish Hermite-Hadamard type integral inequalities for these convexities.
Keywords: *convexity, logarithmically convexity, Hermite-Hadamard inequality.*

1 Preliminaries

A function $f : I \to \mathbb{R}$ is said to be convex on I if the inequality

$$f(\lambda u + (1-\lambda)v) \leq \lambda f(u) + (1-\lambda) f(v) \tag{1}$$

holds for all $u, v \in I$ and $\lambda \in [0,1]$. We say that f is concave if $-f$ is convex.

In this section we give some necessary definitions which are used throughout this paper. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping and $u, v \in I$ with $u < v$. The following double inequality

$$f\left(\frac{u+v}{2}\right) \leq \frac{1}{v-u}\int_u^v f(z)\,dz \leq \frac{f(u)+f(v)}{2} \tag{2}$$

is known in the literature as Hadamard's inequality (or Hermite-Hadamard inequality) for convex mapping. Note that some of the classical inequalities for means can be derived from (2) for appropriate particular selections of the mapping f. If f is a positive concave function, then the inequalities are reversed. For some results which generalize, improve and extend the inequality (2) see [2, 3, 6, 8, 9, 10, 11, 12, 14] and the references therein.

Definition 1.1 *[7] The function $f : I \to \mathbb{R}$ is said to be (α, β)-convex function if for all $(\alpha, \beta) \in [0,1]^2$ and $\lambda \in [0,1]$ we have*

$$f(\lambda u + (1-\lambda) v) \leq \lambda^{\alpha} f(u) + (1-\lambda)^{\beta} f(v).$$

The function $f : I \subset \mathbb{R} \to [0, \infty)$ is said to be log-convex or multiplicative convex if $\log f$ is convex, or, equivalently, if for all $u, v \in I$ and $\lambda \in [0,1]$, one has the inequality

$$f(\lambda u + (1-\lambda) v) \leq [f(u)]^{\lambda} [f(v)]^{(1-\lambda)}.$$

In [1], Akdemir and Tunç defined the class of s-logarithmically convex functions in the first sense as the following:

Definition 1.2 *[1] A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex in the first sense if*

$$f(\alpha u + \beta v) \leq [f(u)]^{\alpha^s} [f(v)]^{\beta^s} \tag{3}$$

for some $s \in (0,1]$, where $u, v \in I$ and $\alpha^s + \beta^s = 1$.

s-logarithmically convex functions in the second sense was defined in [13] as follows:

Definition 1.3 *[13] A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be s-logarithmically convex in the second sense if*

$$f(\lambda u + (1-\lambda) v) \leq [f(u)]^{\lambda^s} [f(v)]^{(1-\lambda)^s} \tag{4}$$

for some $s \in (0,1]$, where $u, v \in I$ and $\lambda \in [0,1]$.

It can be easily checked for $s = 1$, in Definition 1.2 or inequality (4), then f becomes the ordinary logarithmically convex function on I.

Lemma 1.4 *[3, Lemma 2.1]Let $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$. If $f' \in L[a,b]$, then the following equality holds:*

$$\frac{f(a) + f(b)}{2} - \frac{1}{b-a} \int_a^b f(x)\, dx = \frac{b-a}{2} \int_0^1 (1-2\lambda) f'(\lambda a + (1-\lambda) b)\, d\lambda.$$

The main purpose of this paper is to define new classes of convex functions, which is called the (β, α)-logarithmically convex functions in the first sense and second sense. Some new Hermite-Hadamard inequalities to obtain for these two new extensions of logarithmically convex functions.

2 New Definitions

Motivated by Definitions 1.2 and 1.3, now we introduce concepts of (β,α)-logarithmically convex functions in the first sense and second sense.

Definition 2.1 *A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be (β,α)-logarithmically convex in the first sense if*

$$f\left(tx + (1-t)\, y\right) \leq \left[f\left(x\right)\right]^{t^{\beta}} \left[f\left(y\right)\right]^{\left(1-t^{\alpha}\right)} \tag{5}$$

for some $(\beta,\alpha) \in (0,1]^2$, where $x, y \in I$ and $t \in [0,1]$. We denote by $L_1^{\beta,\alpha}(I)$ the set of all (β,α)-logarithmically convex in the first sense functions on I.

Remark 2.2 *In Definition 2.1, if we take*

i) *$\beta = \alpha = 1$, then f is the standard logarithmically convex function on I.*

ii) *$\beta = \alpha = s$, then f is the s-logarithmically convex function in the first sense on I.*

Definition 2.3 *A function $f : I \subset \mathbb{R}_0 \to \mathbb{R}_+$ is said to be (β,α)-logarithmically convex in the second sense if*

$$f\left(tx + (1-t)\, y\right) \leq \left[f\left(x\right)\right]^{t^{\beta}} \left[f\left(y\right)\right]^{(1-t)^{\alpha}} \tag{6}$$

for some $(\beta,\alpha) \in (0,1]^2$, where $x, y \in I$ and $t \in [0,1]$. We denote by $L_2^{\beta,\alpha}(I)$ the set of all (β,α)-logarithmically convex in the second sense functions on I.

Remark 2.4 *In Definition 2.3, if we take*

i) *$\beta = \alpha = 1$, then f is the standard logarithmically convex function on I.*

ii) *$\beta = \alpha = s$, then f is the s-logarithmically convex function in the second sense on I.*

Definition 2.5 *Let $f : [0,b] = I \to \mathbb{R}$ be a function, $(\beta,\alpha) \in (0,1]^2$. Then f is said to be (β,α)-Godunova-Levin-*log*-convex functions in the first sense if the inequality*

$$f\left(ta + (1-t)\, b\right) \leq \left[f\left(a\right)\right]^{\frac{1}{t^{\beta}}} \left[f\left(b\right)\right]^{\frac{1}{1-t^{\alpha}}}$$

*holds for all $a, b \in I$ and $t \in (0,1)$. It can be easily that for $(\beta,\alpha) \in \{(1,1), (s,s), (\alpha,\alpha)\}$ one obtains the following classes of functions: Godunova-Levin-*log*-convex function, s-Godunova-Levin-*log*-convex function in the first sense, α-Godunova-Levin-*log*-convex function in the first sense.*

Definition 2.6 *Let $f : [0,b] = I \to \mathbb{R}$ be a function $(\beta,\alpha) \in (0,1]^2$. Then f is said to be (β,α)-Godunova-Levin-*log*-convex functions in the second sense if the inequality*

$$f\left(ta + (1-t)\, b\right) \leq \left[f\left(a\right)\right]^{\frac{1}{t^{\beta}}} \left[f\left(b\right)\right]^{\frac{1}{(1-t)^{\alpha}}}$$

*holds for all $a, b \in I$ and $t \in (0,1)$. It can be easily that for $(\beta,\alpha) \in \{(1,1),(s,s),(\alpha,\alpha)\}$ one obtains the following classes of functions: Godunova-Levin-*log*-convex function, s-Godunova-Levin-*log*-convex function in the second sense, α-Godunova-Levin-*log*-convex function in the second sense.*

Lemma 2.7 *(See [13]) If $0 < \mu \leq 1 \leq \eta$, $0 < \alpha, s \leq 1$, then*

$$\mu^{\alpha^s} \leq \mu^{s\alpha} \text{ and } \eta^{\alpha^s} \leq \eta^{\alpha s+1-s}. \tag{7}$$

Lemma 2.8 *Let $t \in [0,1]$. Then*

$$\int_0^1 |1-2t|\, k^t dt = \left[\frac{k-1}{\ln k} - 2\left(\frac{\sqrt{k}-1}{\ln k}\right)^2\right] = M(k;\beta,\alpha) \tag{8}$$

where $k = \frac{|f'(a)|^\beta}{|f'(b)|^\alpha}$.

Proof: Proof is directly clear via integrating by parts.

3 Main Results

Theorem 3.1 *Let $f : I \subset \mathbb{R} \to \mathbb{R}_+$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$ and $f \in L[a,b]$. If f is (β,α)-logarithmically convex in the second sense, $(\beta,\alpha) \in (0,1]^2$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \begin{cases} L\left([f(a)]^\beta, [f(b)]^\alpha\right) & , \quad 0 < f(a), f(b) \leq 1 \\ [f(b)]^{1-\alpha} L\left([f(a)]^\beta, [f(b)]^\alpha\right) & , \quad 0 < f(a) \leq 1 \leq f(b) \\ [f(a)]^{1-\beta} L\left([f(a)]^\beta, [f(b)]^\alpha\right) & , \quad 0 \leq f(b) \leq 1 \leq f(a) \\ [f(a)]^{1-\beta}[f(b)]^{1-\alpha} L\left([f(a)]^\beta, [f(b)]^\alpha\right) & , \quad 1 \leq f(a), f(b) \end{cases}$$

where L is logarithmic mean.

Proof: Since $f \in L_2^{\beta,\alpha}(I)$, we have

$$f(ta+(1-t)b) \leq [f(a)]^{t^\beta}[f(b)]^{(1-t)^\alpha} \tag{9}$$

for all $t \in [0,1]$. Integrating this inequality on $[0,1]$, we get

$$\int_0^1 f(ta+(1-t)b)\,dt = \frac{1}{b-a}\int_a^b f(x)dx \leq \int_0^1 [f(a)]^{t^\beta}[f(b)]^{(1-t)^\alpha}\,dt.$$

From (7), if $0 < f(a), f(b) \leq 1$, we get

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq [f(b)]^{\alpha}\int_0^1 \left(\frac{[f(a)]^{\beta}}{[f(b)]^{\alpha}}\right)^t dt \\ &= [f(b)]^{\alpha}\left[\frac{\frac{[f(a)]^{\beta}}{[f(b)]^{\alpha}}-1}{\ln\left(\frac{[f(a)]^{\beta}}{[f(b)]^{\alpha}}\right)}\right] \\ &= \frac{[f(a)]^{\beta}-[f(b)]^{\alpha}}{\ln[f(a)]^{\beta}-\ln[f(b)]^{\alpha}} = L\left([f(a)]^{\beta},[f(b)]^{\alpha}\right). \end{aligned}$$

If $0 < f(a) \leq 1 \leq f(b)$, it is easy to see that

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq [f(b)]\int_0^1 \left(\frac{[f(a)]^{\beta}}{[f(b)]^{\alpha}}\right)^t dt \\ &= [f(b)]^{1-\alpha} L\left([f(a)]^{\beta},[f(b)]^{\alpha}\right). \end{aligned}$$

If $0 \leq f(b) \leq 1 \leq f(a)$, it is easy to see that

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq [f(a)]^{1-\beta}[f(b)]^{\alpha}\int_0^1 \left(\frac{[f(a)]^{\beta}}{[f(b)]^{\alpha}}\right)^t dt \\ &= [f(a)]^{1-\beta} L\left([f(a)]^{\beta},[f(b)]^{\alpha}\right). \end{aligned}$$

If $1 \leq f(a), f(b)$, so we reach

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)dx &\leq [f(a)]^{1-\beta}[f(b)]\int_0^1 \left(\frac{[f(a)]^{\beta}}{[f(b)]^{\alpha}}\right)^t dt \\ &= [f(a)]^{1-\beta}[f(b)]^{1-\alpha} L\left([f(a)]^{\beta},[f(b)]^{\alpha}\right). \end{aligned}$$

The proof is completed by combining the above four inequality.

Remark 3.2 ***i)*** *In Theorem 3.1, if we take $\beta = \alpha = 1$, then we have (see [4])*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq L([f(a)],[f(b)]).$$

ii) *In Theorem 3.1, if we choose $\beta = \alpha = s$, then we obtain following inequality*

$$\frac{1}{b-a}\int_a^b f(x)dx \leq \begin{cases} L([f(a)]^s,[f(b)]^s) & , \quad 0 < f(a), f(b) \leq 1 \\ [f(b)]^{1-s} L([f(a)]^s,[f(b)]^s) & , \quad 0 < f(a) \leq 1 \leq f(b) \\ [f(a)]^{1-s} L([f(a)]^s,[f(b)]^s) & , \quad 0 \leq f(b) \leq 1 \leq f(a) \\ [f(a)]^{1-s}[f(b)]^{1-s} L([f(a)]^s,[f(b)]^s) & , \quad 1 \leq f(a), f(b) \end{cases}$$

Theorem 3.3 *Let $f: I \subset \mathbb{R} \to \mathbb{R}_{+}$ be a differentiable mapping on I°, $a, b \in I^{\circ}$ with $a < b$ and $f \in L[a, b]$. If $|f'| \in L_2^{\beta,\alpha}(I)$, $(\beta, \alpha) \in (0,1]^2$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \begin{cases} |f'(b)|^{\alpha} M(k;\beta,\alpha) & , \quad 0 < f'(a), f'(b) \leq 1 \\ |f'(b)| M(k;\beta,\alpha) & , \quad 0 < f'(a) \leq 1 \leq f'(b) \\ |f'(a)|^{1-\beta} |f'(b)|^{\alpha} M(k;\beta,\alpha) & , \quad 0 \leq f'(b) \leq 1 \leq f'(a) \\ |f'(a)|^{1-\beta} |f'(b)| M(k;\beta,\alpha) & , \quad 1 \leq f'(a), f'(b) \end{cases}$$

where $M(k;\beta,\alpha)$ and k are given by Lemma 2.8.

Proof: As $|f'| \in L_2^{\beta,\alpha}(I)$, using Lemma 2.7 and 2.8 respectively, additionally if we take $0 < f'(a), f'(b) \leq 1$, then

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \int_0^1 |1-2t|\, f'(ta+(1-t)b)dt \\ &\leq \int_0^1 |1-2t|\, |f'(a)|^{t^{\beta}} |f'(b)|^{(1-t)^{\alpha}} dt \\ &\leq \int_0^1 |1-2t|\, |f'(a)|^{\beta t} |f'(b)|^{\alpha(1-t)} dt \\ &= |f'(b)|^{\alpha} \int_0^1 |1-2t| \left[\frac{|f'(a)|^{\beta}}{|f'(b)|^{\alpha}}\right]^t dt \\ &= |f'(b)|^{\alpha} \int_0^1 |1-2t|\, k^t dt \\ &= |f'(b)|^{\alpha} M(k;\beta,\alpha). \end{aligned}$$

If we take $0 < f'(a) \leq 1 \leq f'(b)$, we obtain

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \int_0^1 |1-2t|\, f'(ta+(1-t)b)dt \\
&\leq \int_0^1 |1-2t|\, |f'(a)|^{t^\beta}\, |f'(b)|^{(1-t)\alpha}\, dt \\
&\leq \int_0^1 |1-2t|\, |f'(a)|^{\beta t}\, |f'(b)|^{\alpha(1-t)+1-\alpha}\, dt \\
&= |f'(b)| \int_0^1 |1-2t| \left[\frac{|f'(a)|^\beta}{|f'(b)|^\alpha}\right]^t dt \\
&= |f'(b)|\, M(k;\beta,\alpha).
\end{aligned}$$

If we obtain $0 \leq f'(b) \leq 1 \leq f'(a)$, we get

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \int_0^1 |1-2t|\, f'(ta+(1-t)b)dt \\
&\leq \int_0^1 |1-2t|\, |f'(a)|^{t^\beta}\, |f'(b)|^{(1-t)^\alpha}\, dt \\
&\leq \int_0^1 |1-2t|\, |f'(a)|^{\beta t+1-\beta}\, |f'(b)|^{\alpha(1-t)}\, dt \\
&= |f'(a)|^{1-\beta}\, |f'(b)|^{\alpha} \int_0^1 |1-2t| \left[\frac{|f'(a)|^\beta}{|f'(b)|^\alpha}\right]^t dt \\
&= |f'(a)|^{1-\beta}\, |f'(b)|^{\alpha}\, M(k;\beta,\alpha).
\end{aligned}$$

If we take $1 \leq f'(a), f'(b)$, we get

$$\begin{aligned}
\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \int_0^1 |1-2t|\, f'(ta+(1-t)b)dt \\
&\leq \int_0^1 |1-2t|\, |f'(a)|^{t^\beta} |f'(b)|^{(1-t)\alpha}\, dt \\
&\leq \int_0^1 |1-2t|\, |f'(a)|^{\beta t+1-\beta} |f'(b)|^{\alpha(1-t)+1-\alpha}\, dt \\
&= |f'(a)|^{1-\beta} |f'(b)| \int_0^1 |1-2t| \left[\frac{|f'(a)|^\beta}{|f'(b)|^\alpha}\right]^t dt \\
&= |f'(a)|^{1-\beta} |f'(b)|\, M(k;\beta,\alpha).
\end{aligned}$$

We reach required result.

Remark 3.4 ***i)*** *In Theorem 3.3, if we take $\beta=\alpha=1$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq |f'(b)|\, M(k;1,1),$$

ii) *In Theorem 3.3, if we take $\beta=\alpha=s$, then we have*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \begin{cases} |f'(b)|^s M(k;s,s) &, \quad 0<f'(a), f'(b) \leq 1 \\ |f'(b)|\, M(k;s,s) &, \quad 0<f'(a) \leq 1 \leq f'(b) \\ |f'(a)|^{1-s} |f'(b)|^s M(k;s,s) &, \quad 0 \leq f'(b) \leq 1 \leq f'(a) \\ |f'(a)|^{1-s} |f'(b)|\, M(k;s,s) &, \quad 1 \leq f'(a), f'(b) \end{cases}$$

where $M(k;\beta,\alpha)$ is as in Lemma 2.8.

Theorem 3.5 *Under the conditions of Theorem 3.3, if $|f'|^q \in L_2^{\beta,\alpha}(I)$, $(\beta,\alpha) \in (0,1]^2$ for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \left(\frac{1}{p+1}\right)^{1/p} \times$$

$$\begin{cases} \left[L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right)\right]^{1/q} &, \quad 0<f'(a), f'(b) \leq 1 \\ \left[|f'(b)|^{1-\alpha} L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right)\right]^{1/q} &, \quad 0<f'(a) \leq 1 \leq f'(b) \\ \left[|f'(a)|^{1-\beta} L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right)\right]^{1/q} &, \quad 0 \leq f'(b) \leq 1 \leq f'(a) \\ \left[|f'(a)|^{1-\beta} |f'(b)|^{1-\alpha} L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right)\right]^{1/q} &, \quad 1 \leq f'(a), f'(b) \end{cases}$$

where $M(k;\beta,\alpha)$ and k are given by Lemma 2.8.

Proof: Under the assumptions, using Hölder inequality, we obtain

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| &\leq \int_0^1 |1-2t|\left|f'(ta+(1-t)b)\right|dt \\ &\leq \left(\int_0^1 |1-2t|^p\,dt\right)^{1/p}\left(\int_0^1 \left|f'(ta+(1-t)b)\right|^q dt\right)^{1/q} \\ &\leq \left(\frac{1}{p+1}\right)^{1/p}\left(\int_0^1 |f'(a)|^{qt^\beta}\left|f'(b)\right|^{q(1-t)^\alpha} dt\right)^{1/q}.\end{aligned}$$

For $0<f'(a), f'(b)\leq 1$, we get

$$\begin{aligned}\int_0^1 |f'(a)|^{qt^\beta}\left|f'(b)\right|^{q(1-t)^\alpha} dt &\leq \int_0^1 |f'(a)|^{\beta qt}\left|f'(b)\right|^{\alpha q(1-t)} dt \\ &= \left|f'(b)\right|^{\alpha q}\int_0^1\left[\frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}}\right]^t dt \\ &= \left|f'(b)\right|^{\alpha q}\left[\frac{\frac{[f'(a)]^{\beta q}}{[f'(b)]^{\alpha q}}-1}{\ln\left(\frac{[f'(a)]^{\beta q}}{[f'(b)]^{\alpha q}}\right)}\right] \\ &= \frac{[f'(a)]^{\beta q}-[f'(b)]^{\alpha q}}{\ln[f'(a)]^{\beta q}-\ln[f'(b)]^{\alpha q}} \\ &= L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right).\end{aligned}$$

For $0<f'(a)\leq 1\leq f'(b)$, we get

$$\begin{aligned}\int_0^1 |f'(a)|^{qt^\beta}\left|f'(b)\right|^{q(1-t)^\alpha} dt &\leq \int_0^1 |f'(a)|^{\beta qt}\left|f'(b)\right|^{\alpha q(1-t)+1-\alpha} dt \\ &= \left|f'(b)\right|^{\alpha q+1-\alpha}\int_0^1\left[\frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}}\right]^t dt \\ &= \left|f'(b)\right|^{1-\alpha}\frac{[f'(a)]^{\beta q}-[f'(b)]^{\alpha q}}{\ln[f'(a)]^{\beta q}-\ln[f'(b)]^{\alpha q}} \\ &= \left|f'(b)\right|^{1-\alpha} L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right).\end{aligned}$$

For $0 \leq f'(b) \leq 1 \leq f'(a)$, we get

$$\begin{aligned}
\int_0^1 |f'(a)|^{qt^\beta} |f'(b)|^{q(1-t)^\alpha} dt &\leq \int_0^1 |f'(a)|^{\beta q t+1-\beta} |f'(b)|^{\alpha q(1-t)} dt \\
&= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q} \int_0^1 \left[\frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}}\right]^t dt \\
&= |f'(a)|^{1-\beta} \frac{[f'(a)]^{\beta q} - [f'(b)]^{\alpha q}}{\ln [f'(a)]^{\beta q} - \ln [f'(b)]^{\alpha q}} \\
&= |f'(a)|^{1-\beta} L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right).
\end{aligned}$$

For $1 \leq f'(a), f'(b)$, we get

$$\begin{aligned}
\int_0^1 |f'(a)|^{qt^\beta} |f'(b)|^{q(1-t)^\alpha} dt &\leq \int_0^1 |f'(a)|^{\beta q t+1-\beta} |f'(b)|^{\alpha q(1-t)+1-\alpha} dt \\
&= |f'(a)|^{1-\beta} |f'(b)|^{\alpha q+1-\alpha} \int_0^1 \left[\frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}}\right]^t dt \\
&= |f'(a)|^{1-\beta} |f'(b)|^{1-\alpha} \frac{[f'(a)]^{\beta q} - [f'(b)]^{\alpha q}}{\ln [f'(a)]^{\beta q} - \ln [f'(b)]^{\alpha q}} \\
&= |f'(a)|^{1-\beta} |f'(b)|^{1-\alpha} L\left([f'(a)]^{\beta q}, [f'(b)]^{\alpha q}\right).
\end{aligned}$$

The proof is completed.

Theorem 3.6 *Under the conditions of Theorem 3.3, if $|f'|^q \in L_2^{\beta,\alpha}(I)$, for $q \geq 1$ and $(\beta, \alpha) \in (0,1]^2$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,dx\right| \leq \frac{1}{2^{(q-1)/q}} \times$$

$$\begin{cases}
\left[|f'(b)|^{\alpha q} M(k^q; \beta q, \alpha q)\right]^{1/q} &, \quad 0 < f'(a), f'(b) \leq 1 \\
\left[|f'(b)|^{\alpha q+1-\alpha} M(k^q; \beta q, \alpha q)\right]^{1/q} &, \quad 0 < f'(a) \leq 1 \leq f'(b) \\
\left[|f'(a)|^{1-\beta} |f'(b)|^{\alpha q} M(k^q; \beta q, \alpha q)\right]^{1/q} &, \quad 0 \leq f'(b) \leq 1 \leq f'(a) \\
\left[|f'(a)|^{1-\beta} |f'(b)|^{\alpha q+1-\alpha} M(k^q; \beta q, \alpha q)\right]^{1/q} &, \quad 1 \leq f'(a), f'(b)
\end{cases}$$

where $M(k; \beta, \alpha)$ and k are defined by Lemma 2.8.

Proof: By virtue of Definition 2.3, Lemma 2.7 and 2.8, and Hölder inequality, $q \geq 1$, we have

$$\begin{aligned}
&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\,dx\right| \\
\leq\ & \int_0^1 |1-2t|\,|f'(ta+(1-t)b)|\,dt \\
\leq\ & \left(\int_0^1 |1-2t|\,dt\right)^{(q-1)/q}\left(\int_0^1 |1-2t|\,|f'(ta+(1-t)b)|^q\,dt\right)^{1/q} \\
\leq\ & \frac{1}{2^{(q-1)/q}}\left(\int_0^1 |1-2t|\,|f'(a)|^{qt^\beta}\,|f'(b)|^{q(1-t)^\alpha}\,dt\right)^{1/q}
\end{aligned}$$

If $0 < f'(a), f'(b) \leq 1$, we obtain

$$\begin{aligned}
\int_0^1 |1-2t|\,|f'(a)|^{qt^\beta}\,|f'(b)|^{q(1-t)^\alpha}\,dt &\leq \int_0^1 |1-2t|\,|f'(a)|^{\beta qt}\,|f'(b)|^{\alpha q(1-t)}\,dt \\
&= |f'(b)|^{\alpha q}\int_0^1 |1-2t|\left[\frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}}\right]^t dt \\
&= |f'(b)|^{\alpha q}\int_0^1 |1-2t|\,k^{qt}dt \\
&= |f'(b)|^{\alpha q}\,M(k^q;\beta q,\alpha q).
\end{aligned}$$

If $0 < f'(a) \leq 1 \leq f'(b)$, we get

$$\begin{aligned}
\int_0^1 |1-2t|\,|f'(a)|^{qt^\beta}\,|f'(b)|^{q(1-t)^\alpha}\,dt &\leq \int_0^1 |1-2t|\,|f'(a)|^{\beta qt}\,|f'(b)|^{\alpha q(1-t)+1-\alpha}\,dt \\
&= |f'(b)|^{\alpha q+1-\alpha}\int_0^1 |1-2t|\left[\frac{|f'(a)|^{\beta q}}{|f'(b)|^{\alpha q}}\right]^t dt \\
&= |f'(b)|^{\alpha q+1-\alpha}\int_0^1 |1-2t|\,k^{qt}dt \\
&= |f'(b)|^{\alpha q+1-\alpha}\,M(k^q;\beta q,\alpha q).
\end{aligned}$$

If $0 \leq f'(b) \leq 1 \leq f'(a)$, hence

$$\begin{aligned} & \int_0^1 |1-2t| \left|f'(a)\right|^{qt^{\beta}} \left|f'(b)\right|^{q(1-t)^{\alpha}} dt \\ \leq & \int_0^1 |1-2t| \left|f'(a)\right|^{\beta qt+1-\beta} \left|f'(b)\right|^{\alpha q(1-t)} dt \\ = & \left|f'(a)\right|^{1-\beta} \left|f'(b)\right|^{\alpha q} \int_0^1 |1-2t| \left[\frac{\left|f'(a)\right|^{\beta q}}{\left|f'(b)\right|^{\alpha q}}\right]^t dt \\ = & \left|f'(a)\right|^{1-\beta} \left|f'(b)\right|^{\alpha q} \int_0^1 |1-2t| \, k^{qt} dt \\ = & \left|f'(a)\right|^{1-\beta} \left|f'(b)\right|^{\alpha q} M\left(k^q; \beta q, \alpha q\right). \end{aligned}$$

If $1 \leq f'(a), f'(b)$, then

$$\begin{aligned} & \int_0^1 |1-2t| \left|f'(a)\right|^{qt^{\beta}} \left|f'(b)\right|^{q(1-t)^{\alpha}} dt \\ \leq & \int_0^1 |1-2t| \left|f'(a)\right|^{\beta qt+1-\beta} \left|f'(b)\right|^{\alpha q(1-t)+1-\alpha} dt \\ = & \left|f'(a)\right|^{1-\beta} \left|f'(b)\right|^{\alpha q+1-\alpha} \int_0^1 |1-2t| \left[\frac{\left|f'(a)\right|^{\beta q}}{\left|f'(b)\right|^{\alpha q}}\right]^t dt \\ = & \left|f'(a)\right|^{1-\beta} \left|f'(b)\right|^{\alpha q+1-\alpha} \int_0^1 |1-2t| \, k^{qt} dt \\ = & \left|f'(a)\right|^{1-\beta} \left|f'(b)\right|^{\alpha q+1-\alpha} M\left(k^q; \beta q, \alpha q\right). \end{aligned}$$

We reach desired result.

4 Open Problem

It is a well-known fact that if f is a convex function on the interval $I \subset \mathbb{R}$, then the Hadamard's inequality retains for the convex functions. As a matter of fact, it has been demonstrated lots of this type inequalities for several convex functions. Therefore, there is one questions as follows:

Under what conditions, the composition $f \circ g$ or fg are $(\beta, \alpha)-$logarithmically convex function on I? Can we prove Hadamard type inequalities for $f \circ g$ or fg.

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