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Solving Fractional Order Logistic Equation by Approximate Analytical Methods

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Abstract

In this paper, we consider Adomian Decomposition Method (ADM),and Variational Iteration Method (VIM), Homotopy Analysis Method (HAM), Homotopy Analysis Transform Method (HATM) to solve the Fractional-Order Logistic Equation (FOLE). Comparisons between the approximate analytical solution and exact solution of FOLE are shown.

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1 Introduction

The logistic growth model for continuous version [18] is

$$\frac{dP(t)}{dt} = \mu P(t)\left(1 - \frac{P(t)}{K}\right), \tag{1.1}$$

where μ is the growth rate and K is the carrying capacity. Multiplying both side of (1.1) by $\frac{1}{K}$ and defining $y = P/K$ the logistic growth model, becomes

$$\frac{dy(t)}{dt} = \mu y(t)(1 - y(t)). \tag{1.2}$$

The fractional order logistic equation (FOLE) has been discussed in the literature [3, 6, 7, 17]. A detailed study of existence, uniqueness, stability and approximate solutions of FOLE can be found in [6, 22, 24]. The paper is organized as follows. Section 2 contains the basic of fractional Calculus. Section 3 outlines the basic idea of the ADM, HAM, HATM and VIM. Section 4 deals with an application of the ADM, HAM, HATM and VIM to solve the fractional-order logistic growth model. and Section 5 contains the conclusions.

2 Fractional calculus

Well-known definitions of a fractional derivative of order $\alpha > 0$ have been given by Riemann, Liouville, Grunwald, Letnikov and Caputo [1, 4, 15, 20] and are based on generalized functions. The most commonly used definitions are those of Riemann and Liouville and Caputo. Here we give some basic definitions and properties of fractional calculus theory.

Definition 2.1. A real function $f(t)$, $t > 0$, is said to be in the space C_μ, $\mu \in R$, if there exists a real number $p > \mu$ such that $f(t) = t^p f_1(t)$, where $f_1(t) \in C[0, \infty)$, and it is said to be in the space C_μ^m iff $f^m \in C_m$, $m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\alpha \geq 0$ for a function $f \in C_\mu$, $\mu \geq -1$, is defined as

$$J_0^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, t > 0,$$

$$J^0 f(t) = f(t).$$

It has the following properties. For $f \in C_\mu$, $\mu \geq -1$, $\alpha, \beta \geq 0$, and $\gamma > -1$,

1. $J^\alpha J^\beta f(t) = J^{\alpha+\beta} f(t)$,
2. $J^\alpha J^\beta f(t) = J^\beta J^\alpha f(t)$,
3. $J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} t^{\alpha+\gamma}$.

Definition 2.3. The fractional derivative of $f(t)$ in the Caputo sense is defined as

$$D^\alpha f(t) = J^{m-\alpha} D^m f(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau)\, d\tau,$$

for $m-1 < \alpha \leq m, m \in \mathbb{N}, t > 0, f \in C_\mu^m, \mu \geq -1$, then

$$D^\alpha J^\alpha f(t) = f(t)$$

$$J^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} f^k(0^+) \frac{t^k}{k!}.$$

3 Fractional-Order Logistic Equation

In this section we apply the ADM, VIM, HAM and HATM to solve the fractional-order logistic equation. Consider

$$D^\alpha y(t) = \mu y(t)(1 - y(t)), \quad t > 0, \mu > 0, 0 < \alpha \leq 1, \tag{3.3}$$

with the initial condition

$$y(0) = y_0. \tag{3.4}$$

The exact solution of this equation for $\alpha = 1$, is

$$y(t) = \frac{y_0}{y_0 + (1 - y_0)e^{-\mu t}}. \tag{3.5}$$

3.1 Basic idea of ADM

We present the basic idea of the ADM [11] in this section by considering the following nonlinear ordinary differential equation

$$D^\alpha (y(t)) + R(y(t)) + N(y(t)) = g(t), \quad \alpha > 0 \tag{3.6}$$

subject to the initial value

$$y^{(k)}(0) = c_k, \quad k = 0, 1, 2, \cdots, n-1, \quad n-1 < \alpha < n \tag{3.7}$$

where R is the remaining linear operator, which might include other fractional derivatives operator $D^{\nu}(\nu < \alpha)$, N represent a nonlinear operator and $g(t)$ is a given continuous function. Now, applying j^{α} to both the sides of (3.6), we get

$$y(t) = \sum_{k=0}^{[\alpha]} c_k \frac{t^k}{k!} + J^{\alpha} g(t) - J^{\alpha} R(y(t)) - J^{\alpha} N(y(t)). \tag{3.8}$$

We employ the Adomian decomposition method to solve equations (3.7)–(3.8). Let

$$y = \sum_{m=0}^{\infty} y_m, \tag{3.9}$$

and

$$N(y) = \sum_{m=0}^{\infty} A_m, \tag{3.10}$$

where A_m are Adomin polynomials which depend upon y. In view of Equations (3.9)–(3.10), (3.8) takes the form

$$\sum_{m=0}^{\infty} y_m = \sum_{k=0}^{[\alpha]} c_k \frac{t^k}{k!} + J^{\alpha} g(t) - J^{\alpha} R(y(t)) - J^{\alpha} \sum_{m=0}^{\infty} A_m(y). \tag{3.11}$$

We set

$$y_0(t) = \sum_{k=0}^{[\alpha]} c_k \frac{t^k}{k!} + J^{\alpha} g(t); \tag{3.12}$$

$$y_m = -J^{\alpha} R(y(t)) - J^{\alpha} \sum_{m=0}^{\infty} A_m(y), m = 0, 1, \cdots \tag{3.13}$$

In order to determine the Adomian polynomials, we introduce a parameter λ and (3.10) becomes

$$N\left(\sum_{m=0}^{\infty} y_m \lambda^m\right) = \sum_{m=0}^{\infty} A_m \lambda^m. \tag{3.14}$$

Let $y_{\lambda}(t) = \sum_{m=0}^{\infty} y_m \lambda^m$. Then

$$A_m = \frac{1}{m!} \left[\frac{d^m}{d\lambda^m} N_{\lambda}(y)\right]_{\lambda=0}, \tag{3.15}$$

where

$$N_{\lambda}(y) = N(y_{\lambda}). \tag{3.16}$$

In view of (3.15) and (3.16), we get

$$\begin{aligned} A_m &= \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[Ny_\lambda\right]_{\lambda=0} \\ &= \frac{1}{m!}\frac{d^m}{d\lambda^m}\left[N(\sum_{m=0}^{\infty} y_m\lambda^m)\right]_{\lambda=0} \\ &= \left[\frac{1}{m!}\frac{d^m}{d\lambda^m}N(\sum_{m=0}^{\infty} y_m\lambda^m)\right]_{\lambda=0}. \end{aligned} \tag{3.17}$$

Hence, (3.12)–(3.13) and (3.17) lead to the following recurrence relations

$$y_0(t) = \sum_{k=0}^{[\alpha]} c_k + J^\alpha g(t), \quad y_m(t) = -J^\alpha R(y(t)) - J^\alpha \left[\frac{1}{m!}\frac{d^m}{d\lambda^m}N(\sum_{m=0}^{\infty} y_m\lambda^m)\right]_{\lambda=0} \tag{3.18}$$

We can approximate the solution y by the truncated series

$$\phi_k = \sum_{m=0}^{k-1} y_m, \quad lim_{k\to\infty}\phi_k = y(t).$$

3.2 Basic idea of HAM

The principles of the HAM and its applicability for various kinds of differential equations are given in [2, 8, 12, 13, 14, 23, 26, 27]. For convenience, we will present a review of the HAM [13]. To describe the basic idea of the standard HAM [5], we consider the nonlinear differential equation

$$\mathcal{N}[y(t)] = 0, \quad t \geq 0, \tag{3.14}$$

where $\mathcal{N}$ is nonlinear differential operator and $y(t)$ is an unknown function. Liao [12] constructed the so-called zeroth-order deformation equation :

$$(1-q)\mathcal{L}[\phi(t;q) - y_0(t)] = qhH(t)\mathcal{N}[\phi(t;q)], \tag{3.15}$$

where $q \in [0,1]$ is an embedding parameter , $h \neq 0$ is an auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\mathcal{L}$ ia an auxiliary linear operator, $\phi(t;q)$ is an unknown function , and $y_0(t)$ is an initial guess for $y(t)$ which satisfies the initial conditions. It should be emphasized that one has great freedom in choosing the initial guess $y_0(t)$, $\mathcal{L}$, h and $H(t)$. Obviously, when $q = 0$ and $q = 1$, the following relations hold respectively

$$\phi(t;0) = y_0(t), \qquad \phi(t;1) = y_1(t).$$

Expanding $\phi(t;q)$ in Taylor series with respect to q, one has

$$\phi(t;q) = y_0(t) + \sum_{m=1}^{\infty} y_m(t)q^m, \tag{3.16}$$

where

$$y_m(t) = \frac{1}{m!}\frac{\partial^m \phi(t;q)}{\partial q^m}|_{q=0}. \tag{3.17}$$

We assume that the auxiliary parameter h, the auxiliary function $H(t)$, the initial approximation $y_0(t)$ and the auxiliary linear operator $\mathcal{L}$ are selected such that the series (3.16) converges at $q = 1$, and one has

$$y(t) = y_0(t) + \sum_{m=1}^{\infty} y_m(t). \tag{3.18}$$

We can deduced the governing equation from the zero order deformation equation by define the vector

$$\overrightarrow{y_n} = \{y_0(t), y_1(t), y_2(t), \ldots, y_n(t)\}. \tag{3.19}$$

Differentiating (3.15) m-times with respect to q, then setting $q = 0$ and dividing them by $m!$, we have, using (3.17) , the so-called mth-order deformation equation

$$\mathcal{L}[y_m(t) - \chi_m y_{m-1}(t)] = \hbar H(t) R_m(\overrightarrow{y}_{m-1}(t)), \quad m = 1, 2, 3, ..., n, \tag{3.20}$$

where

$$R_m(\overrightarrow{y}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathcal{N}[\phi(t;q)]}{\partial q^{m-1}}|_{q=0}, \tag{3.21}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1 \\ 1, & m > 1 \end{cases} \tag{3.22}$$

3.3 Basic idea of VIM

To illustrate the basic concept of the variational iteration method, we consider the following general nonlinear equation

$$D^\alpha y(t) + Ny(t) = g(t), \tag{3.33}$$

According to the variational iteration method [9, 10, 25], we can construct a correction functional in the form

$$y_{m+1}(t) = y_0(t) + \int_0^t \lambda(s)\left(D^\alpha(y_m(s) + N(\tilde{y}_m(s)) - g(s)\right) ds, \tag{3.34}$$

where $y_0(x)$ is an initial approximation with possible unknowns, $\lambda(s)$ is a Lagrange multiplier which can be identified optimally via variational theory, the subscript m denotes the m-th approximation, and $\tilde{y}_m$ is considered as a restricted variation [10, 9], i.e. $\delta\tilde{y}_m = 0$. It is shown that this method is very effective and easy for linear problems as its exact solution can be obtained by only one iteration because $\lambda(s)$ can be exactly identified. To solve (3.33) by the VIM, we must first evaluate $\lambda(s)$ that will be identified optimally via integration by parts. Then the successive approximation $y_m(t), m = 0, 1, \cdots$, of the solution $y(t)$ will be readily obtained upon using $\lambda(s)$ and $y_0(t)$. The zeroth approximation y_0 may be any function that satisfies at least the initial and boundary conditions with $\lambda(s)$ determined. Then successive approximations $y_m(t), m = 0, 1, 2, \cdots$ follows immediately, and consequently the exact solution may be arrived since: $y = lim_{m\to\infty} y_m$.

3.4 Basic idea of HATM

In this section, we introduce an approximate analytical method, namely the HATM, which is a combination of the HAM and the LDM [28, 29, 30, 31, 32]. This scheme is simple to apply to linear and nonlinear fractional differential equations and requires less computational effort compared with other exiting methods.

3.4.1 Laplace Transform

Let $f(t)$ be defined for $0 \le t < \infty$. Then, when the improper integral exists, the Laplace transform $F(s)$ of $f(t)$, written symbolically as $F(s) = \mathcal{L}\{f(t)\}$, is defined by

$$F(s) = \int_0^\infty e^{-st} f(t)dt.$$

Lemma If $m-1 < \alpha \le m, m \in \mathbb{N}$, then the Laplace transform of the fractional derivative $D^\alpha f(t)$ is

$$\mathcal{L}(D^\alpha f(t)) = s^\alpha F(s) - \sum_{k=0}^{m-1} f^{(k)}(0^+) s^{\alpha-k-1}, \quad t > 0, \tag{3.35}$$

where $F(s)$ is the Laplace transform of $f(t)$.

Apply the Laplace transform to both sides of Equation (3.6) we obtain

$$\mathcal{L}\left(D^\alpha(t)\right) + \mathcal{L}\left(Ry(t) + Ny(t)\right) = \mathcal{L}g(t). \tag{3.36}$$

Using (3.35) we then have

$$\mathcal{L}y(t) - \frac{1}{s^\alpha} \sum_{i=0}^{m-1} y^{(i)}(0) s^{\alpha-i-1} - \frac{1}{s^\alpha}\left(\mathcal{L}\left(Ry(t) + \mathcal{N}y(t)\right) - \mathcal{L}g(t)\right) = 0. \tag{3.37}$$

We define the nonlinear operator

$$\mathcal{N}\left[\phi(t;q)\right] = \mathcal{L}\left[\phi(t;q)\right] - \frac{1}{s^{\alpha}}\sum_{i=0}^{m-1}\phi(t;q)^{(i)}(0)s^{\alpha-i-1} - \frac{1}{s^{\alpha}}\left(\mathcal{L}\left(R\phi(t;q)+\mathcal{N}\phi(t;q)\right)-\mathcal{L}g(t)\right). \tag{3.38}$$

In similar procedure with HAM, we can obtain the m-th-order deformation equation.

4 Applications

In this section we apply the ADM, VIM, HAM and HATM to solve FOLE. Consider

$$D^{\alpha}y(t) = \mu y(t)(1-y(t)), \quad t>0, \mu>0, 0<\alpha\leq 1, \tag{4.1}$$

with the initial condition

$$y(0) = y_0. \tag{4.2}$$

The exact solution of this equation for $\alpha = 1$, is

$$y(t) = \frac{y_0}{y_0+(1-y_0)e^{-\mu t}}. \tag{4.3}$$

4.1 The ADM for FOLE

In this section, we apply the ADM to solve (7.24). According to the Adomian decomposition

$$y(t) = \sum_{i=0}^{m-1} y^{(i)}\frac{t^i}{i!} + J^{\alpha}\mu(y-y^2), \tag{4.4}$$

$$y_0(t) = \sum_{i=0}^{m-1} y^{(i)}\frac{t^i}{i!}, \tag{4.5}$$

$$y_{i+1}(t) = J^{\alpha}\mu(y-y^2). \tag{4.6}$$

The Adomian polynomials are as follows
$A_0 = \mu(y_0-y_0^2)$, $\quad A_1 = \mu(y_1-2y_0y_1)$, $\quad A_2 = \frac{1}{2}\mu\left(-2y_1^2+2y_2-4y_0y_2\right)$, $\quad \cdots$
Now if we take $y_0(t) = y(0) = y_0$, we obtain

$$y_1(t) = -(y_0^2-y_0)\mu\frac{t^{\alpha}}{\Gamma(1+\alpha)}, \tag{4.7}$$

$$y_2(t) = \left(2y_0^3-3y_0^2+y_0\right)\mu^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \tag{4.8}$$

$$\vdots$$

4.2 The HAM for FOLE

In this section, we apply the HAM to solve (7.24). According to the explanations in section 3.2 we have $y_0(t) = y_0$ and have the m-th-order deformation

$$y_m(t) = \chi_m y_{m-1}(t) + \hbar J^\alpha[R_m(\overrightarrow{y}_{m-1}(t))], \tag{4.9}$$

where

$$R_m(\overrightarrow{y}_{m-1}(t)) = D^\alpha y_{m-1}(t) - \mu y_{m-1}(t) + \mu \sum_{i=0}^{m-1} y_i(t)\ y_{m-1-i}(t). \tag{4.10}$$

The first two terms of the HAM series solution take following form:-

$$y_1(t) = \mu h(y_0 - y_0^2)\frac{t^\alpha}{\Gamma(\alpha+1)}, \tag{4.11}$$

and

$$y_2(t) = (1+h)y_1 + \mu^2 h^2(y_0^2 - y_0)(2y_0 - 1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}. \tag{4.12}$$

Hence, according to equation (3.18), the approximate solution is:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \cdots \tag{4.13}$$

The series (4.13) contains the auxiliary parameter h. The result (4.13) is the same approximate solution is obtained in [19, 24]. Similar to [21] we can obtain the value of h so that the solution series is convergent for $-2 < h < 0$. The obtimal value of h is -0.834348 [7].

4.3 The VIM for FOLE

In this section, we apply the VIM to solve (7.24). According to the variational iteration method, formula (3.34) for (7.24) can be expressed in the following form:

$$y_m(t) = y_{m-1}(t) + J^\alpha \lambda(s)\left(D^\alpha y_{m-1}(s) + \mu y_{m-1}(s) - \mu y_{m-1}^2(s)\right). \tag{4.14}$$

We can find the value of $\lambda(s)$ as in [9, 10, 16]. This value is $\lambda(s) = -1$. Substituting this value of $\lambda(s)$ into (4.14), we obtain

$$y_m(t) = y_{m-1}(t) - J^\alpha\left(D^\alpha y_{m-1}(s) + \mu y_{m-1}(s) - \mu y_{m-1}^2(s)\right). \tag{4.15}$$

Finally the exact solution is obtained by

$$y(t) = lim_{m\to\infty} y_m(t). \tag{4.16}$$

With the initial approximation $y_0(t) = y_0$, and using the iteration (4.15), we can directly obtain the components of the solution. The first two components of the solution $y(t)$ by using (4.15) of the fractional-order logistic equation are given by

$$y_2(t) = y_1(t) + (6 + \mu t(3 + 2y_0(-3 + (y_0 - 1)\mu\ t))\Gamma(2-\alpha))\,\frac{y_0^2 - y_0}{6\Gamma(2-\alpha)}t$$
$$+\frac{y_0^2 - y_0}{\Gamma(2-\alpha)(\alpha-2)}\mu\ t^{2-\alpha}, \tag{4.17}$$

$$\vdots$$

4.4 The HATM for FOLE

In this section, we apply the HATM to solve (7.24). The Laplace transform to both sides of equation (7.24) is

$$\mathcal{L}\{y(t)\} - \frac{1}{s}y(0) + \frac{1}{s^\alpha}\mathcal{L}\{\mu y(t)\} - \frac{1}{s^\alpha}\mathcal{L}\{\mu y^2(t)\} = 0. \tag{4.19}$$

So as the procedure in section 3.4 we get

$$y_m(t) = \chi_m y_{m-1}(t) + h\mathcal{L}^{-1}\{R_m(\overrightarrow{y}_{m-1}(t))\}, \quad m = 1, 2, 3, ..., n, \tag{4.20}$$

where

$$R_m(\overrightarrow{y}_{m-1}(t)) = \mathcal{L}\{y_{m-1}(t)\} - \frac{1}{s}(1-\chi_m)y(0) + \frac{\mu}{s^\alpha}\mathcal{L}\{y_{m-1}(t)\} - \frac{\mu}{s^\alpha}\mathcal{L}\{\sum_{i=0}^{m-1} y_i(t)y_{m-1-i}(t)\}. \tag{4.21}$$

Consequently, the first tow terms of the HATM series approximate solution with $y_0(t) = y_0$ are

$$y_1(t) = \frac{h\mu(-y_0 + y_0^2)}{\Gamma(1+\alpha)}t^\alpha, \tag{4.22}$$

$$y_2(t) = y_1(t)\left(1 + h\left(1 + \frac{\mu(2y_0 - 1)\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}t^\alpha\right)\right). \tag{4.23}$$

5 A Comparison Results

Now through the plotting the previous approximate analytical solutions with the exact solution we can compare which the solution is better. Of course We've deliberately to keep an equal number of approximations in the four methods. We use only the 2-term approximate solution. But in the VIM we obtained the approximate analytical solution by $(y(t) = y_2(t))$. Figure 1, shows that the error between the exact solution $y(t)$ of (7.24) and the approximate analytic solution using ADM, HAM, VIM and HATM for $\alpha = 1, \mu = y_0 = 0.5, h = -0.834348$. From this figure we can conclude that solution obtained by VIM is very close to the exact solution than the solutions obtained by ADM, HAM and HATM.

6 Conclusion

In this paper ADM, VIM, HAM and HATM methods have used to solve FOLE. The efficiency and accuracy of these methods is obvious from the graph, when compared with the exact solution. Comparisons of ADM, VIM, HAM and HATM methods with exact solution have been shown by graphs which show the efficiency of the methods and we find that VIM results better than ADM, HAM and HATM.

7 Open Problem

As future work, we will extend this work to find the analytic approximation for the Fisher equation. i.e.

$$\frac{\partial^\alpha y}{\partial t^\alpha} = \frac{\partial^\beta y}{\partial x^\beta} + \mu y(1-y), (x,t) \in (-\infty, \infty) \times (0, \infty), \tag{7.24}$$

where y is a function of x and t, and $0 < \alpha \leq 1, 0 < \beta \leq 2$.

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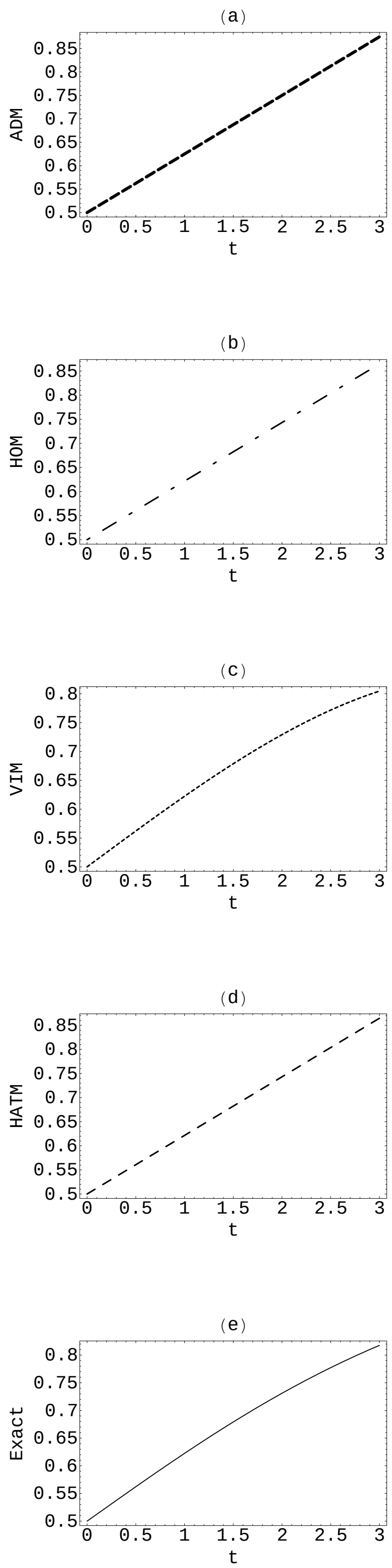
(a)
ADM
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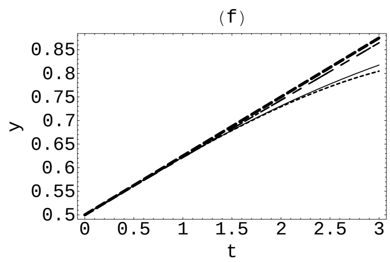

Figure 1: Comparison of the approximate analytical solutions obtained by ADM, HAM, VIM, and HATM for the (7.24)at $\alpha = 1,\ \mu = y_0 = 0.5, h = -0.834348$