Some new results of symmetric derivative on time scales

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Abstract

In this paper, we present some new results of symmetric derivative on time scales by using induction.

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1 Introduction

Symmetric derivative (Schwarz derivative) is very useful in a large number of problems. Particularly, in the theory of trigonometric series, applications of such properties are well known [1]. Recently, in [5], the authors introduced the notation of symmetric derivative on time scales. The theory of time scales, which has received a lot of attention, was introduced by Hilger in his PHD thesis in [6] in order to unify continuous and discrete analysis.

The aim of this paper is to present some new results of symmetric derivative on time scales based on reference [5].
2 Notations and lemmas

2.1 Notations

A time scale $T$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : T \to T$ are defined by

$$
\sigma(t) = \inf \{ s \in T : s > t \},
$$

$$
\rho(t) = \sup \{ s \in T : s < t \},
$$

where the supremum of the empty set is defined to be the infimum of $T$. A point $t \in T$ is said to be right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$, and $t \in T$ with $t > \inf T$ is said to be left-scattered if $\rho(t) < t$ and left-dense if $\rho(t) = t$. A point $t \in T$ is dense if it is right and left dense; isolated if it is right and left scattered. A function $g : T \to \mathbb{R}$ is said to be rd-continuous provided $g$ is continuous at right-dense points and has finite left-sided limits at left-dense points in $T$. The graininess function $\mu$ for a time scale $T$ is defined by $\mu(t) = \sigma(t) - t$, and for every function $f : T \to \mathbb{R}$ the notation $f^\sigma$ means the composition $f \circ \sigma$.

We also need the set $T^-\kappa$ and $T^+\kappa$ which are derived from the time scale $T$ as follows: If $T$ has a left-scattered maximum $m$, then $T^-\kappa = T - \{m\}$. Otherwise, $T^-\kappa = T$. If $\inf T$ is finite and right-scattered, then $T^+\kappa = T - \{\inf T\}$. Otherwise, $T^+\kappa = T$. We set $T^-\kappa \cap T^+\kappa$. For more, the readers may refer to [2].

**Definition 2.1** [5, Definition 3.1] We say that a function $f : T \to \mathbb{R}$ is symmetric continuous at $t \in T$. If for any $\varepsilon > 0$, there exists a neighborhood $U_t \subseteq T$ of $t$ such that for all $s \in U_t$ for which $2t - s \in U_t$ one has $|f(s) - f(2t - s)| \leq \varepsilon$.

**Definition 2.2** [5, Definition 3.4] Let $f : T \to \mathbb{R}$ and $t \in T^-\kappa$. The symmetric derivative at $t \in T$, denoted by $f^\Diamond(t)$, is the real number (provide it exists) with the property that, for any $\varepsilon > 0$, there exists a neighborhood $U \subseteq T$ of $t$ such that

$$
|f^\sigma(t) - f(s) + f(2t - s) - f^\rho(t) - f^\Diamond(t)\sigma(t) + 2t - 2s - \rho(t)||
$$

$$
\leq \varepsilon |\sigma(t) + 2t - 2s - \rho(t)|
$$

for all $s \in U$ for which $2t - s \in U$.

2.2 Lemmas

**Lemma 2.3** [5, Theorem 3.5] Let $f : T \to \mathbb{R}$ and $t \in T^-\kappa$. The following holds:

(i) Function $f$ has at most one symmetric derivative at $t$. 

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(ii) If $f$ is symmetric differentiable at $t$, then $f$ is symmetric continuous at $t$.

(iii) If $f$ is continuous at $t$ and $t$ is not dense, then $f$ is symmetric differentiable at $t$ with $f^{\diamond}(t) = \frac{f'(t)-f'(t)}{\sigma(t)-\rho(t)}$.

(iv) If $t$ is dense, then $f$ is symmetric differentiable at $t$ if and only if the limit $\lim_{s \to t} \frac{f(2t-s)-f(s)}{2t-2s}$ exists as a finite number. In this case $f^{\diamond}(t) = \lim_{s \to t} \frac{f(2t-s)-f(s)}{2t-2s}$.

Lemma 2.4 [5, Theorem 3.11] Let $f, g : \mathbb{T} \to \mathbb{R}$ be two symmetric differentiable functions at $t \in \mathbb{T}$ and $\lambda \in \mathbb{R}$. The following holds:

(i) Function $f + g$ is symmetric differentiable at $t$ with $(f + g)^{\diamond}(t) = f^{\diamond}(t) + g^{\diamond}(t)$.

(ii) Function $\lambda f$ is symmetric differentiable at $t$ with $(\lambda f)^{\diamond}(t) = \lambda f^{\diamond}(t)$.

(iii) If $f$ and $g$ are continuous at $t$, then Functions $fg$ is symmetric differentiable at $t$ with $(fg)^{\diamond}(t) = f^{\diamond}(t)g^{\diamond}(t) + f^{\sigma}(t)g^{\rho}(t)$.

(iv) If $f$ is continuous at $t$ and $f^{\sigma}(t)f^{\rho}(t) \neq 0$, then $\frac{1}{f}$ is symmetric differentiable at $t$ with $\left(\frac{1}{f}\right)^{\diamond}(t) = \frac{-f^{\diamond}(t)}{f^{\sigma}(t)f^{\rho}(t)}$.

(v) If $f$ and $g$ are continuous at $t$ and $g^{\sigma}(t)g^{\rho}(t) \neq 0$, then $\frac{f}{g}$ is symmetric differentiable at $t$ with $\left(\frac{f}{g}\right)^{\diamond}(t) = \frac{f^{\diamond}(t)g^{\sigma}(t)-f^{\sigma}(t)g^{\diamond}(t)}{g^{\sigma}(t)g^{\rho}(t)}$.

3 Main results

Theorem 3.1 Let $\alpha$ be a constant and $n \in \mathbb{N}$.

(i) For $f$ defined by $f(t) = (t - \alpha)^n$, we have

$$f^{\diamond}(t) = \sum_{k=0}^{n-1} (\sigma(t) - \alpha)^k(\rho(t) - \alpha)^{n-1-k}. \quad (3.1)$$

(ii) For $g$ defined by $g(t) = \frac{1}{(t-\alpha)^n}$, we have

$$g^{\diamond}(t) = -\sum_{k=0}^{n-1} \frac{1}{(\sigma(t) - \alpha)^{n-k}(\rho(t) - \alpha)^{k+1}} \quad (3.2)$$

provided $(\rho(t) - \alpha)(\sigma(t) - \alpha) \neq 0$.

Proof. We will prove (3.1) by induction. If $n = 1$, then $f(t) = t - \alpha$ and $f^{\diamond}(t) = 1$ holds. If $n = 2$, then $f(t) = (t-\alpha)^2$ and $f^{\diamond}(t) = (\sigma(t) - \alpha) + (\rho(t) - \alpha)$ holds.
Now, we assume that (3.1) holds for $f(t) = (t - \alpha)^n$. Using Lemma 2.2, we have

$$[(t - \alpha)^{n+1}]^{\hat{}'} = [(t - \alpha)(t - \alpha)^n]^{\hat{}} = (t - \alpha)^{\hat{}}(\sigma(t) - \alpha)^n + (\rho(t) - \alpha) [(t - \alpha)^n]^{\hat{}'}$$

$$= (\sigma(t) - \alpha)^n + \sum_{k=0}^{n-1} (\sigma(t) - \alpha)^k(\rho(t) - \alpha)^{n-k}$$

$$= \sum_{k=0}^{n} (\sigma(t) - \alpha)^k(\rho(t) - \alpha)^{n-k}.$$

To (3.2), using Lemma 2.2 and (i) of Theorem 3.1, it can easily obtain

$$g^{\hat{}}(t) = \left(\frac{1}{f}\right)^{\hat{}}(t) = \frac{-f^{\hat{}}(t)}{f^\sigma(t)f^\rho(t)}$$

$$= -\sum_{k=0}^{n-1} \frac{(\sigma(t) - \alpha)^k(\rho(t) - \alpha)^{n-1-k}}{(\sigma(t) - \alpha)^n(\rho(t) - \alpha)^n}$$

$$= -\sum_{k=0}^{n-1} \frac{1}{(\sigma(t) - \alpha)^{n-k}(\rho(t) - \alpha)^{k+1}}.$$

**Definition 3.2** For a function $f : \mathbb{T} \to \mathbb{R}$, we shall talk about the second derivative $f^{\hat{}}^{\hat{}}(t)$ provided $f^{\hat{}}(t)$ is symmetric differentiable on $\mathbb{T}^{\infty} = (\mathbb{T}^{\infty})^\kappa$ with derivative $f^{\hat{}}^{\hat{}}(t) = (f^{\hat{}}(t))^{\hat{}}$. Similarly, we define higher order symmetric derivatives $f^{\hat{}}^n : \mathbb{T}^{\infty} \to \mathbb{R}$. Finally, for $t \in \mathbb{T}$, we denote $\sigma^2(t) = \sigma(\sigma(t))$, $\rho^2(t) = \rho(\rho(t))$, $\sigma^n(t)$ and $\rho^n(t)$ are defined accordingly.

**Theorem 3.3** Let $f, g : \mathbb{T} \to \mathbb{R}$ be two functions whose $n$ symmetric derivative exists at $t \in \mathbb{T}^{\infty}$. Let $S_k^{(n)}$ and $C_k^{(n)}$ be the set consisting of all possible string of length $n \in \mathbb{N}$, containing exactly $k$ times $\rho$ and $n - k$ times $\hat{\hat{}}$, and $k$ times $\hat{}$ and $n - k$ times $\sigma$ each other. If $f^\land$ and $g^\lor$ exist for all $\land \in S_k^{(n)}, \lor \in C_k^{(n)}$, then

$$(fg)^{\hat{}}^{\hat{}} = \sum_{k=0}^{n} \sum_{\land \in S_k^{(n)}} \sum_{\lor \in C_k^{(n)}} (f^\land g^\lor). \tag{3.3}$$

**Proof.** We will prove (3.3) by mathematical induction. First, if $n = 1$, then (3.3) holds.
Next, we assume that (3.3) is true for \( n \in \mathbb{N} \), then

\[
(fg)^{(n+1)} = \left( \sum_{k=0}^{n} \left( \sum_{\wedge \in S^{(n)}} \left( f^\wedge \cdot g^\vee \right) \right) \right) = \sum_{k=0}^{n} \left( \sum_{\wedge \in S^{(n)}} \left( f^\wedge \cdot g^\vee + f^\wedge \cdot g^\vee \right) \right)
\]

\[
= \sum_{k=0}^{n} \sum_{\wedge \in S^{(n)}} \left( f^\wedge \cdot g^\vee \right) + \sum_{k=0}^{n-1} \sum_{\wedge \in S^{(n)}} \left( f^\wedge \cdot g^\vee \right)
\]

\[
= \sum_{\wedge \in S^{(n+1)}} \left( f^\wedge \cdot g^\vee \right) + \sum_{k=1}^{n} \sum_{\wedge \in S^{(n+1)}} \left( f^\wedge \cdot g^\vee \right) + \sum_{k=1}^{n} \sum_{\wedge \in S^{(n) \cup C^{(n)}}} \left( f^\wedge \cdot g^\vee \right)
\]

\[
= \sum_{k=0}^{n+1} \sum_{\wedge \in S^{(n+1)}} \left( f^\wedge \cdot g^\vee \right)
\]

(3.4)

So that (3.3) holds for \( n + 1 \), by the principle of induction. Then (3.3) holds for all \( n \in \mathbb{N} \).

**Theorem 3.4** Let \( f : [a, b] \to \mathbb{R} \) be continuous function. If \( f^{\wedge \cdot \wedge} (t) = 0 \) and \( t \in [a, b] \) is dense, then \( f(t) \) is a linear function.

**Proof.** For any given positive number \( \varepsilon \), we set

\[
\varphi(t) = f(t) - \left[ f(a) + \frac{f(b) - f(a)}{b - a} (t - a) \right] + \varepsilon(t - a)(t - b).
\]

(3.5)

It is easily known that \( \varphi(t) \) is continuous in \([a, b]\) and \( \varphi(a) = \varphi(b) = 0 \). Since \( t \) is dense, we have \( \sigma(t) = t = \rho(t) \). Easy computation yields \( \varphi^{\wedge \cdot \wedge} (t) = 2\varepsilon \).

Next, we will prove \( \varphi(t) \leq 0 \). If \( \varphi(t) > 0 \), then exist a local maximum in \( x_0 \in (a, b) \) such that

\[
\frac{\varphi(x_0 + h) - 2\varphi(x_0) + \varphi(x_0 - h)}{h^2} \leq 0.
\]
Using Lemma 2.1, we have $\varphi^{\diamond\diamond}(x_0) \leq 0$. It is a contradiction. On the other hand, Taking

$$
\psi(t) = -\left\{ f(t) - \left[ f(a) + \frac{f(b) - f(a)}{b-a}(t-a) \right] \right\} + \varepsilon(t-a)(t-b). \quad (3.6)
$$

Similarly method yield $\psi(t) \leq 0$. Hence, we obtain

$$
\left| f(t) - \left[ f(a) + \frac{f(b) - f(a)}{b-a}(t-a) \right] \right| \leq \varepsilon |(t-a)(t-b)|. \quad (3.7)
$$

By virtue of random of $\varepsilon$, the proof is completed.

4 Example

**Definition 4.1** Factorial function $t^{(k)}$ and binomial coefficient $\binom{\alpha}{\beta}$ by

$$
t^{(k)} = \frac{\Gamma(t+1)}{\Gamma(t-k+1)} \quad (4.1)
$$

and

$$
\binom{\alpha}{\beta} = \frac{\alpha^{(\beta)}}{\Gamma(\beta+1)}. \quad (4.2)
$$

**Example 4.2** Assume $\alpha, k \in \mathbb{C}$ and $\diamond$ is symmetric differentiation with respect to $t$ on time scale $T = \mathbb{Z}$. Show that

(i) $$(t + \alpha)^{(k)} \diamond = \left( k + \frac{k(1-k)}{2(t+\alpha)} \right) (t + \alpha)^{(k-1)}.$$

(ii) $$(\alpha^t) \diamond = \left( \frac{\alpha^2 - 1}{2\alpha} \right) \alpha^t.$$

(iii) $$(\frac{t}{\alpha}) \diamond = \left( 1 + \frac{1-\alpha}{2t} \right) \left( \frac{t}{\alpha} - 1 \right).$$
Proof. Since $T = Z$, then we have $\sigma(t) = t + 1, \rho(t) = t - 1$. Using Lemma 2.1, it can easily obtain

$$
\left( (t + \alpha)^{(k)} \right) ^\Diamond = \frac{(t + 1 + \alpha)^{(k)} - (t - 1 + \alpha)^{(k)}}{2} = \frac{1}{2} \frac{\Gamma(t + \alpha)}{\Gamma(t - k + \alpha)}
$$

$$
= \frac{k + k(1 - k)}{2(t + \alpha)} \frac{\Gamma(t + \alpha + 1)}{\Gamma(t - k + \alpha + 2)} = \frac{k + k(1 - k)}{2(t + \alpha)} (t + \alpha)^{(k-1)},
$$

$$
(\alpha^t) ^\Diamond = \frac{\alpha^{t+1} - \alpha^{t-1}}{2} = \left( \frac{\alpha^2 - 1}{2\alpha} \right) \alpha^t,
$$

and

$$
\left( \frac{t}{\alpha} \right) ^\Diamond = \left( \frac{t^{(\alpha)}}{\Gamma(\alpha + 1)} \right) ^\Diamond = \left( 1 + \frac{1 - \alpha}{2t} \right) t^{(\alpha-1)} \frac{t^{(\alpha-1)}}{\Gamma(\alpha)} = \left( 1 + \frac{1 - \alpha}{2t} \right) \left( \frac{t}{\alpha - 1} \right).
$$

5 Open Problem

Let $f_1, f_2, \cdots, f_m : T \to \mathbb{R}$ be some of these functions whose $n$ symmetric derivative exists at $t \in T^\kappa$. Compute

$$
(f_1 f_2 \cdots f_m) ^\Diamond^n (5.1)
$$

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References


