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Some new results of symmetric derivative on time scales

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Abstract

In this paper, we present some new results of symmetric derivative on time scales by using induction.

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1 Introduction

Symmetric derivative (Schwarz derivative) is very useful in a large number of problems. Particularly, in the theory of trigonometric series, applications of such properties are well known[1]. Recently, in [5], the authors introduced the notation of symmetric derivative on time scales. The theory of time scales, which has received a lot of attention, was introduced by Hilger in his PHD thesis in [6] in order to unify continuous and discrete analysis.

The aim of this paper is to present some new results of symmetric derivative on time scales based on reference [5].

2 Notations and lemmas

2.1 Notations

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf \{s \in \mathbb{T} : s > t\},$$

$$\rho(t) = \sup \{s \in \mathbb{T} : s < t\},$$

where the supremum of the empty set is defined to be the infimum of $\mathbb{T}$. A point $t \in \mathbb{T}$ is said to be right-scattered if $\sigma(t) > t$ and right-dense if $\sigma(t) = t$, and $t \in \mathbb{T}$ with $t > \inf \mathbb{T}$ is said to be left-scattered if $\rho(t) < t$ and left-dense if $\rho(t) = t$. A point $t \in \mathbb{T}$ is dense if it is right and left dense; isolated if it is right and left scattered. A function $g : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous provided g is continuous at right-dense points and has finite left-sided limits at left-dense points in $\mathbb{T}$. The graininess function μ for a time scale $\mathbb{T}$ is defined by $\mu(t) = \sigma(t) - t$, and for every function $f : \mathbb{T} \to \mathbb{R}$ the notation f^σ means the composition $f \circ \sigma$. We also need the set $\mathbb{T}^\kappa$ and $\mathbb{T}_\kappa$ which are derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^\kappa = \mathbb{T} - \{m\}$. Otherwise, $\mathbb{T}^\kappa = \mathbb{T}$. If $\inf \mathbb{T}$ is finite and right- scattered, then $\mathbb{T}_\kappa = \mathbb{T} - \{\inf \mathbb{T}\}$. Otherwise, $\mathbb{T}_\kappa = \mathbb{T}$. We set $\mathbb{T}^\kappa_\kappa = \mathbb{T}_\kappa \cap \mathbb{T}^\kappa$. For more, the readers may refer to [2].

Definition 2.1 *[5, Definition 3.1] We say that a function $f : \mathbb{T} \to \mathbb{R}$ is symmetric continuous at $t \in \mathbb{T}$. If for any $\varepsilon > 0$, there exists a neighborhood $U_t \subseteq \mathbb{T}$ of t such that for all $s \in U_t$ for which $2t - s \in U_t$ one has $|f(s) - f(2t - s)| \leq \varepsilon$.*

Definition 2.2 *[5, Definition 3.4] Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa_\kappa$. The symmetric derivative at $t \in \mathbb{T}$, denoted by $f^\diamondsuit(t)$, is the real number (provide it exists) with the property that, for any $\varepsilon > 0$, there exists a neighborhood $U \subseteq \mathbb{T}$ of t such that*

$$\begin{aligned}&\left|f^\sigma(t) - f(s) + f(2t - s) - f^\rho(t) - f^\diamondsuit(t)[\sigma(t) + 2t - 2s - \rho(t)]\right| \\ &\leq \varepsilon \left|\sigma(t) + 2t - 2s - \rho(t)\right|\end{aligned}$$

for all $s \in U$ for which $2t - s \in U$.

2.2 Lemmas

Lemma 2.3 *[5, Theorem 3.5] Let $f : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^\kappa_\kappa$. The following holds:*

(i)Function f has at most one symmetric derivative at t.

(ii)If f is symmetric differentiable at t, then f is symmetric continuous at t.

(iii)If f is continuous at t and t is not dense, then f is symmetric differentiable at t with $f^{\diamondsuit}(t) = \frac{f^\sigma(t)-f^\rho(t)}{\sigma(t)-\rho(t)}$.

(iv)If t is dense, then f is symmetric differentiable at t if and only if the limit $\lim\limits_{s\to t} \frac{f(2t-s)-f(s)}{2t-2s}$ exists as a finite number. In this case $f^{\diamondsuit}(t) = \lim\limits_{s\to t} \frac{f(2t-s)-f(s)}{2t-2s} = \lim\limits_{h\to 0} \frac{f(t+h)-f(t-h)}{2h}$.

Lemma 2.4 *[5, Theorem 3.11] Let $f, g : \mathbb{T} \to \mathbb{R}$ be two symmetric differentiable functions at $t \in \mathbb{T}^\kappa_\kappa$ and $\lambda \in \mathbb{R}$. The following holds:*

(i)Function $f+g$ is symmetric differentiable at t with $(f+g)^{\diamondsuit}(t) = f^{\diamondsuit}(t)+g^{\diamondsuit}(t)$.

(ii)Function λf is symmetric differentiable at t with $(\lambda f)^{\diamondsuit}(t) = \lambda f^{\diamondsuit}(t)$.

(iii)If f and g are continuous at t, then Functions fg is symmetric differentiable at t with $(fg)^{\diamondsuit}(t) = f^{\diamondsuit}(t)g^\sigma(t) + f^\rho(t)g^{\diamondsuit}(t)$.

(iv)If f is continuous at t and $f^\sigma(t)f^\rho(t) \neq 0$, then $\frac{1}{f}$ is symmetric differentiable at t with $\left(\frac{1}{f}\right)^{\diamondsuit}(t) = \frac{-f^{\diamondsuit}(t)}{f^\sigma(t)f^\rho(t)}$.

(v)If f and g are continuous at t and $g^\sigma(t)g^\rho(t) \neq 0$, then $\frac{f}{g}$ is symmetric differentiable at t with $\left(\frac{f}{g}\right)^{\diamondsuit}(t) = \frac{f^{\diamondsuit}(t)g^\rho(t)-f^\rho(t)g^{\diamondsuit}(t)}{g^\sigma(t)g^\rho(t)}$.

3 Main results

Theorem 3.1 *Let α be a constant and $n \in \mathbb{N}$.*

(i) For f defined by $f(t) = (t-\alpha)^n$, we have

$$f^{\diamondsuit}(t) = \sum_{k=0}^{n-1} (\sigma(t)-\alpha)^k(\rho(t)-\alpha)^{n-1-k}. \tag{3.1}$$

(ii)For g defined by $g(t) = \frac{1}{(t-\alpha)^n}$, we have

$$g^{\diamondsuit}(t) = -\sum_{k=0}^{n-1} \frac{1}{(\sigma(t)-\alpha)^{n-k}(\rho(t)-\alpha)^{k+1}} \tag{3.2}$$

provided $(\rho(t)-\alpha)(\sigma(t)-\alpha) \neq 0$.

Proof. We will prove (3.1) by induction. If $n = 1$, then $f(t) = t-\alpha$ and $f^{\diamondsuit}(t) = 1$ holds. If $n = 2$, then $f(t) = (t-\alpha)^2$ and $f^{\diamondsuit}(t) = (\sigma(t)-\alpha)+(\rho(t)-\alpha)$ holds.

Now, we assume that (3.1) holds for $f(t) = (t-\alpha)^n$. Using Lemma 2.2, we have

$$\begin{aligned}
&\left[(t-\alpha)^{n+1}\right]^{\diamondsuit} = [(t-\alpha)(t-\alpha)^n]^{\diamondsuit} \\
&= (t-\alpha)^{\diamondsuit}(\sigma(t)-\alpha)^n + (\rho(t)-\alpha)\,[(t-\alpha)^n]^{\diamondsuit} \\
&= (\sigma(t)-\alpha)^n + \sum_{k=0}^{n-1}(\sigma(t)-\alpha)^k(\rho(t)-\alpha)^{n-k} \\
&= \sum_{k=0}^{n}(\sigma(t)-\alpha)^k(\rho(t)-\alpha)^{n-k}.
\end{aligned}$$

To (3.2), using Lemma 2.2 and (i) of Theorem 3.1, it can easily obtain

$$\begin{aligned}
g^{\diamondsuit}(t) &= \left(\frac{1}{f}\right)^{\diamondsuit}(t) = \frac{-f^{\diamondsuit}(t)}{f^{\sigma}(t)f^{\rho}(t)} \\
&= -\frac{\sum\limits_{k=0}^{n-1}(\sigma(t)-\alpha)^k(\rho(t)-\alpha)^{n-1-k}}{(\sigma(t)-\alpha)^n(\rho(t)-\alpha)^n} \\
&= -\sum_{k=0}^{n-1}\frac{1}{(\sigma(t)-\alpha)^{n-k}(\rho(t)-\alpha)^{k+1}}.
\end{aligned}$$

Definition 3.2 *For a function $f : \mathbb{T} \to \mathbb{R}$, we shall talk about the second derivative $f^{\diamondsuit\diamondsuit}(t)$ provided $f^{\diamondsuit}(t)$ is symmetric differentiable on $\mathbb{T}^{\kappa^2} = (\mathbb{T}^{\kappa})^{\kappa}$ with derivative $f^{\diamondsuit\diamondsuit}(t) = (f^{\diamondsuit}(t))^{\diamondsuit}$. Similarly, we define higher order symmetric derivatives $f^{\diamondsuit^n} : \mathbb{T}^{\kappa^n} \to \mathbb{R}$. Finally, for $t \in \mathbb{T}$, we denote $\sigma^2(t) = \sigma(\sigma(t))$, $\rho^2(t) = \rho(\rho(t))$, $\sigma^n(t)$ and $\rho^n(t)$ are defined accordingly.*

Theorem 3.3 *Let $f, g : \mathbb{T} \to \mathbb{R}$ be two functions whose n symmetric derivative exists at $t \in \mathbb{T}^{\kappa}_{\kappa}$. Let $S_k^{(n)}$ and $C_k^{(n)}$ be the set consisting of all possible string of length $n \in \mathbb{N}$, containing exactly k times ρ and $n-k$ times $\diamondsuit$, and k times $\diamondsuit$ and $n-k$ times σ each other. If $f^{\wedge}$ and $g^{\vee}$ exist for all $\wedge \in S_k^{(n)}, \vee \in C_k^{(n)}$, then*

$$(fg)^{\diamondsuit^n} = \sum_{k=0}^{n}\sum_{\substack{\wedge\in S_k^{(n)}\\ \vee\in C_k^{(n)}}}\left(f^{\wedge}g^{\vee}\right). \tag{3.3}$$

Proof. We will prove (3.3) by mathematical induction. First, if $n = 1$, then (3.3) holds.

Next, we assume that (3.3) is true for $n \in \mathbb{N}$, then

$$
\begin{aligned}
(fg)^{\diamondsuit(n+1)} &= \left(\sum_{k=0}^{n} \sum_{\substack{\wedge \in S_k^{(n)} \\ \vee \in C_k^{(n)}}} (f^{\wedge} g^{\vee}) \right)^{\diamondsuit} = \sum_{k=0}^{n} \left(\sum_{\substack{\wedge \in S_k^{(n)} \\ \vee \in C_k^{(n)}}} \left(f^{\wedge} g^{\vee\sigma} + f^{\wedge\rho} g^{\vee\diamondsuit} \right) \right) \\
&= \sum_{k=1}^{n+1} \sum_{\substack{\wedge \in S_{k-1}^{(n)} \\ \vee \in C_{k-1}^{(n)}}} (f^{\wedge\diamondsuit} g^{\vee\sigma}) + \sum_{k=0}^{n-1} \sum_{\substack{\wedge \in S_k^{(n)} \\ \vee \in C_k^{(n)}}} (f^{\wedge\rho} g^{\vee\diamondsuit}) \\
&= \sum_{\substack{\wedge \in S_n^{(n)} \\ \vee \in C_n^{(n)}}} \left(f^{\wedge\diamondsuit} g^{\vee\sigma} \right) + \sum_{k=1}^{n} \left(\sum_{\substack{\wedge \in S_{k-1}^{(n)} \\ \vee \in C_{k-1}^{(n)}}} (f^{\wedge\diamondsuit} g^{\vee\sigma}) + \sum_{\substack{\wedge \in S_k^{(n)} \\ \vee \in C_k^{(n)}}} (f^{\wedge\rho} g^{\vee\diamondsuit}) \right) + \sum_{\substack{\wedge \in S_0^{(n)} \\ \vee \in C_0^{(n)}}} (f^{\wedge\rho} g^{\vee\diamondsuit}) \\
&= \sum_{\substack{\wedge \in S_{n+1}^{(n+1)} \\ \vee \in C_{n+1}^{(n+1)}}} (f^{\wedge} g^{\vee}) + \sum_{k=1}^{n} \sum_{\substack{\wedge \in S_k^{(n+1)} \\ \vee \in C_k^{(n+1)}}} (f^{\wedge} g^{\vee}) + \sum_{\substack{\wedge \in S_0^{(n+1)} \\ \vee \in C_0^{(n+1)}}} (f^{\wedge} g^{\vee}) \\
&= \sum_{k=0}^{n+1} \sum_{\substack{\wedge \in S_k^{(n+1)} \\ \vee \in C_k^{(n+1)}}} (f^{\wedge} g^{\vee}) .
\end{aligned} \tag{3.4}
$$

So that (3.3) holds for $n+1$, by the principle of induction. Then (3.3) holds for all $n \in \mathbb{N}$.

Theorem 3.4 *Let $f : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be continuous function. If $f^{\diamondsuit\diamondsuit}(t) = 0$ and $t \in [a,b]_{\mathbb{T}^{\kappa}_{\kappa}}$ is dense, then $f(t)$ is a linear function.*

Proof. For any given positive number ε, we set

$$\varphi(t) = f(t) - \left[f(a) + \frac{f(b) - f(a)}{b - a}(t - a) \right] + \varepsilon(t - a)(t - b). \tag{3.5}$$

It is easily known that $\varphi(t)$ is continuous in $[a,b]$ and $\varphi(a) = \varphi(b) = 0$. Since t is dense, we have $\sigma(t) = t = \rho(t)$. Easy computation yields $\varphi^{\diamondsuit\diamondsuit}(t) = 2\varepsilon$.

Next, we will prove $\varphi(t) \leq 0$. If $\varphi(t) > 0$, then exist a local maximum in $x_0 \in (a,b)$ such that

$$\frac{\varphi(x_0 + h) - 2\varphi(x_0) + \varphi(x_0 - h)}{h^2} \leq 0.$$

Using Lemma 2.1, we have $\varphi^{\diamondsuit\diamondsuit}(x_0) \leq 0$. It is a contradiction. On the other hand, Taking

$$\psi(t) = -\left\{f(t) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(t-a)\right]\right\} + \varepsilon(t-a)(t-b). \tag{3.6}$$

Similarly method yield $\psi(t) \leq 0$. Hence, we obtain

$$\left|f(t) - \left[f(a) + \frac{f(b)-f(a)}{b-a}(t-a)\right]\right| \leq \varepsilon\left|(t-a)(t-b)\right|. \tag{3.7}$$

By virtue of random of ε, the proof is completed.

4 Example

Definition 4.1 *Factorial function $t^{(k)}$ and binomial coefficient $\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$ by*

$$t^{(k)} = \frac{\Gamma(t+1)}{\Gamma(t-k+1)} \tag{4.1}$$

and

$$\begin{pmatrix}\alpha\\ \beta\end{pmatrix} = \frac{\alpha^{(\beta)}}{\Gamma(\beta+1)}. \tag{4.2}$$

Example 4.2 *Assume $\alpha, k \in \mathbb{C}$ and $\diamondsuit$ is symmetric differentiation with respect to t on time scale $\mathbb{T} = \mathbb{Z}$. Show that*

(i) $$\left((t+\alpha)^{(k)}\right)^{\diamondsuit} = \left(k + \frac{k(1-k)}{2(t+\alpha)}\right)(t+\alpha)^{(k-1)}.$$

(ii) $$\left(\alpha^t\right)^{\diamondsuit} = \left(\frac{\alpha^2-1}{2\alpha}\right)\alpha^t.$$

(iii) $$\begin{pmatrix}t\\ \alpha\end{pmatrix}^{\diamondsuit} = \left(1 + \frac{1-\alpha}{2t}\right)\begin{pmatrix}t\\ \alpha-1\end{pmatrix}.$$

Proof. Since $\mathbb{T} = \mathbb{Z}$, then we have $\sigma(t) = t+1, \rho(t) = t-1$. Using Lemma 2.1, it can easily obtain

$$\begin{aligned}\left((t+\alpha)^{(k)}\right)^{\diamondsuit} &= \frac{(t+1+\alpha)^{(k)} - (t-1+\alpha)^{(k)}}{2} = \frac{1}{2}\frac{\Gamma(t+\alpha)}{\Gamma(t-k+\alpha)} \\ &= \left(k + \frac{k(1-k)}{2(t+\alpha)}\right)\frac{\Gamma(t+\alpha+1)}{\Gamma(t-k+\alpha+2)} = \left(k + \frac{k(1-k)}{2(t+\alpha)}\right)(t+\alpha)^{(k-1)}, \\ \left(\alpha^t\right)^{\diamondsuit} &= \frac{\alpha^{t+1} - \alpha^{t-1}}{2} = \left(\frac{\alpha^2-1}{2\alpha}\right)\alpha^t,\end{aligned}$$

and

$$\binom{t}{\alpha}^{\diamondsuit} = \left(\frac{t^{(\alpha)}}{\Gamma(\alpha+1)}\right)^{\diamondsuit} = \left(1 + \frac{1-\alpha}{2t}\right)\frac{t^{(\alpha-1)}}{\Gamma(\alpha)} = \left(1 + \frac{1-\alpha}{2t}\right)\binom{t}{\alpha-1}.$$

5 Open Problem

Let $f_1, f_2, \cdots, f_m : \mathbb{T} \to \mathbb{R}$ be some of these functions whose n symmetric derivative exists at $t \in \mathbb{T}^{\kappa}_{\kappa}$. Compute

$$(f_1 f_2 \cdots f_m)^{\diamondsuit^n} \tag{5.1}$$

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References

[1] A. Zygmund: *Trigonometric series*, reprint of the 1979 edition, Cambridge Mathematical Library, Cambridge Univ. Press, Cambridge, (1988).

[2] M. Bohner and A. Peterson: *Dynamic equations on time scales: An introduction with applications*, Birkhäuser, Boston, (2001).

[3] M. Bohner and A. Peterson: *Advances in dynamic equations on time scales*, Birkhäuser, Boston, (2003).

[4] V. Kac and P. Cheung: *Quantum calculus*, Universitext, Springer, New York, (2002).

[5] Artur M. C., Brito Da Cruz, Natalia Martins and Delfim F. M. Torres: *symmetric differentiation on time scales*, Appl. Math. Lett. **26** (2013), no. 2, 264-269.

[6] B. Aulbach and S. Hilger: *A united approach to continuous and discrete dynamics, in Qualitative theory of differential equations (Szeged,1988)*,37-56, Colloq. Math. Soc. János Bolyai, 53 North-Holland, Amsterdam, (1990).