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Weber's Inhomogeneous Differential Equation with Initial and Boundary Conditions

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Abstract

Weber's inhomogeneous differential equation is analyzed and solved in this work when initial and boundary data are given. In the process, a new parametric function that represents an extension to the Nield-Kuznetsov function is introduced.

Keywords: *Weber's equation, parametric Nield-Kuznetsov function.*

1 Introduction

The well-known Weber's homogeneous differential equation has been reported in the literature in many forms, (*cf*. [7, 11, 12] and the references therein) among which is the following equation:

$$\text{(1)} \qquad \frac{d^2y}{dx^2} + (ax^2 + bx + c)y = 0$$

where a, b, c are scalars and the independent variable, x, is in general complex.

Equation (1) has the following three distinctive forms, whose mathematical and physical applications have been discussed in the pioneering work of Temme, [12], and are listed below with their solutions, [7, 12]. All solutions are entire functions of x and entire functions of the parameters a or ν, appearing in the equations below, [12].

The equation:

(2) $$\frac{d^2y}{dx^2} - (\frac{x^2}{4} + a)y = 0$$

possesses solutions given by the functions $U(-a, \mp ix)$. For real values of x, two linearly independent, numerically satisfactory pair of solutions of (2) are $U(a, x)$, $V(a, x)$ when $x > 0$, or $U(a, -x)$, $V(a, -x)$ when $x < 0$.

The equation:

(3) $$\frac{d^2y}{dx^2} + (\nu + \frac{1}{2} - \frac{x^2}{4})y = 0$$

possesses the solutions $D_\nu(\mp x)$, where $D_\nu(x) = U(-\frac{1}{2} - \nu, x)$.

Finally, the equation:

(4) $$\frac{d^2y}{dx^2} + (\frac{x^2}{4} - a)y = 0$$

possesses the solutions $W(a, \mp x)$, where x and a are real numbers. These solutions represent a linearly independent, numerically satisfactory pair of solutions for all $x \in \Re$.

Our interest in this work is equation (4) due to its validity in the real domain and its usefulness in modelling practical situations in the real plain. In particular, we are interested in solving the following inhomogeneous form of equation (4), namely:

(5) $$\frac{d^2y}{dx^2} + (\frac{x^2}{4} - a)y = f(x)$$

where the forcing function, $f(x)$, is in general a function of x. Equation (5) can be shown to have applications in the study of flow through porous layers, where the Brinkman equation with variable permeability (which governs the flow in the transition layer, [8]) reduces to Airy's equation, or Weber's equation, [1], [2], [5], [6]. We provide in this work a unified approach to initial and boundary value problems involving Weber's equation, and introduce parametric forms of the Nield-Kuznetsov functions, together with method of computation. We present an example of each type of problem and obtain its solution.

2 Solution Methodology

In order to obtain the general solution to (5), we first express the solution to the homogeneous part of (5) as:

(6) $$y_c = c_1W(a, x) + c_2W(a, -x)$$

where c_1 and c_2 are arbitrary constants. The numerically satisfactory pair of solutions, $W(a,\mp x)$, are linearly independent with a Wronskian given by, [4, 11, 12]:

(7) $$\mathfrak{W}(W(a,x),W(a,-x))=1.$$

Based on the method of variation of parameters, we assume the particular solution of the form:

(8) $$y_p = u_1 W(a,x) + u_2 W(a,-x)$$

where u_1 and u_2 are expressed in the following forms:

(9) $$u_1 = -\int_0^x f(t)W(a,-x)dt$$

(10) $$u_2 = \int_0^x f(t)W(a,x)dt\ .$$

The particular solution (8) can thus be expressed as:

(11) $$y_p = -\left[W(a,x)\int_0^x f(t)W(a,-x)dt - W(a,-x)\int_0^x f(t)W(a,x)dt \right]$$

and the following two cases arise, depending on $f(x)$.

Case 1: If $f(x) = R =$ constant, then

(12) $$y_p = -R\left[W(a,x)\int_0^x W(a,-x)dt - W(a,-x)\int_0^x W(a,x)dt \right].$$

In equation (12), we recognize the expression in brackets as a parametric Nield-Kuznetsov function (after Nield and Kuznetsov, [8], who first introduced this type of integral function), that depends on parameter a. Denoting the parametric Nield-Kuznetsov function by $N_w(a,x)$, we define:

(13) $$N_w(a,x) = W(a,x)\int_0^x W(a,-t)dt - W(a,-x)\int_0^x W(a,t)dt$$

and write particular solution (12) as:

(14) $$y_p = -RN_w(a,x)\ .$$

General solution to (5) when $f(x) = R =$ constant is given by the sum of solutions (6) and (14), namely

(15) $$y = c_1 W(a,x) + c_2 W(a,-x) - RN_w(a,x).$$

Case 2: If $f(x)$ is a variable function of x then:

assuming that $f(x)$ can be written as the derivative of a function, $F(x)$, namely $f(x) = F'(x)$, we can write (13) as:

(16) $$y_p = W(a,-x)\int_0^x F'(t)W(a,t)dt - W(a,x)\int_0^x F'(t)W(a,-t)dt\ .$$

Upon integration by parts, equation (16) can be written as:

(17) $$y_p = -\left\{W(a,-x)\int_0^x F(t)W'(a,t)dt + W(a,x)\int_0^x F(t)W'(a,-t)dt\right\}.$$

The integral expression in (17) is reminiscent of the $Ki(x)$ function that was introduced in [6] in connection with the Nield-Kuznetsov function $Ni(x)$.

Due to the dependence of the integral expression (17) on parameter a, we define:

(18) $$K_w(a,x) = W(a,-x)\int_0^x F(t)W'(a,t)dt + W(a,x)\int_0^x F(t)W'(a,-t)dt\ .$$

Accordingly, we can express the particular solution (17) as

(19) $$y_p = -K_w(a,x)$$

and the general solution to (5) thus takes the form

(20) $$y = c_1 W(a,x) + c_2 W(a,-x) - K_w(a,x).$$

3 Initial Value and Boundary Value Problem

3.1 Values of $W(a,\mp x)$, $W'(a,\mp x)$ $N_w(a,x)$, $N'_w(a,x)$ at $x = 0$, $a = 0$

Values of $W(a,0)$ and $W'(a,0)$ have been reported in the literature, [7, 12], and take the following expressions, respectively:

(21) $$W(a,0) = \frac{1}{(2)^{3/4}} \left| \frac{\Gamma(\frac{1}{4} + \frac{ia}{2})}{\Gamma(\frac{3}{4} + \frac{ia}{2})} \right|^{1/2}$$

(22) $$W'(a,0) = -\frac{1}{(2)^{1/4}} \left| \frac{\Gamma(\frac{3}{4} + \frac{ia}{2})}{\Gamma(\frac{1}{4} + \frac{ia}{2})} \right|^{1/2} .$$

Using equations (13), (21) and (22), we develop the following expressions for $N_w(a,x)$ and $N'_w(a,x)$ at $a = 0$ and at $x = 0$. From (13), we obtain:

(23) $$N_w(a,0) = 0$$

(24) $$N_w(0,x) = W(0,x)\int_0^x W(0,-x)dt - W(0,-x)\int_0^x W(0,x)dt$$

and the following expression for $N'_w(a,x)$:

(25) $$N'_w(a,x) = W'(a,x)\int_0^x W(a,-t)dt + W'(a,-x)\int_0^x W(a,t)dt.$$

Values of $N'_w(a,0)$ and $N'_w(0,x)$ are obtained from (25) as:

(26) $$N'_w(a,0) = 0$$

and

(27) $$N'_w(0,x) = W'(0,x)\int_0^x W(0,-t)dt + W'(0,-x)\int_0^x W(0,t)dt .$$

The above values are important in the formulation and solution of initial value and boundary value problems, discussed in the next subsection.

3.2 Initial and Boundary Conditions

Initial value problem associated with Weber's differential equation, when $f(x) = R$, is comprised of solving (5) subject to the initial conditions:

(28) $$y(0) = \alpha; y'(0) = \beta$$

where α and β are known constants.

Using (28) in (20), we obtain the following values for the arbitrary constants, c_1 and c_2:

$$c_1 = \frac{\alpha W'(a,0) + \beta W(a,0)}{2W(a,0)W'(a,0)} \tag{29}$$

$$c_2 == \frac{\alpha W'(a,0) - \beta W(a,0)}{2W(a,0)W'(a,0)} \tag{30}$$

and solution to the initial value problem takes the form:

$$\begin{aligned} y &= \left(\frac{\alpha W'(a,0) + \beta W(a,0)}{2W(a,0)W'(a,0)} \right) W(a,x) \\ &+ \left(\frac{\alpha W'(a,0) - \beta W(a,0)}{2W(a,0)W'(a,0)} \right) W(a,-x) - RN_w(a,x) \end{aligned} \tag{31}$$

Boundary value problem associated with Weber's differential equation, with $f(x) = R$, is comprised of solving (5) subject to the boundary conditions:

$$y(x_1) = y_1; y(x_2) = y_2 \tag{32}$$

where $y_1, y_2, x_1 \neq x_2 \in \Re$.

Using conditions (32) in general solution (20), we obtain the following expressions for the arbitrary constants, c_1 and c_2:

$$c_1 = \frac{y_2 W(a,-x_1) - y_1 W(a,-x_2) + RN_w(a,x_2)W(a,-x_1) - RN_w(a,x_1)W(a,-x_2)}{W(a,x_2)W(a,-x_1) - W(a,x_1)W(a,-x_2)} \tag{33}$$

$$c_2 = \frac{y_2 W(a,x_1) - y_1 W(a,x_2) + RN_w(a,x_2)W(a,x_1) - RN_w(a,x_1)W(a,x_2)}{W(a,-x_2)W(a,x_1) - W(a,-x_1)W(a,x_2)} \tag{34}$$

and solution to the boundary value problem can thus be expressed as:

$$\begin{aligned} y &= -RN_w(a,x) + \\ &\left(\frac{y_2 W(a,-x_1) - y_1 W(a,-x_2) + RN_w(a,x_2)W(a,-x_1) - RN_w(a,x_1)W(a,-x_2)}{W(a,x_2)W(a,-x_1) - W(a,x_1)W(a,-x_2)} \right) W(a,x) \\ &+ \left(\frac{y_2 W(a,x_1) - y_1 W(a,x_2) + RN_w(a,x_2)W(a,x_1) - RN_w(a,x_1)W(a,x_2)}{W(a,-x_2)W(a,x_1) - W(a,-x_1)W(a,x_2)} \right) W(a,-x) \end{aligned} \tag{35}$$

It is clear that the solutions presented by equations (31) and (35), and the arbitrary constants in equations (29), (30), (33) and (34), involve $W(a,x)$, $W(a,-x)$ and $N_w(a,x)$, which must be evaluated at given values of *a* and *x*. Methods of evaluation are discussed in the following section.

3.3 Computations of Weber Functions

A number of techniques, algorithms and methods have been discussed in the literature to evaluate Weber functions, (*cf.* [3], [4], [9-12] and the references therein). Recent ground-breaking techniques and algorithms have gained impetus with the contributions associated with the names of Temme, Gil, and Segura (*cf.* [4], [9-12] and the references therein). In the current work, we employ one of the techniques described in [12] and is based on Maclaurin series. We summarize below the expressions we used.

(36) $W(a,x) = W(a,0)k_1(a,x) + W'(a,0)k_2(a,x)$

(37) $W'(a,x) = W(a,0)k_1'(a,x) + W'(a,0)k_2'(a,x)$

(38) $W(a,-x) = W(a,0)k_1(a,-x) + W'(a,0)k_2(a,-x)$

(39) $W(a,-x) = W(a,0)k_1(a,x) - W'(a,0)k_2(a,x)$

(40) $W'(a,-x) = W'(a,0)k_2'(a,x) - W(a,0)k_1'(a,x)$

(41) $k_1(a,x) = \sum_{n=0}^{\infty} \rho_n(a) \frac{x^{2n}}{(2n)!}$

(42) $k_1'(a,x) = \sum_{n=1}^{\infty} \rho_n(a) \frac{x^{2n-1}}{(2n-1)!}$

(43) $k_1(a,-x) = \sum_{n=0}^{\infty} \rho_n(a) \frac{(-x)^{2n}}{(2n)!} = k_1(a,x)$

(44) $k_2(a,x) = \sum_{n=0}^{\infty} \eta_n(a) \frac{x^{2n+1}}{(2n+1)!}$

(45) $k_2'(a,x) = \sum_{n=1}^{\infty} \eta_n(a) \frac{x^{2n}}{(2n)!}$

(46) $k_2(a,-x) = \sum_{n=0}^{\infty} \eta_n(a) \frac{(-x)^{2n+1}}{(2n+1)!} = -\sum_{n=0}^{\infty} \eta_n(a) \frac{x^{2n+1}}{(2n+1)!} = -k_2(a,x)$

where

(47) $\rho_{n+2} = a\rho_{n+1} - \frac{1}{2}(n+1)(2n+1)\rho_n$

(48) $\eta_{n+2} = a\eta_{n+1} - \frac{1}{2}(n+1)(2n+3)\eta_n$

(49) $\rho_0(a) = 1\,;\, \rho_1(a) = a$

(50) $\eta_0(a) = 1\,;\, \eta_1(a) = a$.

The above expressions are used in (13) to obtain the following expression for the Nield-Kuznetsov parametric function $N_w(a,x)$:

(51) $$N_w(a,x) = \left[2W(a,0)W'(a,0)w_2(a,x)\right]\left\{\sum_{n=0}^{\infty}\rho_n(a)\frac{x^{2n+1}}{(2n+1)!}\right\}$$

$$-\left[2W'(a,0)W(a,0)w_1(a,x)\right]\left\{\sum_{n=0}^{\infty}\eta_n(a)\frac{x^{2n+2}}{(2n+2)!}\right\} \quad .$$

The above infinite series are approximated in this work by using 50 terms of the series and replacing ∞ by *N=50*.

In order to illustrate the above and provide numerical values to the parameters involved in the initial value problem described in section **3.2**, above, we take *a=0* and *a=1*, $\alpha = 0, \beta = 1$ and $f(x) = R = \pi$. The following Table of values is constructed:

Table 1

Computed Values of $W(a,0)$, c_1, c_2 for the Initial Value Problem

a=0	$W(0,0) = 1.022765672$	$W'(0,0) = -0.4888705334$	$c_1 = -1.022765673$ $c_2 = 1.022765673$
a=1	$W(1,0) = 0.7314810901$	$W'(1,0) = -0.6835446695$	$c_1 = -0.73148109$ $c_2 = 0.73148109$

Solution (31) to the initial value problem is then evaluated and sketched in **Fig.** 1, below.

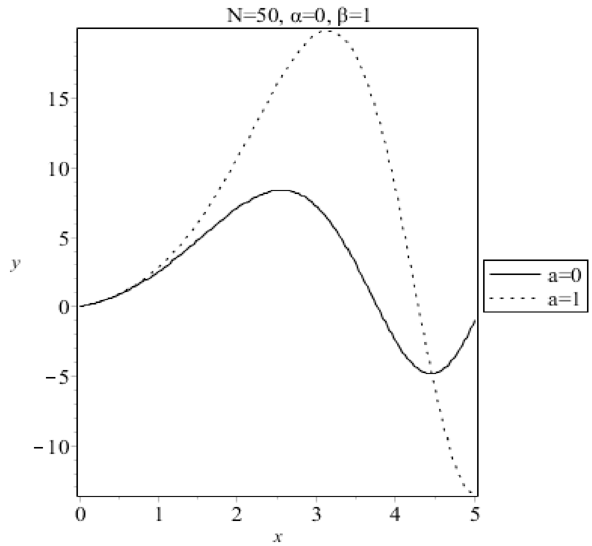

Fig. 1. Solution (31) of the initial value problem

For the boundary value problem, we take *a=0* and *a=10*, $f(x) = R = \pi$. For each value of *a*, obtain the values of c_1 and c_2 for $x_1 = 0, x_2 = 1, y_1 = 0, y_2 = 1$. The following Table is produced.

Table 2

Computed Values of $W(a, \mp x)$, c_1, c_2 for the Boundary Value Problem

a=0	$W(0,-x_1) = 1.022765672$ $W(0,x_1) = 1.022765672$ $W(0,-x_2) = 1.484333833$ $W(0,x_2) = 0.5187721609$	$N(0,x_1) = 0$ $N(0,x_2) = -0.495844891$	$c_1 = 0.57763547$ $c_2 = -0.57763547$
a=10	$W(10,-x_1) = 0.397760438$ $W(10,x_1) = -1.257038036$ $W(10,-x_2) = 9.31341554$ $W(10,x_2) = 0.017205410$	$N(10,x_1) = 0$ $N(10,x_2) = -1.07579847$	$c_1 = 0.25598825$ $c_2 = -0.25598825$

Solution (35) to the boundary value problem is then evaluated and sketched in **Fig.** 2, below, using *N=50* terms.

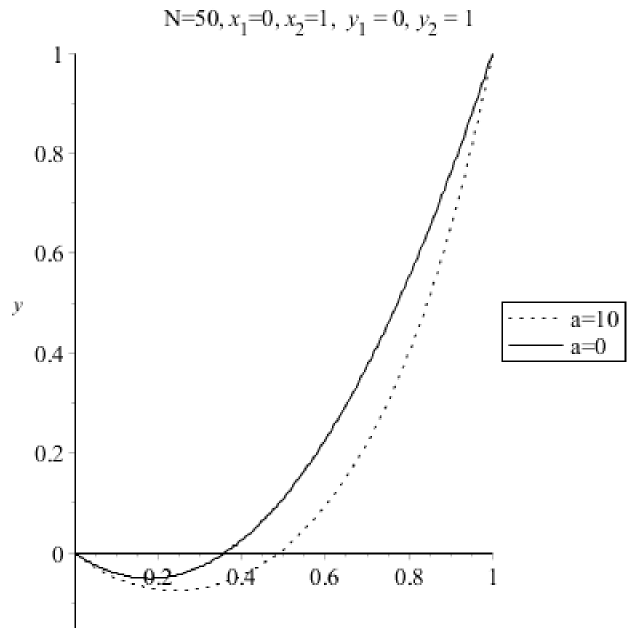

Fig. 2. Solution (35) of the boundary value problem

4 Conclusion

In this work, we considered the initial value and boundary value problems associated with the Weber differential equation. We introduced a parametric form of the Nield-Kuznetsov function and referred to it as $N_w(a, x)$ in order to facilitate a unified expression for the particular solution when the forcing function is constant. For variable forcing function, we introduced a parametric form of the $K_i(x)$ function and identified it as $K_w(a, x)$. We expressed the particular solution, when the forcing function is variable, in terms of $K_w(a, x)$.

In order to show numerical and graphical solutions to the initial and boundary value problems, we relied on an existing, state-of-the-art procedure (using Maclaurin series) to compute the Weber functions. In order to compute $N_w(a, x)$, we introduced a series expression for this parametric Nield-Kuznetsov function in terms of Weber functions and their series expressions.

5 Open Problem

The current work involves real arguments of the Weber functions. In the general analysis of Weber equation and associated functions with complex arguments, the introduction of a Nield-Kuznetsov type function is still challenging and requires further consideration.

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