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Certain Subclass of $p-$Valent Starlike and Convex Uniformly Functions Defined by Convolution

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Abstract

The aim of this paper is to obtain coefficient estimates, distortion theorems, convex linear combinations and radii of close-to convexity, starlikenees and convexity for functions belonging to the subclass $TS_{p,\lambda}(f, g; \alpha, \beta)$ ***of p-valent*** $\beta-$***uniformaly starlike and convex functions. We consider integral operators and modified Hadamarad products of functions belonging to this subclass.***

Keywords: *Analytic, p-valent, uniformly starlike, uniformly convex, modified Hadamard product.*
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1 Introduction

The class of analytic and p-valent in $\mathbb{U}$ and has the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}, \mathbb{N} = \{1, 2, ...\}) \tag{1}$$

is denoted by $S(p)$. We note that $S(1) = S$. Let $f(z) \in S(p)$ be given by (1) and $g(z) \in S(p)$ be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k, \tag{2}$$

then the Hadamard product $(f * g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z). \tag{3}$$

Denote by $T(p)$ the class of functions of the form:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \quad (a_k \geq 0). \tag{4}$$

Salim et al. [20] and Marouf [16] introduced and studied the following subclasses of $p-$valent functions:

(i) A function $f(z) \in S(p)$ is said to be $p-$valent $\beta-$uniformly starlike of order α if it satisfies the condition:

$$Re\left\{\frac{zf'(z)}{f(z)} - \alpha\right\} \geq \beta\left|\frac{zf'(z)}{f(z)} - p\right| (z \in \mathbb{U}), \tag{5}$$

for some $\alpha(-p \leq \alpha < p)$, $\beta \geq 0$ and for all $z \in \mathbb{U}$.
denote to the class of these functions by $\beta - S_p(\alpha)$.

(ii) A function $f(z) \in S(p)$ is said to be $p-$valent $\beta-$uniformly convex of order α if it satisfies the condition:

$$Re\left\{1 + \frac{zf''(z)}{f'(z)} - \alpha\right\} \geq \beta\left|1 + \frac{zf''(z)}{f'(z)} - p\right| (z \in \mathbb{U}), \tag{6}$$

for some $\alpha(-p \leq \alpha < p)$, $\beta \geq 0$ and for all $z \in \mathbb{U}$. We denote to the class of these functions by $\beta - K_p(\alpha)$.

It follows from (5) and (6) that

$$f(z) \in \beta - K_p(\alpha) \Longleftrightarrow \frac{zf'}{p} \in \beta - S_p(\alpha). \tag{7}$$

For different choices of parameters α, β, p we obtain many subclasses studied earlier see (for example) ([1], [11], [12], [13], [14], [18] and [19])

Definition 1. For $0 \leq \alpha < p$, $p \in \mathbb{N}$, $0 \leq \lambda \leq 1$ and $\beta \geq 0$, let $S_{p,\lambda}(f, g; \alpha, \beta)$ be the subclass of $S(p)$ consisting of functions $f(z)$ of the form (1) and functions $g(z)$ of the form (2) and satisfying the analytic criterion:

$$Re\left\{\frac{(1-\lambda+\frac{\lambda}{p})z(f*g)'(z) + \frac{\lambda}{p}z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \frac{\lambda}{p}z(f*g)'(z)} - \alpha\right\} \tag{8}$$

$$> \beta \left| \frac{(1-\lambda+\frac{\lambda}{p})z(f*g)'(z)+\frac{\lambda}{p}z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\frac{\lambda}{p}z(f*g)'(z)} - p \right| (z \in \mathbb{U}) \tag{9}$$

and

$$TS_{p,\lambda}(f,g;\alpha,\beta) = S_{p,\lambda}(f,g;\alpha,\beta) \cap T(p). \tag{10}$$

We note that:

(i)$TS_{1,\lambda}(f,g;\alpha,\beta) = TS_{\gamma}(f,g;\alpha,\beta)$ (see Aouf et al. [3]);

(ii)$TS_{p,0}(f, z^p + \sum_{k=p+1}^{\infty} \Omega_k z^k; \alpha,\beta) = T_p^{l,m}(\alpha_l,\beta_m,\beta,\alpha)$(see Marouf [16]), where

$$\Omega_k = \frac{(\alpha_1)_{k-p}...(\alpha_l)_{k-p}}{(\beta_1)_{k-p}...(\beta_m)_{k-p}} \frac{1}{(k-p)!} \tag{11}$$

and

$$(a)_k = \frac{\Gamma(a+k)}{\Gamma(a)} = \begin{cases} 1 & k=0 \\ a(a+1)(a+2)...(a+k-1) & k \in \mathbb{N} \end{cases}, \tag{12}$$

$(0 \le \alpha < p, \beta \ge 0, \alpha_i \in \mathbb{C}(i=1,2,...l)$ and $\beta_j \in \mathbb{C}\backslash\{-1,-2,...\}$, $j=1,2,...m, z \in \mathbb{U})$;

(iii)$TS_{p,0}(f, z^p + \sum_{k=p+1}^{\infty} \phi_k(n,\gamma,\delta,p)z^k; \alpha,\beta) = S_p^n(\alpha,\beta,\gamma,\delta)$(see Salim et al.[20]), where

$$\phi_k(n,\gamma,\delta,p) = \left[(1-\gamma)\left(1+(\frac{k}{p}-1)\delta\right) + \gamma C_{n+k-p}^n\right], \tag{13}$$

$$(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \ge 0, \delta \ge 0, -p \le \alpha < p, \beta \ge 0) \tag{14}$$

and

$$C_{n+k-p}^n = \frac{(n+k-p)!}{n!(k-p)!}. \tag{15}$$

(iv)$TS_{p,0}(f, z^p + \sum_{k=1}^{\infty} C(k,n)(1-\gamma k)^n z^{k+p}) = S_p^*(\alpha,\beta,\gamma)$ and $TS_{p,1}(f, z^p + \sum_{k=1}^{\infty} C(k,n)(1-\gamma k)^n z^{k+p}) = C_p(\alpha,\beta,\gamma)$(see Darus and Ibrahim [7, by replacing k by $k-p$]), where

$$C(k,n) = \frac{(n+p)_k}{(1)_k} \tag{16}$$

$(p \in \mathbb{N}, n \in \mathbb{N}_0, 0 \le \gamma < \frac{1}{k}, 0 \le \alpha < p, \beta \ge 0, z \in \mathbb{U})$;

(vi)$TS_{p,0}(f, z^p + \sum_{k=p+n}^{\infty} \Phi(k)z^k) = K(\mu,\gamma,\eta,\alpha,\beta)$(see Khairnar and More [15, with $n=1$]), where

$$\Phi(k) = \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}} \tag{17}$$

$(-\infty < \mu < 1, -\infty < \gamma < 1, \eta \in \mathbb{R}^+, p \in \mathbb{N}, -p \leq \alpha < p, \beta \geq 0, a \in \mathbb{R}, c \in \mathbb{R}\backslash\mathbb{Z}_0, z \in \mathbb{U})$.

Also we note that:

(i)$TS_{p,\lambda}(f, z^p + \sum\limits_{k=p+1}^{\infty} (\frac{k}{p})^n z^k; \alpha, \beta) = TS_{p,\lambda}(n, \alpha, \beta)$

$$= \left\{ \begin{array}{c} f \in T(p) : Re\left\{ \frac{(1-\lambda+\frac{\lambda}{p})z(D_p^n f(z))' + \frac{\lambda}{p} z^2 (D_p^n f(z))''}{(1-\lambda)(D_{\lambda,p}^n f(z)) + \frac{\lambda}{p} z (D_p^n f(z))'} - \alpha \right\} \\ > \beta \left| \frac{(1-\lambda+\frac{\lambda}{p})z(D_p^n f(z))' + \frac{\lambda}{p} z^2 (D_p^n f(z))''}{(1-\lambda)(D_{\lambda,p}^n f(z)) + \frac{\lambda}{p} z (D_p^n f(z))'} - p \right| \end{array} \right\} \tag{18}$$

$(0 \leq \lambda \leq 1, -p \leq \alpha < p; \beta \geq 0, n \in \mathbb{N}_0, z \in \mathbb{U})$, where D_p^n is $p-$valent Sãlãgean operator introduced by ([10] and [4]);

$\sum\limits_{k=p+1}^{\infty} \Omega_k z^k; \alpha, \beta) = TS_{\lambda,q,s}^{p,n}(n, \alpha, \beta)$

$$= \left\{ \begin{array}{c} f \in T(p) : Re\left\{ \frac{(1-\lambda+\frac{\lambda}{p})z(H_{p,q,s}(\alpha_1)f(z))' + \frac{\lambda}{p} z^2 (H_{p,q,s}(\alpha_1)f(z))''}{(1-\lambda)(H_{p,q,s}(\alpha_1)f(z)) + \frac{\lambda}{p} z (H_{p,q,s}(\alpha_1)(z))'} - \alpha \right\} \\ > \beta \left| \frac{(1-\lambda+\frac{\lambda}{p})z(H_{p,q,s}(\alpha_1)f(z))' + \frac{\lambda}{p} z^2 (H_{p,q,s}(\alpha_1)f(z))''}{(1-\lambda)(H_{p,q,s}(\alpha_1)f(z)) + \frac{\lambda}{p} z (H_{p,q,s}(\alpha_1)(z))'} - p \right| (z \in \mathbb{U}), \end{array} \right\} \tag{19}$$

where Ω_k is given by (1.10) and $H_{p,q,s}(\alpha_1)$ is the $p-$valent Dziok-Srivastava operatorintroduced by ([8] and [9], see also [5], [2] and [6]).

Denote by U the unit disc of ....

Remark 1.1 *If* $f \in \mathcal{A}_\zeta^*$, $f(z, \zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then* $R^m f(z, \zeta) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

2 Coefficient estimates

Throughout our present paper, we assume that: $g(z)$ is defined by (2) with $b_k > 0 (k \geq p+1)$, $0 \leq \alpha < p, p \in \mathbb{N}$, $0 \leq \lambda \leq 1$,$\beta \geq 0$ and $z \in \mathbb{U}$.

Theorem 1 *A function* $f(z) \in S_{p,\lambda}(f, g; \alpha, \beta)$ *if*

$$\sum_{k=p+1}^{\infty} \left(\frac{p + \lambda(k-p)}{p} \right) [k(1+\beta) - (\alpha + p\beta)] |a_k| b_k \leq p - \alpha. \tag{20}$$

Proof. It suffices to show that

$$\beta \left| \frac{(1 - \lambda + \frac{\lambda}{p}\ )z(f * g)'(z) + \frac{\lambda}{p} z^2 (f * g)''(z)}{(1-\lambda)(f * g)(z) + \frac{\lambda}{p} z (f * g)'(z)} - p \right| \tag{21}$$

$$-Re\left\{\frac{(1-\lambda+\frac{\lambda}{p}\ )z(f*g)^{'}(z)+\frac{\lambda}{p}z^{2}(f*g)^{''}(z)}{(1-\lambda)\,(f*g)(z)+\frac{\lambda}{p}z(f*g)'(z)}-p\right\}\leq p-\alpha. \tag{22}$$

We have

$$\beta\left|\frac{(1-\lambda+\frac{\lambda}{p}\ )z(f*g)^{'}(z)+\frac{\lambda}{p}z^{2}(f*g)^{''}(z)}{(1-\lambda)\,(f*g)(z)+\frac{\lambda}{p}z(f*g)'(z)}-p\right| \tag{23}$$

$$-Re\left\{\frac{(1-\lambda+\frac{\lambda}{p}\ )z(f*g)^{'}(z)+\frac{\lambda}{p}z^{2}(f*g)^{''}(z)}{(1-\lambda)\,(f*g)(z)+\frac{\lambda}{p}z(f*g)'(z)}-p\right\} \tag{24}$$

$$\leq(1+\beta)\left|\frac{(1-\lambda+\frac{\lambda}{p}\ )z(f*g)^{'}(z)+\frac{\lambda}{p}z^{2}(f*g)^{''}(z)}{(1-\lambda)\,(f*g)(z)+\frac{\lambda}{p}z(f*g)'(z)}-p\right| \tag{25}$$

$$\leq\frac{(1+\beta)\sum\limits_{k=p+1}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)(k-p)\left|a_k\right|b_k}{1-\sum\limits_{k=p+1}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)\left|a_k\right|b_k}. \tag{26}$$

The last expression is bounded above by $(p-\alpha)$ since (20) holds. Hence the proof of Theorem 1 is completed.

Theorem 2 . *A function $f(z)\in TS_{p,\lambda}(f,g;\alpha,\beta)$ if and only if*

$$\sum_{k=p+1}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)\left[k(1+\beta)-(\alpha+p\beta)\right]a_kb_k\leq p-\alpha. \tag{27}$$

Proof. In view of Theorem 1, we need only to prove the necessity. If $f(z)\in TS_{p,\lambda}(f,g;\alpha,\beta)$ and z is real, then

$$\frac{p-\sum\limits_{k=p+1}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)a_kb_kz^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)a_kb_kz^{k-p}}-\alpha\geq\beta\left|\frac{-\sum\limits_{k=p+1}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)(k-p)\ \ a_kb_kz^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)a_kb_kz^{k-p}}\right|. \tag{28}$$

Letting $z\to1^{-}$ along the real axis, we obtain the desired inequality

$$\sum_{k=1+p}^{\infty}\left(\frac{p+\lambda(k-p)}{p}\right)\left[k(1+\beta)-(\alpha+p\beta)\right]a_kb_k\leq p-\alpha \tag{29}$$

and hence the proof of Theorem 2 is completed.

Corollary 1 *If a function $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$. Then*

$$a_k \leq \frac{p-\alpha}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k} \quad (k \geq p+1). \tag{30}$$

The result is sharp for the function

$$f(z) = z^p - \frac{p-\alpha}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k} z^k \quad (k \geq p+1). \tag{31}$$

3 Distortion theorems

Theorem 3 *If $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$, then for $|z| = r < 1$*

$$|f(z)| \geq r^p - \frac{p-\alpha}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} r^{p+1} \tag{32}$$

and

$$|f(z)| \leq r^p + \frac{p-\alpha}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} r^{p+1}, \tag{33}$$

provided that $b_k \geq b_{p+1}$ $(k \geq p+1)$. The equalities in (32) and (33) are attained for the function $f(z)$ given by

$$f(z) = z^p - \frac{p-\alpha}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} z^{p+1}. \tag{34}$$

Proof. Using Theorem 2, we have

$$(p+1+\beta-\alpha)\left(\frac{p+\lambda}{p}\right) b_{p+1} \sum_{k=p+1}^{\infty} a_k \tag{35}$$

$$\leq \sum_{k=p+1}^{\infty} \left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k a_k \leq p-\alpha \tag{36}$$

that is, that

$$\sum_{k=p+1}^{\infty} a_k \leq \frac{p-\alpha}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}}. \tag{37}$$

From (4) and (37), we have

$$|f(z)| \geq r^p - r^{1+p} \sum_{k=p+1}^{\infty} a_k \tag{38}$$

$$\geq r^p - \frac{p-\alpha}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} r^{p+1} \tag{39}$$

and

$$|f(z)| \leq r^p + r^{1+p} \sum_{k=p+1}^{\infty} a_k \tag{40}$$

$$\leq r^p + \frac{p-\alpha}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} r^{p+1}. \tag{41}$$

This completes the proof of Theorem 3.

Theorem 4 *Let* $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$, *then for* $|z| = r < 1$

$$\left|f'(z)\right| \geq p\ r^{p-1} - \frac{(p+1)(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} r^p \tag{42}$$

and

$$\left|f'(z)\right| \leq p\ r^{p-1} + \frac{(p+1)(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} r^p, \tag{43}$$

provided that $b_k \geq b_{p+1}$ $(k \geq p+1)$. *The result is sharp for the function* $f(z)$ *given by (34).*

Proof. From Theorem 2, we have

$$\sum_{k=p+1}^{\infty} k a_k \leq \frac{(p+1)(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}}. \tag{44}$$

The remaining part of the proof is similar to the proof of Theorem 3, then we omit the details.

Now, differentiating both sides of (4) m-times, we have

$$f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k z^{k-m} (k \geq p+1; p \in \mathbb{N}; m < p, m \in \mathbb{N}_0). \tag{45}$$

Theorem 5 *If a function* $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$. *Then for* $|z| = r < 1$, *we have*

$$\left|f^{(m)}(z)\right| \geq \left(\frac{p!}{(p-m)!} - \frac{(p+1)!(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1-m)!(p+1+\beta-\alpha)b_{p+1}} r \right) r^{p-m} \tag{46}$$

and

$$\left|f^{(m)}(z)\right| \leq \left(\frac{p!}{(p-m)!} + \frac{(p+1)!(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1-m)!(p+1+\beta-\alpha)b_{p+1}}r\right)r^{p-m}. \tag{47}$$

The result is sharp for the function $f(z)$ given by (34).

Proof. Using (27), we have

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!}a_k \leq \frac{(p+1)!(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1-m)!(p+1+\beta-\alpha)b_{p+1}}. \tag{48}$$

From (45) and (48), we have

$$\left|f^{(m)}(z)\right| \geq \frac{p!}{(p-m)!}r^{p-m} - r^{p+1-m}\sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!}a_k \tag{49}$$

$$\geq \left(\frac{p!}{(p-m)!} - \frac{(p+1)!(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1-m)!(p+1+\beta-\alpha)b_{p+1}}r\right)r^{p-m} \tag{50}$$

and

$$\left|f^{(m)}(z)\right| \leq \frac{p!}{(p-m)!}r^{p-m} + r^{p+1-m}\sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!}a_k \tag{51}$$

$$\leq \left(\frac{p!}{(p-m)!} + \frac{(p+1)!(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1-m)!(p+1+\beta-\alpha)b_{p+1}}r\right)r^{p-m} \tag{52}$$

This completes the proof of Theorem 5.

Remark 1 *Putting $m = 0$ and $m = 1$, in Theorem 5 we obtain Theorem 3 and Theorem 4, respectively.*

4 Convex linear combinations

Theorem 6 *Let $\eta_\mu \geq 0$ for $\mu = 1, 2, ..., n$ and $\sum_{\mu=1}^{n} \eta_\mu \leq 1$. If the functions $f_\mu(z)$ defined by*

$$f_\mu(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,\mu}z^k \; (a_{k,\mu} \geq 0; \mu = 1, 2, ..., n) \tag{53}$$

are in the class $TS_{p,\lambda}(f,g;\alpha,\beta)$ *for every* $\mu = 1, 2, ..., n$ *then the function* $f(z)$ *defined by*

$$f(z) = z^p - \sum_{k=p+1}^{\infty} \left(\sum_{\mu=1}^{n} \eta_\mu a_{k,\mu} \right) z^k \tag{54}$$

is in the class $TS_{p,\lambda}(f,g;\alpha,\beta)$.

Proof. Since $f_\mu(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$, it follows from Theorem 2 that

$$\sum_{k=p+1}^{\infty} \left(\frac{p+\lambda(k-p)}{p} \right) [k(1+\beta) - (\alpha + p\beta)]\, b_k\ a_{k,\mu} \leq p - \alpha\ , \tag{55}$$

for every $\mu = 1, 2, ..., n$. Hence

$$\sum_{k=p+1}^{\infty} \left(\frac{p+\lambda(k-p)}{p} \right) [k(1+\beta) - (\alpha + p\beta)] \left(\sum_{\mu=1}^{n} \eta_\mu a_{k,\mu} \right) b_k \tag{56}$$

$$= \sum_{\mu=1}^{n} \eta_\mu \left(\sum_{k=p+1}^{\infty} \left(\frac{p+\lambda(k-p)}{p} \right) [k(1+\beta) - (\alpha + p\beta)]\ \ a_{k,\mu} b_k \right) \tag{57}$$

$$\leq (p-\alpha) \sum_{\mu=1}^{n} \eta_\mu \leq p - \alpha. \tag{58}$$

By Theorem 2, it follows that $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$ and hence the proof of Theorem 6 is completed.

Corollary 2 *The class* $TS_{p,\lambda}(f,g;\alpha,\beta)$ *is closed under convex linear combinations.*

Theorem 7 *Let* $f_p(z) = z^p$ *and*

$$f_k(z) = z^p - \frac{p-\alpha}{\left(\frac{p+\lambda(k-p)}{p} \right) [k(1+\beta) - (\alpha + p\beta)]\, b_k} z^k\, (k \geq p+1)\,. \tag{59}$$

Then $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$ *if and only if*

$$f\ (z) = \sum_{k=p+1}^{\infty} \gamma_k\ f_k(z), \tag{60}$$

where $\gamma_k \geq 0$ *and* $\sum\limits_{k=p}^{\infty} \gamma_k = 1$.

Proof. Assume that

$$f\ (z) = \sum_{k=p}^{\infty} \gamma_k\ f_k(z) \tag{61}$$

$$= z^p - \sum_{k=p+1}^{\infty} \frac{(p-\alpha)}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k}\ \gamma_k\ z^k. \tag{62}$$

Then it follows that

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k}{p-\alpha} \cdot \frac{(p-\alpha)}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k}\ \gamma_k \tag{63}$$

$$= \sum_{k=p+1}^{\infty} \gamma_k\ = 1-\gamma_p \le 1. \tag{64}$$

So, by Theorem 2, $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$.

Conversely, assume that the function $f(z)$ belongs to the class $TS_{p,\lambda}(f, g; \alpha, \beta)$. Then

$$a_k \le \frac{p-\alpha}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k}(k \ge p+1). \tag{65}$$

Setting

$$\gamma_k = \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,a_k b_k}{p-\alpha}(k \ge p+1) \tag{66}$$

and

$$\gamma_p = 1 - \sum_{k=p+1}^{\infty} \gamma_k\ , \tag{67}$$

we see that $f(z)$ can be expressed in the form (60). This completes the proof of Theorem 7.

Corollary 3 *The extreme points of the class $TS_{p,\lambda}(f, g; \alpha, \beta)$ are the functions $f_p(z) = z^p$ and*

$$f_k(z) = z^p - \frac{p-\alpha}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k} z^k\,(k \ge p+1)\,. \tag{68}$$

5 Radii of close-to-convexity, starlikeness and convexity

Theorem 8 *Let the function* $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$*. Then*

Corollary 4 *(i)* $f(z)$ *is p-valent close-to-convex of order* δ $(0 \leq \delta < p)$ *in* $|z| < r_1$*, where*

$$r_1 = \inf_{k \geq p+1} \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)(p-\delta)\left[k(1+\beta)-(\alpha+p\beta)\right] b_k}{k(p-\alpha)} \right\}^{\frac{1}{k-p}}. \tag{69}$$

(ii) $f(z)$ *is p-valent starlike of order* δ $(0 \leq \delta < p)$ *in* $|z| < r_2$*, where*

$$r_2 = \inf_{k \geq p+1} \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)(p-\delta)\left[k(1+\beta)-(\alpha+p\beta)\right] b_k}{(k-\delta)(p-\alpha)} \right\}^{\frac{1}{k-p}}. \tag{70}$$

The result is sharp, the extremal function being given by (31).

Proof. (i) We must show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta \quad for\, |z| < r_1, \tag{71}$$

where r_1 is given by (69). Indeed we find from (4) that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}. \tag{72}$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta, \tag{73}$$

if

$$\sum_{k=p+1}^{\infty} \left(\frac{k}{p-\delta} \right) a_k |z|^{k-p} \leq 1. \tag{74}$$

But, by Theorem 2, (74) will be true if

$$\left(\frac{k}{p-\delta} \right) |z|^{k-p} \leq \frac{\left(\frac{p+\lambda(k-p)}{p}\right)\left[k(1+\beta)-(\alpha+p\beta)\right] b_k}{(p-\alpha)}, \tag{75}$$

that is, if

$$|z| \leq \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)(p-\delta)\left[k(1+\beta)-(\alpha+p\beta)\right]b_k}{k(p-\alpha)} \right\}^{\frac{1}{k-p}} \quad (k \geq p+1). \tag{76}$$

the proof of (i) is completed.

(ii) It is sufficient to show that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \delta for \;\; |z| < r_2, \tag{77}$$

where r_2 is given by (70). Indeed we find, again from (4) that

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq \frac{\sum\limits_{k=p+1}^{\infty}(k-p)a_k\,|z|^{k-p}}{1-\sum\limits_{k=p+1}^{\infty}a_k\,|z|^{k-p}}\,. \tag{78}$$

Thus

$$\left|\frac{zf'(z)}{f(z)} - p\right| \leq p - \delta \tag{79}$$

if

$$\sum_{k=p+1}^{\infty}\frac{(k-\delta)}{(p-\delta)}\,|z|^{k-p}\,a_k \leq 1. \tag{80}$$

But, by Theorem 2, (80) will be true if

$$\frac{(k-\delta)}{(p-\delta)}\,|z|^{k-p} \leq \frac{\left(\frac{p+\lambda(k-p)}{p}\right)\left[k(1+\beta)-(\alpha+p\beta)\right]b_k}{(p-\alpha)} \tag{81}$$

that is, if

$$|z| \leq \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)(p-\delta)\left[k(1+\beta)-(\alpha+p\beta)\right]b_k}{(k-\delta)(p-\alpha)} \right\}^{\frac{1}{k-p}} \quad (k \geq p+1)\,. \tag{82}$$

the proof of (ii) is completed.

Corollary 5 *Let $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$. Then $f(z)$ is p-valent convex of order δ $(0 \leq \delta < p)$ in $|z| < r_3$, where*

$$r_3 = \inf_{k \geq p+1}\left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)(p-\delta)\left[k(1+\beta)-(\alpha+p\beta)\right]b_k}{k\;(k-\delta)(p-\alpha)} \right\}^{\frac{1}{k-p}}. \tag{83}$$

The result is sharp, with the extremal function $f(z)$ given by (31).

6 Classes of preserving integral operators

In this section, we discuss some classes preserving integral operators. Consider the following operators

(i) $F(z)$ defined by

$$F(z) = (J_{c,p}f)(z) = \frac{c+p}{z^c} \int_0^z t^{c-1}\ f\ (t)dt \tag{84}$$

$$= z^p - \sum_{k=p+1}^{\infty} d_k z^k (f \in S(p); c > -p), \tag{85}$$

where

$$d_k = \left(\frac{c+p}{c+k}\right) a_k \leq a_k \left(k \geq p+1\right). \tag{86}$$

(ii) $G(z)$ defined by

$$G(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p} dt \tag{87}$$

$$= z^p - \sum_{k=p+1}^{\infty} \frac{1}{k-p+1} a_k z^k \left(k \geq p+1\right). \tag{88}$$

(iii)The Komatu operator [16] defined by

$$\begin{aligned} H(z) &= P_{c,p}^d f(z) = \frac{(c+p)^d}{\Gamma(d) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t^p}\right)^{d-1} f\ (t)dt \\ &= z^p - \sum_{k=p+1}^{\infty} \left(\frac{c+p}{c+k}\right)^d a_k\ z^k \end{aligned}$$

$$(d > 0; c > -p; p \in \mathbb{N}; z \in \mathbb{U}), \tag{89}$$

and

(iv) $I(z)$, which generalizes Jung-Kim-Srivastava integral operator (see [13]) defined by

$$\begin{aligned} I(z) &= Q_{c,p}^d f(z) = \frac{\Gamma(d+c+p)}{\Gamma(c+p)\Gamma(d)} \frac{1}{z^c} \int_0^z t^{c-1} \left(1 - \frac{t}{z}\right)^{d-1} f\ (t)dt \\ &= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(c+k)\Gamma(d+c+p)}{\Gamma(c+p)\Gamma\left(d+c+k\right)} a_k z^k \end{aligned}$$

$$(d > 0; c > -p; z \in \mathbb{U}), \tag{90}$$

In view of Theorem 2, we see that $z^p - \sum_{k=p+1}^{\infty} d_k z^k$ is in the class $TS_{p,\lambda}(f, g; \alpha, \beta)$ as long as $0 \leq d_k \leq a_k$ for all k. In particular, we have

Theorem 9 *Let $f(z) \in TS_p(f, g; \alpha, \beta, \lambda)$ and c be a real number such that $c > -p$. Then the function $F(z)$ defined by (84) also belongs to the class $TS_{p,\lambda}(f, g; \alpha, \beta)$.*

Proof. From (27) and (86), we have the function $F(z)$ is in the class $TS_{p,\lambda}(f, g; \alpha, \beta)$.

Theorem 10 *Let $F(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k$ $(a_k \geq 0)$ be in the class $TS_{p,\lambda}(f, g; \alpha, \beta)$, and let c be a real number such that $c > -p$. Then the function $f(z)$ given by (84) is p-valent in $|z| < R_1$, where*

$$R_1 = \inf_{k \geq p+1} \left\{ \frac{(c+p)\,(p+\lambda(k-p))\ \ [k(1+\beta) - (\alpha + p\beta)]\, b_k}{k(c+k)\ (p-\alpha)} \right\}^{\frac{1}{k-p}}. \tag{91}$$

The result is sharp.

Proof. From (84), we have

$$f(z) = \frac{z^{1-c}\, [z^c F(z)]^{'}}{(c+p)}\ (c > -p) \tag{92}$$

$$= z^p - \sum_{k=p+1}^{\infty} \left(\frac{c+k}{c+p} \right) a_k z^k. \tag{93}$$

In order to obtain the required result, it suffices to show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| < p \ \ \textit{whenever} \ \ |z| < R_1, \tag{94}$$

where R_1 is given by (91). Now

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} \frac{k(c+k)}{(c+p)} a_k \, |z|^{k-p}. \tag{95}$$

Thus $\left| \frac{f'(z)}{z^{p-1}} - p \right| < p$ if

$$\sum_{k=p+1}^{\infty} \frac{k(c+k)}{p\ (c+p)} a_k \, |z|^{k-p} < 1. \tag{96}$$

But Theorem 2 confirms that

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,|a_k|\,b_k}{p-\alpha} \le 1. \tag{97}$$

Hence (96) will be satisfied if

$$\frac{k(c+k)}{(c+p)}\,|z|^{k-p} < \frac{[p+\lambda(k-p)]\,[k(1+\beta)-(\alpha+p\beta)]\,b_k}{p-\alpha}, \tag{98}$$

that is, if

$$|z| < \left\{\frac{(c+p)\,[p+\lambda(k-p)]\,[k(1+\beta)-(\alpha+p\beta)]\,b_k}{k(c+k)\,(p-\alpha)}\right\}^{\frac{1}{k-p}} \quad (k \ge p+1)\,. \tag{99}$$

Therefore, the function $f(z)$ given by (84) is p-valent in $|z| < R_1$. Sharpness of the result follows if we take

$$f(z) = z^p - \frac{(p-\alpha)(c+k)}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\,b_k\,(c+p)} z^k (k \ge p+1) \tag{100}$$

and hence the proof of Theorem 10 is completed.

Theorem 11 *If $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$, then $H(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$.*

Proof. The function $H(z)$ defined by (6.5) is in the class $TS_{p,\lambda}(f,g;\alpha,\beta)$ if

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{[p+\lambda(k-p)]}{p}\right)\;[k(1+\beta)-(\alpha+p\beta)]\,b_k\;\left(\frac{c+p}{c+k}\right)^d}{p-\alpha} a_k \le 1. \tag{101}$$

Since $\frac{c+p}{c+k} \le 1$ for $k \ge p+1$, then

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{[p+\lambda(k-p)]}{p}\right)\;[k(1+\beta)-(\alpha+p\beta)]\,b_k\left(\frac{c+p}{c+k}\right)^d}{p-\alpha} a_k \tag{102}$$

$$\le \sum_{k=p+1}^{\infty} \frac{\left(\frac{[p+\lambda(k-p)]}{p}\right)\;[k(1+\beta)-(\alpha+p\beta)]\;b_k}{p-\alpha} a_k \le 1. \tag{103}$$

Therefore $H(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$.

Theorem 12 *Let $d > 0, c > -p$ and $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$. Then function $H(z)$ defined by (89) is p-valent in $|z| < R_2$, where*

$$R_2 = \inf_{k \geq p+1} \left\{ \frac{(c+k)^d \, (p + \lambda(k-p) \, [k(1+\beta) - (\alpha + p\beta)] \, b_k}{k(c+p)^d \; (p-\alpha)} \right\}^{\frac{1}{k-p}}. \tag{104}$$

The result is sharp for function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\alpha)(c+p)^d}{\left(\frac{p+\lambda(k-p)}{p}\right) [k(1+\beta) - (\alpha + p\beta)] \, b_k \, (c+k)^d} z^k (k \geq p+1). \tag{105}$$

Proof. In order to prove the assertion. it is enough to show that

$$\left| \frac{H'(z)}{z^{p-1}} - p \right| < pin \, |z| < R_2 \tag{106}$$

Now

$$\begin{aligned} \left| \frac{H'(z)}{z^{p-1}} - p \right| &\leq \left| \sum_{k=p+1}^{\infty} k \left(\frac{c+p}{c+k} \right)^d a_k z^{k-p} \right| \\ &\leq \sum_{k=p+1}^{\infty} k \left(\frac{c+p}{c+k} \right)^d a_k \, |z|^{k-p}. \end{aligned}$$

The last inequality is bounded a above by p if

$$\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} \frac{k}{p} \left(\frac{c+p}{c+k} \right)^d a_k \, |z|^{k-p} \leq 1. \tag{107}$$

Since $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$, then in view of Theorem 2, we have

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{[p+\lambda(k-p)]}{p} \right) \; [k(1+\beta) - (\alpha + p\beta)] \; b_k}{p-\alpha} a_k \leq 1. \tag{108}$$

Thus, (107) holds if

$$k \left(\frac{c+p}{c+k} \right)^d |z|^{k-p} \leq \frac{[p + \lambda(k-p)] \, [k(1+\beta) - (\alpha + p\beta)] \; b_k}{p-\alpha}, \tag{109}$$

or

$$|z| \leq \left\{ \frac{(c+k)^d [p + \lambda(k-p)] \, [k(1+\beta) - (\alpha + p\beta)] \; b_k}{k(c+p)^d (p-\alpha)} \right\}^{\frac{1}{k-p}} (k \geq p+1). \tag{110}$$

Therefore, the function $H(z)$ given by (89) is p-valent in $|z| < R_2$. This completes the prove of Theorem 12.

Following similar steps as in the proofs of Theorems 11 and 12, we can prove the following theorems concerning the integral operator $I(z)$ defined by (90).

Theorem 13 *If $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$, then $I(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$,where $I(z)$ is given by (90).*

Theorem 14 *Let $d > 0, c > -p$ and $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$. Then function $I(z)$ defined by (90) is p-valent in $|z| < R_3$, where*

$$R_3 = \inf_{k \geq p+1} \left\{ \frac{\Gamma(c+p)\Gamma\,(d+c+k)\,(p+\lambda(k-p)\,[k(1+\beta)-(\alpha+p\beta)]\,b_k}{k\Gamma(c+k)\Gamma(d+c+p)\;(p-\alpha)} \right\}^{\frac{1}{k-p}}. \tag{111}$$

The result is sharp for function $f(z)$ given by

$$f(z) = z^p - \frac{(p-\alpha)\Gamma(c+k)\Gamma(d+c+p)}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k\Gamma(c+p)\Gamma\,(d+c+k)} z^k (k \geq p+1). \tag{112}$$

7 Modified Hadamard products

Let the functions $f_\mu(z)$ be defined for $\mu = 1, 2, ...m$ by (53). The modified Hadamard products of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 * f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1}a_{k,2}\; z^k. \tag{113}$$

Theorem 15 *Let $f_\mu(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)(\mu = 1,2)$, where $f_\mu(z)(\mu = 1,2)$are in the form (53). Then $(f_1 * f_2)(z) \in TS_{p,\lambda}(f,g;\delta,\beta)$, where*

$$\delta = p - \frac{(p-\alpha)^2(1+\beta)}{\left(\frac{p+\lambda}{p}\right)(p+1-\alpha+\beta)^2 b_{p+1} - (p-\alpha)^2}. \tag{114}$$

The result is sharp for

$$f_\mu(z) = z^p - \frac{(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1-\alpha+\beta)b_{p+1}} z^{p+1}\ (\mu = 1,2). \tag{115}$$

Proof. Employing the technique used earlier by Schild and Silverman [21], we need to find the largest real parameter δ such that

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)\ [k(1+\beta)-(\delta+p\beta)]b_k}{(p-\delta)} a_{k,1}a_{k,2} \leq 1. \tag{116}$$

Since $f_\mu \in TS_{p,\lambda}(f,g;\alpha,\beta)(\mu=1,2)$, we have

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} a_{k,1} \leq 1 \tag{117}$$

and

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} a_{k,2} \leq 1. \tag{118}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} \sqrt{a_{k,1}a_{k,2}} \leq 1. \tag{119}$$

thus it sufficient to show that

$$\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\delta+p\beta)]b_k}{(p-\delta)} a_{k,1}a_{k,2} \tag{120}$$

$$\leq \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} \sqrt{a_{k,1}a_{k,2}} \tag{121}$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-\delta)[k(1+\beta)-(\alpha+p\beta)]}{(p-\alpha)[k(1+\beta)-(\delta+p\beta)]}. \tag{122}$$

Hence, in the light of the inequality (119), it is sufficient to prove that

$$\frac{(p-\alpha)}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k} \leq \frac{(p-\delta)[k(1+\beta)-(\alpha+p\beta)]}{(p-\alpha)[k(1+\beta)-(\delta+p\beta)]}. \tag{123}$$

It follows from (123) that

$$\delta \leq\ p - \frac{(1+\beta)(k-p)(p-\alpha)^2}{[k(1+\beta)-(\alpha+p\beta)]^2\frac{[p+\lambda(k-p)]}{p}b_k-(p-\alpha)^2}. \tag{124}$$

Let

$$M(k) = p - \frac{(1+\beta)(k-p)(p-\alpha)^2}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]^2 b_k - (p-\alpha)^2}, \tag{125}$$

then $M(k)$ is increasing function of $k(k \geq p+1)$. Therefore, we conclude that

$$\delta \leq M(p+1) = p - \frac{(1+\beta)(p-\alpha)^2}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)^2 b_{p+1} - (p-\alpha)^2} \tag{126}$$

and hence the proof of Theorem 15 is completed.

Theorem 16 *Let* $f_1(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$ *and* $f_2(z) \in TS_{p,\lambda}(f, g; \gamma, \beta)$, *where* $f_\mu(z)(\mu = 1, 2)$ *are in the form (53). Then* $(f_1 * f_2)(z) \in TS_{p,\lambda}(f, g; \xi, \beta)$,*where*

$$\xi \leq p - \frac{(1+\beta)(p-\alpha)(p-\gamma)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)(p+1+\beta-\gamma)b_{p+1} - (p-\alpha)(p-\gamma)}. \tag{127}$$

The result is sharp for

$$f_1(z) = z^p - \frac{(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)b_{p+1}} z^{p+1} \tag{128}$$

and

$$f_2(z) = z^{p} - \frac{(p-\gamma)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\gamma)b_{p+1}} z^{p+1}. \tag{129}$$

Proof. We need to find the largest real parameter ξ such that

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\xi+p\beta)]b_k}{(p-\xi)} a_{k,1}a_{k,2} \leq 1. \tag{130}$$

Since $f_1(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$, we readily see that

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} a_{k,1} \leq 1 \tag{131}$$

and $f_2(z) \in TS_{p,\lambda}(f, g; \gamma, \beta)$, we have

$$\sum_{k=2}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\gamma+p\beta)]b_k}{(p-\gamma)} a_{k,2} \leq 1. \tag{132}$$

By the Cauchy-Schwarz inequality we have

$$\sum_{k=2}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\gamma+p\beta)]^{\frac{1}{2}} b_k}{\sqrt{p-\alpha}\sqrt{p-\gamma}} \sqrt{a_{k,1}a_{k,2}} \leq 1 \tag{133}$$

thus it is sufficient to show that

$$\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\xi+p\beta)]b_k}{(p-\xi)} a_{k,1}a_{k,2} \tag{134}$$

$$\leq \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\gamma+p\beta)]^{\frac{1}{2}} b_k}{\sqrt{p-\alpha}\sqrt{p-\gamma}} \sqrt{a_{k,1}a_{k,2}} \tag{135}$$

or, equivalently, that

$$\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-\xi)[k(1+\beta)-(\alpha+p\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\gamma+p\beta)]^{\frac{1}{2}}}{\sqrt{p-\alpha}\sqrt{p-\gamma}[k(1+\beta)-(\xi+p\beta)]}. \tag{136}$$

Hence, in the light of the inequality (133), it is sufficient to prove that

$$\frac{\sqrt{p-\alpha}\sqrt{p-\gamma}}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\gamma+p\beta)]^{\frac{1}{2}} b_k} \tag{137}$$

$$\leq \frac{(p-\xi)\ [k(1+\beta)-(\alpha+p\beta)]^{\frac{1}{2}}\ [k(1+\beta)-(\gamma+p\beta)]^{\frac{1}{2}}}{\sqrt{p-\alpha}\ \sqrt{p-\gamma}\ [k(1+\beta)-(\xi+p\beta)]}. \tag{138}$$

It follows from (137) that

$$\xi \leq\ p - \frac{(1+\beta)(k-p)(p-\alpha)(p-\gamma)}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\gamma+p\beta)]b_k-(p-\alpha)(p-\gamma)}. \tag{139}$$

Let

$$A(k) = p - \frac{(1+\beta)(k-p)\ (p-\alpha)\ (p-\gamma)}{\left(\frac{p+\lambda(k-p)}{p}\right)\ [k(1+\beta)-(\alpha+p\beta)][k(1+\beta)-(\gamma+p\beta)]b_k-(p-\alpha)(p-\gamma)}, \tag{140}$$

then $A(k)$ is increasing function of $k(k \geq p+1)$. Therefore, we conclude that

$$\xi\ \leq A(p{+}1) = p - \frac{(1+\beta)(p-\alpha)(p-\gamma)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)(p+1+\beta-\gamma)b_{p+1}-(p-\alpha)(p-\gamma)} \tag{141}$$

and hence the proof of Theorem 16 is completed.

Theorem 17 *Let $f_\mu(z) \in TS_{p,\lambda}(f,g\ ;\alpha,\beta)(\mu=1,2,3)$, where $f_\mu(z)(\mu=1,2,3)$ are in the form (53). Then $(f_1*f_2*f_3)(z) \in TS_{p,\lambda}(f,g\ ;\tau,\beta)$, where*

$$\tau\partial p-\frac{(1+\beta)(p-\alpha)^3}{\left(\frac{p+\lambda}{p}\right)^2(p+1+\beta-\alpha)^3b_{p+1}^2-(p-\alpha)^3}. \tag{142}$$

The result is best possible for functions $f_\mu(z)(\mu=1,2,3)$ given by (115).

Proof. From Theorem 15, we have $(f_1*f_2)(z)\ \in TS_{p,\lambda}(f,g\ ;\delta,\beta)$, where δ is given by (114). Now, using Thoerem 16, we get $(\ f_1*f_2*f_3\ )(z) \in TS_{p,\lambda}(f,g\ ;\tau,\beta)$, where

$$\tau\le p-\frac{(1+\beta)(p-\alpha)\ (p-\delta)(k-p)}{\left(\frac{1+\lambda(k-p)}{p}\right)\ [k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\delta+\beta)]b_k-(p-\alpha)(p-\delta)}. \tag{143}$$

Now defining the function $B(k)$ by

$$B(k)=p-\frac{(1+\beta)(p-\alpha)(p-\delta)(k-p)}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+\beta)][k(1+\beta)-(\delta+\beta)]b_k-(p-\alpha)(p-\delta)}. \tag{144}$$

We see that $B(k)$ is increasing function of $k(k\ge p+1)$. Therefore, we conclude that

$$\tau\le B(p+1)\ =p-\frac{(1+\beta)(p-\alpha)(p-\delta)}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)(p+1+\beta-\delta)b_{p+1}-(p-\alpha)(p-\delta)}, \tag{145}$$

substituting from (7.2), we have

$$\tau=p-\frac{(1+\beta)(p-\alpha)^3}{\left(\frac{p+\lambda}{p}\right)^2(p+1+\beta-\alpha)^3b_{p+1}^2-(p-\alpha)^3} \tag{146}$$

and hence the proof of Theorem 17 is completed.

Theorem 18 *Let $f_\mu(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)(\mu=1,2)$, where $f_\mu(z)(\mu=1,2)$ are in the form (53). Then*

$$h\ (z)=z^p-\sum_{k=p+1}^{\infty}(a_{k,1}^2+a_{k,2}^2)z^k \tag{147}$$

belongs to the class $TS_{p,\lambda}(f,g;\varphi,\beta)$*, where*

$$\varphi \le p - \frac{2(1+\beta)(p-\alpha)^2}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)^2 b_{p+1} - 2(p-\alpha)^2}. \tag{148}$$

The result is sharp for functions $f_\mu(z)(\mu=1,2)$ *defined by (115).*

Proof. By using Theorem 16, we obtain

$$\sum_{k=p+1}^{\infty}\left\{\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)}\right\}^2 a_{k,1}^2 \tag{149}$$

$$\le \left\{\sum_{k=p+1}^{\infty}\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)}a_{k,1}\right\}^2 \le 1, \tag{150}$$

and

$$\sum_{k=2}^{\infty}\left\{\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)}\right\}^2 a_{k,2}^2 \tag{151}$$

$$\le \left\{\sum_{k=2}^{\infty}\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)}a_{k,2}\right\}^2 \le 1. \tag{152}$$

It follows form (149) and (151) that

$$\sum_{k=2}^{\infty}\frac{1}{2}\left\{\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\ b_k}{(p-\alpha)}\right\}^2 (a_{k,1}^2+a_{k,2}^2) \le 1. \tag{153}$$

Therefore, we need to find the largest φ such that

$$\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\varphi+p\beta)]b_k}{(p-\varphi)} \tag{154}$$

$$\le \frac{1}{2}\left\{\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]\ b_k}{(p-\alpha)}\right\}^2, \tag{155}$$

that is

$$\varphi \le\ p - \frac{2(1+\beta)(k-p)(p-\alpha)^2}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k - 2(p-\alpha)^2} \tag{156}$$

Let

$$H(k) = \ p - \frac{2(1+\beta)(k-p)(p-\alpha)^2}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]^2 b_k - 2(p-\alpha)^2}, \tag{157}$$

then $H(k)$ is increasing function of $k(k \geq p+1)$. Therefore, we conclude that

$$\varphi \leq H(p+1) \ = p - \frac{2(1+\beta)(p-\alpha)^2}{\left(\frac{p+\lambda}{p}\right)(p+1+\beta-\alpha)^2 b_{p+1} - 2(p-\alpha)^2} \ , \tag{158}$$

and hence the proof of Theorem 18 is completed.

Remark 2 *(i) Putting* $g(z) = z^p + \sum_{k=1}^{\infty} C(k,n)(1-\gamma k)^n z^k$, *where* $C(k,n)$ *given by (16) by replacing* k *by* $k-p$, $\lambda = 0$ *and* $\lambda = 1$, *respectively, in Theorems 16, 17 and* **??**, *respectively, we obtain modified Hadamard products for the classes* $S_p^*(\alpha,\beta,\lambda)$ *and* $C_p(\alpha,\beta,\lambda)$, *respectively, defined in the introduction.*

(ii) Putting $\lambda = 0$ *and* $b_k = \Omega_k$, *where* Ω_k *is given by(11) and* $0 \leq \alpha < p, \beta \geq 0, \alpha_i \in \mathbb{C}(i = 1,2,...l)$, $\beta_j \in \mathbb{C}\backslash\{-1,-2,...\}(j = 1,2,...m)$*in Theorems 16, 17 and* **??** *respectively we modified the results obtained by Marouf [16, Theorems 4, 3 and 5, respectively].*

(iii) Putting $\lambda = 0$ *and* $b_k = \phi_k(n.\lambda.\delta,p)$, *where* ϕ_k*is given by (13) and* $(n \in \mathbb{N}_0, \lambda \geq 0, \delta \geq 0, -p \leq \alpha < p, \beta \geq 0)$*in Theorems 16, 17 and* **??**, *respectively, we obtain the results obtained by Salim et al. [20, Theorems 3, 2 and 4, respectively].*

(iiv) Putting $\lambda = 0$ *and* $b_k = \Phi(k)$,*where* $\Phi(k)$ *is given by (17) and* $(-\infty < \mu < 1, -\infty < \gamma < 1, \eta \in \mathbb{R}^+, p \in \mathbb{N}, -p \leq \alpha < p, \beta \geq 0, a \in \mathbb{R}, c \in \mathbb{R}\backslash\mathbb{Z}, z \in \mathbb{U})$,*in Theorems 16, 17 and* **??**, *respectively, we obtain the results obtained by Khairnar and More[15, Thoerems 4.2, 4.1 and 5.1, respectively, with* $n = 1$*].*

Remark 3 *Specializing the parameters* $p, \lambda, \alpha, \beta$ *and function* $g(z)$*in our results,we obtain new resultes associated to the subclasses* $TS_{p,\lambda}(n,\alpha,\beta)$ *and* $TS_{\lambda,q,s}^{p,n}(n,\alpha,\beta)$ *defined in the introduction.*

8 Open problem

The authors suggest to obtain the same properties of the class consisting of functions $f(z)$, $g(z) \in T(p)$ and satisfying the subordinating condition:

$$\frac{(1-\gamma+\frac{\gamma}{p})z(f*g)'(z)+\frac{\gamma}{p}z^2(f*g)''(z)}{(1-\gamma)(f*g)(z)+\frac{\gamma}{p}z(f*g)'(z)} \tag{159}$$

$$-\left|\frac{(1-\gamma+\frac{\gamma}{p})z(f*g)'(z)+\frac{\gamma}{p}z^2(f*g)''(z)}{(1-\gamma)\,(f*g)(z)+\frac{\gamma}{p}z(f*g)'(z)}-p\right| \prec p\frac{1+Az}{1+Bz}(z\in\mathbb{U}). \quad (160)$$

where $\prec$ denotes the subordinate.

References

[1] H. A. Al-Kharsani and S. S. Al-Hajiry, A note on certain inequalities for $p-$valent functions, J. Ineq. Pure Appl. Math., 9(2008), no. 3, Art. 90, 1-6.

[2] M. K. Aouf and H. E. Darwish, Subclasses of analytic functions associated with the generalized hypergeometric function, Comput. Math. Appl., 57(2009), 356-366.

[3] M. K. Aouf, R. M. El-Ashwah and S. M. EL-Deeb, Certain subclasses of uniformly starlike and convex functions defined by convolution, Acta Math. Paedagogicae Nyr., 26(2010), 55-70.

[4] M. K. Aouf and A. O. Mostafa, On a subclass of $n-p$ valent prestarlike functions, Comput. Math. Appl., 55(2008), no. 4, 851-861.

[5] M. K. Aouf and G. Murugusundaramoorthy, On a subclass of uniformly convex functions defined by the Dziok-Srivastava operator, Austral. J. Math. Anal. Appl., 5(2008), on. 1, Art. 3, 1-17.

[6] M. K. Aouf, A. A. Shamandy, A. O. Mostafa and E. A. Adwan, Some properties for certain class of analytic functions defined by Dziok-Srivastava operator with complex order, Proc. Pakistan Acad. Sci., 48(2011), no. 3, 185-194.

[7] M. Darus and R. W. Ibrahim, On a differential operator for multivalent functions, Studia. Univ. Babes-Bolyai Math., (2011), 125-134.

[8] J.Dziok and H. Srivastava, Classes of analytic functions with the generalized hypergeometric function, Appl. Math. Comput., 103(1999),1-13.

[9] J.Dziok and H. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, Integral Transform. Spec. Funct., 14(2003), 7-18.

[10] M. Kamali and H. Orhan, On a subclass of certain starlike functions with negative cofficients, Bull. Korean Math. Soc., 41(2004), no. 1, 53-71.

[11] S. Kanas, Uniformly $\alpha-$ convex functions, Internat. J. Appl. Math., 1(1999), no. 3, 305-310.

[12] S. Kanas and H.M. Srivastava, Linear operators associated with k-uniformly convex functions, Integral Transform. Spec. Fund., 9 (2000), 121-132.

[13] S. Kanas and A. Wisniowska, Conic regions and k-starlike functions, Rev. Roumaine Math. Pures Appl., 45 (2000), 647-657.

[14] S. Kanas and A. Wisniowska, Conic regions and k-uniform convexity, J. Comput. Appl. Math., 105(1999), 327-336.

[15] S. M. Khairnar and M. More, On a subclass of Multivalent β- uniformly starlike and convex function defined by a linear operator, Laeng Int. J. of App. Math., (2009), 39-46.

[16] M. S. Marouf, A subclass of multivalent uniformly convex functions associated with Deziok-Srivastava linear operator, Int. J. Math. Analysis, 3(2009), no. 22, 1087-1100.

[17] G. Murugusundaramoorthy and N. Magesh, On application of second order differential inequalities based on linear and integral operators. Int. J. Math. Sci. Engg. Appl., 2(2008), no. 1, 105-114.

[18] S.Owa, On certain classes of p-valent functions with negative coefficients, Simon Stevin, 59(1985), 385-402.

[19] B. A. Patel and N. K. Thakare, On convex and extreme point of p-valent starlike and convex classes wuth application, Bull. Math. Soc. Sci. Math. R. S. Roumanie (N. S.) 27(1983), no. 75, 145-160.

[20] T. O. Salim, M. S. Marouf and J. M. Shenan, A subclass of multivalent uniformly convex functions associated with generalized Sălăgean and Rusheweyh differential operators, Acta Univ. Apulensis, 26(2011), 289-300.

[21] A. Schild and H. Silverman, Convolution of univalent functions with negative coefficients, Ann. Univ. Mariae Curie-Sklodwska Sect. A, 29(1975), 99-106.