Certain Subclass of $p-$Valent Starlike and Convex Uniformly Functions Defined by Convolution

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Abstract

The aim of this paper is to obtain coefficient estimates, distortion theorems, convex linear combinations and radii of close-to-convexity, starlikeness and convexity for functions belonging to the subclass $TS_{p,\lambda}(f, g; \alpha, \beta)$ of $p$-valent $\beta-$uniformly starlike and convex functions. We consider integral operators and modified Hadamard products of functions belonging to this subclass.

Keywords: Analytic, $p$-valent, uniformly starlike, uniformly convex, modified Hadamard product.

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1 Introduction

The class of analytic and $p$-valent in $\mathbb{U}$ and has the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}, \mathbb{N} = \{1, 2, \ldots\})$$

(1)
Certain Subclass of Starlike and Convex Uniformly Functions

is denoted by $S(p)$. We note that $S(1) = S$. Let $f(z) \in S(p)$ be given by (1) and $g(z) \in S(p)$ be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k,$$

(2)

then the Hadamard product $(f \ast g)(z)$ of $f(z)$ and $g(z)$ is defined by

$$(f \ast g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g \ast f)(z).$$

(3)

Denote by $T(p)$ the class of functions of the form:

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k (a_k \geq 0).$$

(4)

Salim et al. [20] and Marouf [16] introduced and studied the following subclasses of $p$-valent functions:

(i) A function $f(z) \in S(p)$ is said to be $p$-valent $\beta$-uniformly starlike of order $\alpha$ if it satisfies the condition:

$$\Re \left\{ \frac{z f'(z)}{f(z)} - \alpha \right\} \geq \beta \left| \frac{z f'(z)}{f(z)} - p \right| (z \in U),$$

for some $\alpha (-p \leq \alpha < p)$, $\beta \geq 0$ and for all $z \in U$. We denote to the class of these functions by $\beta - S_p(\alpha)$.

(ii) A function $f(z) \in S(p)$ is said to be $p$-valent $\beta$-uniformly convex of order $\alpha$ if it satisfies the condition:

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} - \alpha \right\} \geq \beta \left| 1 + \frac{zf''(z)}{f'(z)} - p \right| (z \in U),$$

for some $\alpha (-p \leq \alpha < p)$, $\beta \geq 0$ and for all $z \in U$. We denote to the class of these functions by $\beta - K_p(\alpha)$.

It follows from (5) and (6) that

$$f(z) \in \beta - K_p(\alpha) \iff \frac{zf'(z)}{p} \in \beta - S_p(\alpha).$$

(7)

For different choices of parameters $\alpha, \beta, p$ we obtain many subclasses studied earlier see (for example) ([1], [11], [12], [13], [14], [18] and [19])

**Definition 1.** For $0 \leq \alpha < p$, $p \in \mathbb{N}$, $0 \leq \lambda \leq 1$ and $\beta \geq 0$, let $S_{p, \lambda}(f, g; \alpha, \beta)$ be the subclass of $S(p)$ consisting of functions $f(z)$ of the form (1) and functions $g(z)$ of the form (2) and satisfying the analytic criterion:

$$\Re \left\{ \frac{(1 - \lambda + \frac{\alpha}{p}) z (f \ast g)'(z) + \frac{\alpha}{p} z^2 (f \ast g)''(z)}{(1 - \lambda) (f \ast g)(z) + \frac{\alpha}{p} z (f \ast g)'(z)} - \alpha \right\}$$

(8)
\[
\sum_{k=1}^{\infty} p^k \in C(x) \quad \text{(iv)}
\]

\[
\Omega_k = \frac{(\alpha_1)_{k-p} \ldots (\alpha_l)_{k-p}}{(\beta_1)_{k-p} \ldots (\beta_m)_{k-p}} \frac{1}{(k-p)!}
\]

and

\[
(a)_k = \frac{\Gamma(a + k)}{\Gamma(a)} = \begin{cases} 1 & k = 0 \\ a(a + 1)(a + 2) \ldots (a + k - 1) & k \in \mathbb{N} \end{cases}
\]

\[
0 \leq \alpha < p, \beta \geq 0, \alpha_i \in \mathbb{C} (i = 1,2,\ldots,l) \text{ and } \beta_j \in \mathbb{C} \setminus \{-1,-2,\ldots\}, \quad j = 1,2,\ldots,m, z \in \mathbb{U};
\]

\[
(i) TS_{1,\lambda}(f, g; \alpha, \beta) = TS_{\gamma}(f, g; \alpha, \beta) \quad \text{(see Aouf et al. [3])};
\]

\[
(ii) TS_{p,0}(f, z^p + \sum_{k=p+1}^{\infty} \Omega_k z^k; \alpha, \beta) = T^{l,m}_p(\alpha_l, \beta_m, \beta, \alpha) \quad \text{(see Marouf [16])}, \text{ where}
\]

\[
\phi_k(n, \gamma, \delta, p) = \left[ (1 - \gamma) \left( 1 + \left( \frac{k}{p} - 1 \right) \delta \right) + \gamma C^m_{n+k-p} \right],
\]

\[
(n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma \geq 0, \delta \geq 0, -p \leq \alpha < p, \beta \geq 0)
\]

and

\[
C^m_{n+k-p} = \frac{(n+k-p)!}{n!(k-p)!}.
\]

\[
(iv) TS_{p,0}(f, z^p + \sum_{k=1}^{\infty} C(k, n)(1 - \gamma k)^n z^{k+p}) = S_p^*(\alpha, \beta, \gamma) \quad \text{and } TS_{p,1}(f, z^p + \sum_{k=1}^{\infty} C(k, n)(1 - \gamma k)^n z^{k+p}) = C_p(\alpha, \beta, \gamma) \quad \text{(see Darus and Ibrahim [7, by replacing k by k - p])}, \text{ where}
\]

\[
C(k, n) = \frac{(n+p)_k}{(1)_k}
\]

\[
(p \in \mathbb{N}, n \in \mathbb{N}_0, 0 \leq \gamma < \frac{1}{k}, 0 \leq \alpha < p, \beta \geq 0, z \in \mathbb{U});
\]

\[
(vi) TS_{p,0}(f, z^p + \sum_{k=p+n}^{\infty} \Phi(k) z^k) = K(\mu, \gamma, \eta, \alpha, \beta) \quad \text{(see Khairnar and More [15, with n = 1])}, \text{ where}
\]

\[
\Phi(k) = \frac{(a)_{k-p}(1+p)_{k-p}(1+p+\eta-\gamma)_{k-p}}{(c)_{k-p}(1+p-\gamma)_{k-p}(1+p+\eta-\mu)_{k-p}}
\]
$(-\infty < \mu < 1, -\infty < \gamma < 1, \eta \in \mathbb{R}^+, p \in \mathbb{N}, -p \leq \alpha < p, \beta \geq 0, a \in \mathbb{R}, c \in \mathbb{R}\setminus \mathbb{Z}_0, z \in \mathbb{U})$.

Also we note that:

(i) $TS_{p,\lambda}(f, z^p + \sum_{k=p+1}^{\infty} \left(\frac{z}{p}\right)^k, \alpha, \beta) = TS_{p,\lambda}(n, \alpha, \beta)$

\[
\begin{align*}
&\sum_{k=0}^{\infty} \left(\frac{\lambda}{p}\right)^k \sum_{j=0}^{n} \left(\frac{\lambda}{p}\right)^j (\lambda^{n-j} - 1) z^j,
\end{align*}
\]

\[
\begin{align*}
&0 \leq \lambda \leq 1, \quad -p \leq \alpha < p, \quad \beta \geq 0, \quad n \in \mathbb{N}_0, \quad z \in \mathbb{U},
\end{align*}
\]

where $D_p^n$ is $p$–valent Sălăgean operator introduced by ([10] and [4]);

\[
\begin{align*}
\sum_{k=p+1}^{\infty} \Omega_k z^k; \alpha, \beta = TS_{p,\alpha,\beta}^n(n, \alpha, \beta)
\end{align*}
\]

\[
\begin{align*}
&\begin{cases}
 f \in T(p) : Re \left\{ (1 - \lambda + \frac{\lambda}{p}) z (D_p^n f(z))' + \frac{1}{p} z^2 (D_p^n f(z))'' \right\} - \alpha \\
> \beta \left| (1 - \lambda + \frac{\lambda}{p}) z (D_p^n f(z))' + \frac{1}{p} z^2 (D_p^n f(z))'' \right| - p (z \in \mathbb{U}),
\end{cases}
\end{align*}
\]

where $\Omega_k$ is given by (1.10) and $H_{p,q,s}(\alpha_1)$ is the $p$–valent Dziok-Srivastava operator introduced by ([8] and [9], see also [5], [2] and [6]).

Denote by $U$ the unit disc of $\ldots$

**Remark 1.1** If $f \in \mathcal{A}^\ast_{\mathcal{C}},$ $f(z, \zeta) = z + \sum_{j=2}^{\infty} \alpha_j (\zeta) z^j,$ then $R^m f(z, \zeta) = z + \sum_{j=2}^{\infty} C_{m+j-1}^m \alpha_j (\zeta) z^j,$ $z \in U, \zeta \in \mathcal{U}.$

### 2 Coefficient estimates

Throughout our present paper, we assume that: $g(z)$ is defined by (2) with $b_k > 0(k \geq p + 1), 0 \leq \alpha < p, p \in \mathbb{N}, 0 \leq \lambda \leq 1, \beta \geq 0$ and $z \in \mathbb{U}.$

**Theorem 1** A function $f(z) \in S_{p,\lambda}(f, g; \alpha, \beta)$ if

\[
\begin{align*}
\sum_{k=p+1}^{\infty} \left( \frac{p + \lambda(k - p)}{p} \right) \left[ k(1 + \beta) - (\alpha + p\beta) \right] |a_k| b_k \leq p - \alpha.
\end{align*}
\]

**Proof.** It suffices to show that

\[
\begin{align*}
\beta \left| (1 - \lambda - \frac{\lambda}{p}) z (f \ast g)'(z) + \frac{1}{p} z^2 (f \ast g)''(z) \right| - p
\end{align*}
\]

\[
\begin{align*}
\left| (1 - \lambda)(f \ast g)(z) + \frac{1}{p} z (f \ast g)''(z) \right|
\end{align*}
\]
\[- \text{Re} \left\{ \frac{(1 - \lambda + \frac{1}{p}) z(f * g)'(z) + \frac{1}{p} z^2 (f * g)''(z)}{(1 - \lambda) (f * g)(z) + \frac{1}{p} z(f * g)'(z)} - p \right\} \leq p - \alpha. \] (22)

We have
\[
\beta \left( \frac{(1 - \lambda + \frac{1}{p}) z(f * g)'(z) + \frac{1}{p} z^2 (f * g)''(z)}{(1 - \lambda) (f * g)(z) + \frac{1}{p} z(f * g)'(z)} - p \right) \leq (1 + \beta) \left( \frac{(1 - \lambda + \frac{1}{p}) z(f * g)'(z) + \frac{1}{p} z^2 (f * g)''(z)}{(1 - \lambda) (f * g)(z) + \frac{1}{p} z(f * g)'(z)} - p \right) \]
\[
\leq (1 + \beta) \sum_{k=p+1}^{\infty} \frac{(p + \lambda(k - p))}{p} (k - p) |a_k| b_k \]
\[
1 - \sum_{k=p+1}^{\infty} \frac{(p + \lambda(k - p))}{p} |a_k| b_k . \] (26)

The last expression is bounded above by \((p - \alpha)\) since (20) holds. Hence the proof of Theorem 1 is completed.

**Theorem 2.** A function \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \) if and only if
\[
\sum_{k=p+1}^{\infty} \left( \frac{p + \lambda(k - p)}{p} \right) [k(1 + \beta) - (\alpha + p\beta)] |a_k| b_k \leq p - \alpha . \] (27)

**Proof.** In view of Theorem 1, we need only to prove the necessity. If \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \) and \( z \) is real, then
\[
p - \sum_{k=p+1}^{\infty} \left( \frac{p + \lambda(k - p)}{p} \right) a_k b_k z^{k-p} \geq \beta \left( 1 - \sum_{k=p+1}^{\infty} \left( \frac{p + \lambda(k - p)}{p} \right) a_k b_k z^{k-p} \right) \]
\[
1 - \sum_{k=p+1}^{\infty} \left( \frac{p + \lambda(k - p)}{p} \right) a_k b_k z^{k-p} \geq \beta . \] (28)

Letting \( z \to 1^- \) along the real axis, we obtain the desired inequality
\[
\sum_{k=1+p}^{\infty} \left( \frac{p + \lambda(k - p)}{p} \right) [k(1 + \beta) - (\alpha + p\beta)] |a_k| b_k \leq p - \alpha \] (29)

and hence the proof of Theorem 2 is completed.
Certain Subclass of Starlike and Convex Uniformly Functions

Corollary 1  If a function $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$. Then

$$ a_k \leq \frac{p - \alpha}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1 + \beta) - (\alpha + p\beta)]b_k} (k \geq p + 1). \quad (30) $$

The result is sharp for the function

$$ f(z) = z^p - \frac{p - \alpha}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1 + \beta) - (\alpha + p\beta)]b_k} z^k \quad (k \geq p + 1). \quad (31) $$

3  Distortion theorems

Theorem 3  If $f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta)$, then for $|z| = r < 1$

$$ |f(z)| \geq r^p - \frac{p - \alpha}{\left(\frac{p+\lambda}{p}\right)(p + 1 + \beta - \alpha)b_{p+1}} r^{p+1} \quad (32) $$

and

$$ |f(z)| \leq r^p + \frac{p - \alpha}{\left(\frac{p+\lambda}{p}\right)(p + 1 + \beta - \alpha)b_{p+1}} r^{p+1}, \quad (33) $$

provided that $b_k \geq b_{p+1} \,(k \geq p+1)$. The equalities in (32) and (33) are attained for the function $f(z)$ given by

$$ f(z) = z^p - \frac{p - \alpha}{\left(\frac{p+\lambda}{p}\right)(p + 1 + \beta - \alpha)b_{p+1}} z^{p+1}. \quad (34) $$

Proof. Using Theorem 2, we have

$$ (p + 1 + \beta - \alpha) \left(\frac{p + \lambda}{p}\right)b_{p+1} \sum_{k=p+1}^{\infty} a_k \quad (35) $$

$$ \leq \sum_{k=p+1}^{\infty} \left(\frac{p + \lambda(k-p)}{p}\right)[k(1 + \beta) - (\alpha + p\beta)] b_k a_k \leq p - \alpha \quad (36) $$

that is, that

$$ \sum_{k=p+1}^{\infty} a_k \leq \frac{p - \alpha}{\left(\frac{p+\lambda}{p}\right)(p + 1 + \beta - \alpha)b_{p+1}}. \quad (37) $$

From (4) and (37), we have

$$ |f(z)| \geq r^p - r^{1+p} \sum_{k=p+1}^{\infty} a_k \quad (38) $$
\[ \geq r^p - \frac{p - \alpha}{\binom{p+\lambda}{p}} \frac{1}{(p + 1 + \beta - \alpha)b_{p+1}} r^{p+1} \]  
and
\[ |f(z)| \leq r^p + r^{1+p} \sum_{k=p+1}^{\infty} a_k \]
\[ \leq r^p + \frac{p - \alpha}{\binom{p+\lambda}{p}} \frac{1}{(p + 1 + \beta - \alpha)b_{p+1}} r^{p+1} . \]

This completes the proof of Theorem 3.

**Theorem 4** Let \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \), then for \( |z| = r < 1 \)
\[ \left| f'(z) \right| \geq p \left( r^{p-1} - \frac{(p + 1)(p - \alpha)}{\binom{p+\lambda}{p}} \frac{1}{(p + 1 + \beta - \alpha)b_{p+1}} \right) r^p \]  
and
\[ \left| f'(z) \right| \leq p \left( r^{p-1} + \frac{(p + 1)(p - \alpha)}{\binom{p+\lambda}{p}} \frac{1}{(p + 1 + \beta - \alpha)b_{p+1}} \right) r^p , \]

provided that \( b_k \geq b_{p+1} \) \( (k \geq p+1) \). The result is sharp for the function \( f(z) \) given by (34).

**Proof.** From Theorem 2, we have
\[ \sum_{k=p+1}^{\infty} k a_k \leq \frac{(p + 1)(p - \alpha)}{\binom{p+\lambda}{p}} \frac{1}{(p + 1 + \beta - \alpha)b_{p+1}} . \]

The remaining part of the proof is similar to the proof of Theorem 3, then we omit the details.

Now, differentiating both sides of (4) \( m \)-times, we have
\[ f^{(m)}(z) = \frac{p!}{(p-m)!} z^{p-m} - \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k k^{k-m} (k \geq p+1; p \in \mathbb{N}; m < p, m \in \mathbb{N}_0). \]

**Theorem 5** If a function \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \). Then for \( |z| = r < 1 \), we have
\[ \left| f^{(m)}(z) \right| \geq \left( \frac{p!}{(p-m)!} - \frac{(p + 1)! (p - \alpha)}{\binom{p+\lambda}{p}} \frac{1}{(p + 1 + \beta - \alpha)b_{p+1}} \right) r^{p-m} \]
and

$$|f^{(m)}(z)| \leq \left( \frac{p!}{(p-m)!} + \frac{(p+1)!(p-\alpha)}{(p+\lambda)\frac{p}{p} (p+1-m)!(p+1+\beta-\alpha)b_{p+1}} \right) r^{p-m}.$$  \hspace{1cm} (47)

The result is sharp for the function $f(z)$ given by (34).

**Proof.** Using (27), we have

$$\sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k \leq \frac{(p+1)!(p-\alpha)}{(p+\lambda)\frac{p}{p} (p+1-m)!(p+1+\beta-\alpha)b_{p+1}}.$$  \hspace{1cm} (48)

From (45) and (48), we have

$$|f^{(m)}(z)| \geq \frac{p!}{(p-m)!} r^{p-m} - r^{p+1-m} \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k \hspace{1cm} (49)$$

\[ \geq \left( \frac{p!}{(p-m)!} - \frac{(p+1)!(p-\alpha)}{(p+\lambda)\frac{p}{p} (p+1-m)!(p+1+\beta-\alpha)b_{p+1}} \right) r^{p-m} \hspace{1cm} (50) \]

and

$$|f^{(m)}(z)| \leq \frac{p!}{(p-m)!} r^{p-m} + r^{p+1-m} \sum_{k=p+1}^{\infty} \frac{k!}{(k-m)!} a_k \hspace{1cm} (51)$$

\[ \leq \left( \frac{p!}{(p-m)!} + \frac{(p+1)!(p-\alpha)}{(p+\lambda)\frac{p}{p} (p+1-m)!(p+1+\beta-\alpha)b_{p+1}} \right) r^{p-m} \hspace{1cm} (52) \]

This completes the proof of Theorem 5.

**Remark 1** Putting $m = 0$ and $m = 1$, in Theorem 5 we obtain Theorem 3 and Theorem 4, respectively.

### 4 Convex linear combinations

**Theorem 6** Let $\eta_\mu \geq 0$ for $\mu = 1, 2, \ldots, n$ and $\sum_{\mu=1}^{n} \eta_\mu \leq 1$. If the functions $f_\mu(z)$ defined by

$$f_\mu(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,\mu} z^k \quad (a_{k,\mu} \geq 0; \mu = 1, 2, \ldots, n)$$  \hspace{1cm} (53)

For the next page of the document, please refer to the original source or extract the text from it.
are in the class \( TS_{p,\lambda}(f, g; \alpha, \beta) \) for every \( \mu = 1, 2, \ldots, n \) then the function \( f(z) \) defined by

\[
f(z) = z^p - \sum_{k=p+1}^{\infty} \left( \sum_{\mu=1}^{n} \eta_\mu a_{k,\mu} \right) z^k
\]

is in the class \( TS_{p,\lambda}(f, g; \alpha, \beta) \).

**Proof.** Since \( f_\mu(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \), it follows from Theorem 2 that

\[
\sum_{k=p+1}^{\infty} \left( \frac{p + \lambda (k - p)}{p} \right) [k(1 + \beta) - (\alpha + p \beta)] b_k a_{k,\mu} \leq p - \alpha ,
\]

for every \( \mu = 1, 2, \ldots, n \). Hence

\[
\sum_{k=p+1}^{\infty} \left( \frac{p + \lambda (k - p)}{p} \right) [k(1 + \beta) - (\alpha + p \beta)] \left( \sum_{\mu=1}^{n} \eta_\mu a_{k,\mu} \right) b_k
\]

\[
= \sum_{\mu=1}^{n} \eta_\mu \left( \sum_{k=p+1}^{\infty} \left( \frac{p + \lambda (k - p)}{p} \right) [k(1 + \beta) - (\alpha + p \beta)] a_{k,\mu} b_k \right)
\]

\[
\leq (p - \alpha) \sum_{\mu=1}^{n} \eta_\mu \leq p - \alpha.
\]

By Theorem 2, it follows that \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \) and hence the proof of Theorem 6 is completed.

**Corollary 2** The class \( TS_{p,\lambda}(f, g; \alpha, \beta) \) is closed under convex linear combinations.

**Theorem 7** Let \( f_p(z) = z^p \) and

\[
f_k(z) = z^p - \frac{p - \alpha}{\left( \frac{p + \lambda (k - p)}{p} \right) [k(1 + \beta) - (\alpha + p \beta)] b_k} z^k \quad (k \geq p + 1)
\]

Then \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \) if and only if

\[
f(z) = \sum_{k=p+1}^{\infty} \gamma_k f_k(z),
\]

where \( \gamma_k \geq 0 \) and \( \sum_{k=p}^{\infty} \gamma_k = 1 \).
Proof. Assume that
\[ f(z) = \sum_{k=p}^{\infty} \gamma_k f_k(z) \]  
\[ = z^p - \sum_{k=p+1}^{\infty} \left( \frac{p+\lambda(k-p)}{p} \right) \frac{(p-\alpha)}{k(1+\beta)-(\alpha+p\beta)} b_k \gamma_k \]  
\[ = z^p - \sum_{k=p+1}^{\infty} \left( \frac{p+\lambda(k-p)}{p} \right) \frac{(p-\alpha)}{k(1+\beta)-(\alpha+p\beta)} b_k \gamma_k \]  
Then it follows that
\[ \sum_{k=p+1}^{\infty} \left( \frac{p+\lambda(k-p)}{p} \right) \frac{(p-\alpha)}{k(1+\beta)-(\alpha+p\beta)} b_k \gamma_k \]  
\[ = \sum_{k=p+1}^{\infty} \gamma_k = 1 - \gamma_p \leq 1. \]  
So, by Theorem 2, \( f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta) \).
Conversely, assume that the function \( f(z) \) belongs to the class \( TS_{p,\lambda}(f,g;\alpha,\beta) \). Then
\[ a_k \leq \left( \frac{p+\lambda(k-p)}{p} \right) \frac{(p-\alpha)}{k(1+\beta)-(\alpha+p\beta)} b_k \]  
\[ \sum_{k=p+1}^{\infty} \gamma_k = 1 - \gamma_p \leq 1. \]  
Corollary 3 The extreme points of the class \( TS_{p,\lambda}(f,g;\alpha,\beta) \) are the functions \( f_p(z) = z^p \) and
\[ f_k(z) = z^p - \left( \frac{p+\lambda(k-p)}{p} \right) \frac{(p-\alpha)}{k(1+\beta)-(\alpha+p\beta)} b_k \gamma_k z^k \]  
\[ (k \geq p+1). \]
5 Radii of close-to-convexity, starlikeness and convexity

Theorem 8 Let the function $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$. Then

Corollary 4 (i) $f(z)$ is $p$-valent close-to-convex of order $\delta$ $(0 \leq \delta < p)$ in $|z| < r_1$, where

$$r_1 = \inf_{k \geq p+1} \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right) (p-\delta) [k(1+\beta) - (\alpha + p\beta)] b_k}{k(p-\alpha)} \right\}^{\frac{1}{k-p}}. \quad (69)$$

(ii) $f(z)$ is $p$-valent starlike of order $\delta$ $(0 \leq \delta < p)$ in $|z| < r_2$, where

$$r_2 = \inf_{k \geq p+1} \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right) (p-\delta) [k(1+\beta) - (\alpha + p\beta)] b_k}{(k-\delta) (p-\alpha)} \right\}^{\frac{1}{k-p}}. \quad (70)$$

The result is sharp, the extremal function being given by (31).

Proof. (i) We must show that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta \quad \text{for} \quad |z| < r_1, \quad (71)$$

where $r_1$ is given by (69). Indeed we find from (4) that

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} k a_k |z|^{k-p}. \quad (72)$$

Thus

$$\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq p - \delta, \quad (73)$$

if

$$\sum_{k=p+1}^{\infty} \left( \frac{k}{p-\delta} \right) a_k |z|^{k-p} \leq 1. \quad (74)$$

But, by Theorem 2, (74) will be true if

$$\left( \frac{k}{p-\delta} \right) |z|^{k-p} \leq \frac{\left(\frac{p+\lambda(k-p)}{p}\right) [k(1+\beta) - (\alpha + p\beta)] b_k}{(p-\alpha)}, \quad (75)$$
that is, if
\[
|z| \leq \left\{ \left( \frac{p+\lambda(k-p)}{p} \right) \left( p - \delta \right) \frac{k(1 + \beta) - (\alpha + p\beta)}{k(p - \alpha)} \right\}^{\frac{1}{k-p}} (k \geq p + 1). \tag{76}
\]
the proof of (i) is completed.

(ii) It is sufficient to show that
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \text{ for } |z| < r_2, \tag{77}
\]
where \( r_2 \) is given by (70). Indeed we find, again from (4) that
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq \sum_{k=p+1}^{\infty} (k-p) a_k |z|^{k-p} \tag{78}
\]
Thus
\[
\left| \frac{zf'(z)}{f(z)} - p \right| \leq p - \delta \tag{79}
\]
if
\[
\sum_{k=p+1}^{\infty} \frac{(k-\delta)}{(p-\delta)} a_k \leq 1. \tag{80}
\]
But, by Theorem 2, (80) will be true if
\[
\frac{(k-\delta)}{(p-\delta)} |z|^{k-p} \leq \left( \frac{p+\lambda(k-p)}{p} \right) \frac{k(1 + \beta) - (\alpha + p\beta)}{k(p - \alpha)} \tag{81}
\]
that is, if
\[
|z| \leq \left\{ \left( \frac{p+\lambda(k-p)}{p} \right) \left( p - \delta \right) \frac{k(1 + \beta) - (\alpha + p\beta)}{k(p - \alpha)} \right\}^{\frac{1}{k-p}} (k \geq p + 1). \tag{82}
\]
the proof of (ii) is completed.

**Corollary 5** Let \( f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta) \). Then \( f(z) \) is \( p \)-valent convex of order \( \delta \) \((0 \leq \delta < p)\) in \( |z| < r_3 \), where
\[
r_3 = \inf_{k\geq p+1} \left\{ \left( \frac{p+\lambda(k-p)}{p} \right) \left( p - \delta \right) \frac{k(1 + \beta) - (\alpha + p\beta)}{k(k - \delta)(p - \alpha)} \right\}^{\frac{1}{k-p}}. \tag{83}
\]
The result is sharp, with the extremal function \( f(z) \) given by (31).
6 Classes of preserving integral operators

In this section, we discuss some classes preserving integral operators. Consider the following operators

(i) $F(z)$ defined by

\[
F(z) = (J_{c,p}f)(z) = \frac{c + p}{z^c} \int_0^z t^{c-1} f(t)dt
\]

\[
= z^p - \sum_{k=p+1}^{\infty} d_k z^k (f \in S(p); c > -p),
\]

where

\[
d_k = \left(\frac{c + p}{c + k}\right) a_k \leq a_k (k \geq p + 1).
\]

(ii) $G(z)$ defined by

\[
G(z) = z^{p-1} \int_0^z \frac{f(t)}{t^p} dt
\]

\[
= z^p - \sum_{k=p+1}^{\infty} \frac{1}{k-p + 1} a_k z^k (k \geq p + 1).
\]

(iii) The Komatu operator [16] defined by

\[
H(z) = P_{c,p}^d f(z) = \frac{(c + p)^d}{\Gamma(d)z^c} \int_0^z t^{c-1} \left(\log \frac{z}{tp}\right)^{d-1} f(t)dt
\]

\[
= z^p - \sum_{k=p+1}^{\infty} \left(\frac{c + p}{c + k}\right)^d a_k z^k
\]

\[
(d > 0; c > -p; p \in \mathbb{N}; z \in \mathbb{U}),
\]

and

(iv) $I(z)$, which generalizes Jung-Kim-Srivastava integral operator (see [13]) defined by

\[
I(z) = Q_{c,p}^d f(z) = \frac{\Gamma(d + c + p)}{\Gamma(c + p)\Gamma(d)} \frac{1}{z^c} \int_0^z t^{c-1} \left(1 - \frac{t}{z}\right)^{d-1} f(t)dt
\]

\[
= z^p + \sum_{k=p+1}^{\infty} \frac{\Gamma(c + k)\Gamma(d + c + p)}{\Gamma(c + p)\Gamma(d + c + k)} a_k z^k
\]
(90)

In view of Theorem 2, we see that \( z^p - \sum_{k=p+1}^{\infty} a_k z^k \) is in the class \( TS_{p,\lambda}(f, g; \alpha, \beta) \) as long as \( 0 \leq d_k \leq a_k \) for all \( k \). In particular, we have

**Theorem 9** Let \( f(z) \in TS_p(f, g; \alpha, \beta, \lambda) \) and \( c \) be a real number such that \( c > -p \). Then the function \( F(z) \) defined by (84) also belongs to the class \( TS_{p,\lambda}(f, g; \alpha, \beta) \).

**Proof.** From (27) and (86), we have the function \( F(z) \) is in the class \( TS_{p,\lambda}(f, g; \alpha, \beta) \).

**Theorem 10** Let \( F(z) = z^p - \sum_{k=p+1}^{\infty} a_k z^k \) \((a_k \geq 0)\) be in the class \( TS_{p,\lambda}(f, g; \alpha, \beta) \), and let \( c \) be a real number such that \( c > -p \). Then the function \( f(z) \) given by (84) is \( p \)-valent in \( |z| < R_1 \), where

\[
R_1 = \inf_{k \geq p+1} \left\{ \frac{(c+p)(p+\lambda(k-p))}{k(c+k)(p-\alpha)} \right. \left[ k(1+\beta) - (\alpha + p\beta) \right] \frac{1}{b_k} \right\}^{\frac{1}{1-p}}. \tag{91}
\]

The result is sharp.

**Proof.** From (84), we have

\[
f(z) = \frac{z^{1-c} [z^c F(z)]'}{(c+p)} \quad (c > -p) \tag{92}
\]

\[
= z^p - \sum_{k=p+1}^{\infty} \left( \frac{c+k}{c+p} \right) a_k z^k. \tag{93}
\]

In order to obtain the required result, it suffices to show that

\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| < p \text{ whenever } |z| < R_1, \tag{94}
\]

where \( R_1 \) is given by (91). Now

\[
\left| \frac{f'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} \frac{k(c+k)}{(c+p)} a_k |z|^{k-p}. \tag{95}
\]

Thus \( \left| \frac{f'(z)}{z^{p-1}} - p \right| < p \) if

\[
\sum_{k=p+1}^{\infty} \frac{k(c+k)}{p(c+p)} a_k |z|^{k-p} < 1. \tag{96}
\]
But Theorem 2 confirms that
\[
\sum_{k=p+1}^{\infty} \frac{\left( \frac{p+\lambda(k-p)}{p} \right) [k(1 + \beta) - (\alpha + p\beta)] \cdot a_k \cdot b_k}{p - \alpha} \leq 1. \tag{97}
\]

Hence (96) will be satisfied if
\[
\frac{k(c+k)}{(c+p)} |z|^k < \frac{[p + \lambda(k-p)] [k(1 + \beta) - (\alpha + p\beta)] b_k}{p - \alpha}, \tag{98}
\]
that is, if
\[
|z| < \left\{ \frac{(c+p) [p + \lambda(k-p)] [k(1 + \beta) - (\alpha + p\beta)] b_k}{k(c+k) (p - \alpha)} \right\}^{\frac{1}{p - \alpha}} (k \geq p + 1). \tag{99}
\]

Therefore, the function \( f(z) \) given by (84) is \( p \)-valent in \( |z| < R_1 \). Sharpness of the result follows if we take
\[
f(z) = z^p - \frac{(p-\alpha)(c+k)}{\left( \frac{p+\lambda(k-p)}{p} \right) [k(1 + \beta) - (\alpha + p\beta)] b_k (c+p)} z^k (k \geq p + 1) \tag{100}
\]
and hence the proof of Theorem 10 is completed.

**Theorem 11** If \( f(z) \in TS_{p,\lambda}(f,g;\alpha,\beta) \), then \( H(z) \in TS_{p,\lambda}(f,g;\alpha,\beta) \).

**Proof.** The function \( H(z) \) defined by (6.5) is in the class \( TS_{p,\lambda}(f,g;\alpha,\beta) \) if
\[
\sum_{k=p+1}^{\infty} \frac{\left( \frac{p+\lambda(k-p)}{p} \right) [k(1 + \beta) - (\alpha + p\beta)] \cdot b_k \cdot \left( \frac{c+p}{c+k} \right)^d a_k}{p - \alpha} \leq 1. \tag{101}
\]
Since \( \frac{c+p}{c+k} \leq 1 \) for \( k \geq p + 1 \), then
\[
\sum_{k=p+1}^{\infty} \frac{\left( \frac{p+\lambda(k-p)}{p} \right) [k(1 + \beta) - (\alpha + p\beta)] \cdot b_k \cdot \left( \frac{c+p}{c+k} \right)^d a_k}{p - \alpha} \leq \sum_{k=p+1}^{\infty} \frac{\left( \frac{p+\lambda(k-p)}{p} \right) [k(1 + \beta) - (\alpha + p\beta)] \cdot b_k}{p - \alpha} a_k \leq 1. \tag{102}
\]
Therefore \( H(z) \in TS_{p,\lambda}(f,g;\alpha,\beta) \).
Theorem 12 Let \( d > 0, c > -p \) and \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \). Then function \( H(z) \) defined by (89) is \( p \)-valent in \( |z| < R_2 \), where

\[
R_2 = \inf_{k \geq p+1} \left\{ \frac{(c+k)^d (p+\lambda(k-p)) [k(1+\beta) - (\alpha+p\beta)] b_k}{k(c+p)^d (p-\alpha)} \right\}^{\frac{1}{k-p}}.
\] (104)

The result is sharp for function \( f(z) \) given by

\[
f(z) = z^p - \frac{(p-\alpha)(c+p)^d}{\left(\frac{p+\lambda(k-p)}{p}\right)} [k(1+\beta) - (\alpha+p\beta)] b_k (c+k)^d (k \geq p+1).
\] (105)

Proof. In order to prove the assertion, it is enough to show that

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| < p \text{ in } |z| < R_2
\] (106)

Now

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} k \left( \frac{c+p}{c+k} \right)^d a_k z^{k-p} \leq \sum_{k=p+1}^{\infty} k \left( \frac{c+p}{c+k} \right)^d a_k |z|^{k-p}.
\]

The last inequality is bounded above by \( p \) if

\[
\left| \frac{H'(z)}{z^{p-1}} - p \right| \leq \sum_{k=p+1}^{\infty} \frac{k}{p} \left( \frac{c+p}{c+k} \right)^d a_k |z|^{k-p} \leq 1.
\] (107)

Since \( f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \), then in view of Theorem 2, we have

\[
\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right)}{p-\alpha} [k(1+\beta) - (\alpha+p\beta)] b_k \leq 1.
\] (108)

Thus, (107) holds if

\[
k \left( \frac{c+p}{c+k} \right)^d |z|^{k-p} \leq \left( \frac{p+\lambda(k-p)}{p-\alpha} \right) [k(1+\beta) - (\alpha+p\beta)] b_k,
\] (109)

or

\[
|z| \leq \left\{ \frac{(c+k)^d [p+\lambda(k-p)][k(1+\beta) - (\alpha+p\beta)] b_k}{k(c+p)^d (p-\alpha)} \right\}^{1/p} (k \geq p+1).
\] (110)
Therefore, the function $H(z)$ given by (89) is $p$-valent in $|z| < R_2$. This completes the prove of Theorem 12.

Following similar steps as in the proofs of Theorems 11 and 12, we can prove the following theorems concerning the integral operator $I(z)$ defined by (90).

**Theorem 13** If $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$, then $I(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$, where $I(z)$ is given by (90).

**Theorem 14** Let $d > 0$, $c > -p$ and $f(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)$. Then function $I(z)$ defined by (90) is $p$-valent in $|z| < R_3$, where

$$R_3 = \inf_{k \geq p+1} \left\{ \frac{\Gamma(c+p)\Gamma(d+c+k)(p+\lambda(k-p))[k(1+\beta)-(\alpha+p\beta)]b_k}{k\Gamma(c+k)\Gamma(d+c+p)(p-\alpha)} \right\}^{1/p}.$$  

The result is sharp for function $f(z)$ given by

$$f(z) = z^p - \sum_{k=p+1}^{\infty} a_k,1 a_k,2 z^k (k \geq p+1).$$  

(112)

7 Modified Hadamard products

Let the functions $f_{\mu}(z)$ be defined for $\mu = 1, 2, ... m$ by (53). The modified Hadamard products of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1 \ast f_2)(z) = z^p - \sum_{k=p+1}^{\infty} a_{k,1} a_{k,2} z^k.$$  

(113)

**Theorem 15** Let $f_{\mu}(z) \in TS_{p,\lambda}(f, g; \alpha, \beta)(\mu = 1, 2)$, where $f_{\mu}(z)(\mu = 1, 2)$ are in the form (53). Then $(f_1 \ast f_2)(z) \in TS_{p,\lambda}(f, g; \delta, \beta)$, where

$$\delta = p - \frac{(p-\alpha)^2(1+\beta)}{\left(\frac{p+\lambda}{p}\right)(p+1-\alpha+\beta)^2 b_{p+1} - (p-\alpha)^2}.$$  

(114)

The result is sharp for

$$f_{\mu}(z) = z^p - \frac{(p-\alpha)}{\left(\frac{p+\lambda}{p}\right)(p+1-\alpha+\beta) b_{p+1}} z^{p+1} (\mu = 1, 2).$$  

(115)
**Proof.** Employing the technique used earlier by Schild and Silverman [21], we need to find the largest real parameter $\delta$ such that

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right) \left[k(1 + \beta) - (\delta + p\beta)\right] b_k}{(p - \delta)} a_{k,1} a_{k,2} \leq 1. \quad (116)$$

Since $f_\mu \in TS_{p,\lambda}(f, g; \alpha, \beta)(\mu = 1, 2)$, we have

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right) \left[k(1 + \beta) - (\alpha + p\beta)\right] b_k}{(p - \alpha)} a_{k,1} \leq 1 \quad (117)$$

and

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right) \left[k(1 + \beta) - (\alpha + p\beta)\right] b_k}{(p - \alpha)} a_{k,2} \leq 1. \quad (118)$$

By the Cauchy-Schwarz inequality we have

$$\sum_{k=p+1}^{\infty} \frac{\left(\frac{p+\lambda(k-p)}{p}\right) \left[k(1 + \beta) - (\alpha + p\beta)\right] b_k}{(p - \alpha)} \sqrt{a_{k,1} a_{k,2}} \leq 1. \quad (119)$$

thus it sufficient to show that

$$\frac{\left(\frac{p+\lambda(k-p)}{p}\right) \left[k(1 + \beta) - (\delta + p\beta)\right] b_k}{(p - \delta)} a_{k,1} a_{k,2} \leq \frac{\left(\frac{p+\lambda(k-p)}{p}\right) \left[k(1 + \beta) - (\alpha + p\beta)\right] b_k}{(p - \alpha)} \sqrt{a_{k,1} a_{k,2}} \quad (120)$$

or, equivalently, that

$$\sqrt{a_{k,1} a_{k,2}} \leq \frac{(p - \delta)[k(1 + \beta) - (\alpha + p\beta)]}{(p - \alpha)[k(1 + \beta) - (\delta + p\beta)]}. \quad (122)$$

Hence, in the light of the inequality (119), it is sufficient to prove that

$$\frac{(p - \alpha)}{\left(\frac{p+\lambda(k-p)}{p}\right) \left[k(1 + \beta) - (\alpha + p\beta)\right] b_k} \leq \frac{(p - \delta)[k(1 + \beta) - (\alpha + p\beta)]}{(p - \alpha)[k(1 + \beta) - (\delta + p\beta)]}. \quad (123)$$

It follows from (123) that

$$\delta \leq p - \frac{(1 + \beta)(k - p)(p - \alpha)^2}{[k(1 + \beta) - (\alpha + p\beta)]^2 \left(\frac{p+\lambda(k-p)}{p}\right) b_k - (p - \alpha)^2}. \quad (124)$$
Let

\[ M(k) = p - \frac{(1 + \beta)(k - p)(p - \alpha)^2}{\left(\frac{p + \lambda(k-p)}{p}\right)\left[k(1 + \beta) - (\alpha + p\beta)\right]b_k - (p - \alpha)^2}, \]  

(125)

then \( M(k) \) is increasing function of \( k(k \geq p + 1) \). Therefore, we conclude that

\[ \delta \leq M(p + 1) = p - \frac{(1 + \beta)(p - \alpha)^2}{\left(\frac{p + \lambda}{p}\right)(p + 1 + \beta - \alpha) b_{p+1} - (p - \alpha)(p - \gamma)}, \]  

(126)

and hence the proof of Theorem 15 is completed.

**Theorem 16** Let \( f_1(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \) and \( f_2(z) \in TS_{p,\lambda}(f, g; \gamma, \beta) \), where \( f_\mu(z)(\mu = 1, 2) \) are in the form (53). Then \( (f_1 * f_2)(z) \in TS_{p,\lambda}(f, g; \xi, \beta) \), where

\[ \xi \leq p - \frac{(1 + \beta)(p - \alpha)(p - \gamma)}{\left(\frac{p + \lambda}{p}\right)(p + 1 + \beta - \alpha)(p + 1 + \beta - \gamma) b_{p+1} - (p - \alpha)(p - \gamma)}. \]  

(127)

The result is sharp for

\[ f_1(z) = z^p - \frac{(p - \alpha)}{\left(\frac{p + \lambda}{p}\right)(p + 1 + \beta - \alpha) b_{p+1}} z^{p+1} \]  

(128)

and

\[ f_2(z) = z^p - \frac{(p - \gamma)}{\left(\frac{p + \lambda}{p}\right)(p + 1 + \beta - \gamma) b_{p+1}} z^{p+1} \]  

(129)

**Proof.** We need to find the largest real parameter \( \xi \) such that

\[ \sum_{k=1}^{\infty} \frac{\left(\frac{p + \lambda(k-p)}{p}\right)[k(1 + \beta) - (\xi + p\beta)]b_k}{(p - \xi)} a_{k,1} a_{k,2} \leq 1. \]  

(130)

Since \( f_1(z) \in TS_{p,\lambda}(f, g; \alpha, \beta) \), we readily see that

\[ \sum_{k=1}^{\infty} \frac{\left(\frac{p + \lambda(k-p)}{p}\right)[k(1 + \beta) - (\alpha + p\beta)]b_k}{(p - \alpha)} a_{k,1} \leq 1 \]  

(131)

and \( f_2(z) \in TS_{p,\lambda}(f, g; \gamma, \beta) \), we have

\[ \sum_{k=2}^{\infty} \frac{\left(\frac{p + \lambda(k-p)}{p}\right)[k(1 + \beta) - (\gamma + p\beta)]b_k}{(p - \gamma)} a_{k,2} \leq 1. \]  

(132)
By the Cauchy-Schwarz inequality we have
\[
\sum_{k=2}^{\infty} \frac{(p+\lambda(k-p))}{p} \frac{[k(1+\beta) - (\alpha + p\beta)]^{1/2} [k(1+\beta) - (\gamma + p\beta)]^{1/2} b_k}{\sqrt{p-\alpha}\sqrt{p-\gamma}} \sqrt{a_{k,1}a_{k,2}} \leq 1
\] (133)
thus it is sufficient to show that
\[
\frac{(p+\lambda(k-p))}{p} \frac{[k(1+\beta) - (\xi + p\beta)] b_k}{(p-\xi)} a_{k,1}a_{k,2}
\] (134)
\[
\leq \frac{(p+\lambda(k-p))}{p} \frac{[k(1+\beta) - (\alpha + p\beta)]^{1/2} [k(1+\beta) - (\gamma + p\beta)]^{1/2} b_k}{\sqrt{p-\alpha}\sqrt{p-\gamma}} \sqrt{a_{k,1}a_{k,2}}
\] (135)
or, equivalently, that
\[
\sqrt{a_{k,1}a_{k,2}} \leq \frac{(p-\xi)[k(1+\beta) - (\alpha + p\beta)]^{1/2} [k(1+\beta) - (\gamma + p\beta)]^{1/2}}{\sqrt{p-\alpha}\sqrt{p-\gamma}[k(1+\beta) - (\xi + p\beta)]}.
\] (136)
Hence, in the light of the inequality (133), it is sufficient to prove that
\[
\frac{(p+\lambda(k-p))}{p} \frac{[k(1+\beta) - (\alpha + p\beta)]^{1/2} [k(1+\beta) - (\gamma + p\beta)]^{1/2} b_k}{\sqrt{p-\alpha}\sqrt{p-\gamma}}
\] (137)
\[
\leq \frac{(p-\xi)}{\sqrt{p-\alpha}\sqrt{p-\gamma}} \frac{[k(1+\beta) - (\alpha + p\beta)]^{1/2} [k(1+\beta) - (\gamma + p\beta)]^{1/2}}{[k(1+\beta) - (\xi + p\beta)]}.
\] (138)
It follows from (137) that
\[
\xi \leq p - \frac{(1+\beta)(k-p)(p-\alpha)(p-\gamma)}{\frac{(p+\lambda(k-p))}{p} [k(1+\beta) - (\alpha + p\beta)][k(1+\beta) - (\gamma + p\beta)] b_k - (p-\alpha)(p-\gamma)}.
\] (139)
Let
\[
A(k) = p - \frac{(1+\beta)(k-p)(p-\alpha)(p-\gamma)}{\frac{(p+\lambda(k-p))}{p} [k(1+\beta) - (\alpha + p\beta)][k(1+\beta) - (\gamma + p\beta)] b_k - (p-\alpha)(p-\gamma)},
\] (140)
then \(A(k)\) is increasing function of \(k(k \geq p + 1)\). Therefore, we conclude that
\[
\xi \leq A(p+1) = p - \frac{(1+\beta)(p-\alpha)(p-\gamma)}{\frac{(p+\lambda)}{p} (p+1+\beta-\alpha)(p+1+\beta-\gamma) b_{p+1} - (p-\alpha)(p-\gamma)}
\] (141)
and hence the proof of Theorem 16 is completed.
Theorem 17 Let \( f_\mu(z) \in TS_{p,\lambda}(f, g ; \alpha, \beta)(\mu = 1, 2, 3) \), where \( f_\mu(z)(\mu = 1, 2, 3) \) are in the form (53). Then \( (f_1 \ast f_2 \ast f_3)(z) \in TS_{p,\lambda}(f, g ; \tau, \beta) \), where

\[
\tau p - \frac{(1 + \beta)(p - \alpha)^3}{\left( \frac{p + \lambda}{p} \right)^2 (p + 1 + \beta - \alpha)^3 b_{p+1}^2} - (p - \alpha)^3.
\]

(142)

The result is best possible for functions \( f_\mu(z)(\mu = 1, 2, 3) \) given by (115).

Proof. From Theorem 15, we have \( (f_1 \ast f_2)(z) \in TS_{p,\lambda}(f, g ; \delta, \beta) \), where \( \delta \) is given by (114). Now, using Theorem 16, we get \( (f_1 \ast f_2 \ast f_3)(z) \in TS_{p,\lambda}(f, g ; \tau, \beta) \), where

\[
\tau \leq p - \frac{(1 + \beta)(p - \alpha)(p - \delta)(k - p)}{\left( \frac{1 + \lambda(k-p)}{p} \right)} [k(1 + \beta) - (\alpha + \beta)] [k(1 + \beta) - (\delta + \beta)] b_k - (p - \alpha)(p - \delta).
\]

(143)

Now defining the function \( B(k) \) by

\[
B(k) = p - \frac{(1 + \beta)(p - \alpha)(p - \delta)(k - p)}{\left( \frac{p + \lambda(k-p)}{p} \right)} [k(1 + \beta) - (\alpha + \beta)] [k(1 + \beta) - (\delta + \beta)] b_k - (p - \alpha)(p - \delta).
\]

(144)

We see that \( B(k) \) is increasing function of \( k(k \geq p+1) \). Therefore, we conclude that

\[
\tau \leq B(p+1) = p - \frac{(1 + \beta)(p - \alpha)(p - \delta)}{\left( \frac{p + \lambda}{p} \right)} (p + 1 + \beta - \alpha)(p + 1 + \beta - \delta) b_{p+1} - (p - \alpha)(p - \delta),
\]

(145)

substituting from (7.2), we have

\[
\tau = p - \frac{(1 + \beta)(p - \alpha)^3}{\left( \frac{p + \lambda}{p} \right)^2 (p + 1 + \beta - \alpha)^3 b_{p+1}^2} - (p - \alpha)^3.
\]

(146)

and hence the proof of Theorem 17 is completed.

Theorem 18 Let \( f_\mu(z) \in TS_{p,\lambda}(f, g ; \alpha, \beta)(\mu = 1, 2) \), where \( f_\mu(z)(\mu = 1, 2) \) are in the form (53). Then

\[
h(z) = z^p - \sum_{k=p+1}^{\infty} (a_{k,1}^2 + a_{k,2}^2) z^k
\]

(147)
Certain Subclass of Starlike and Convex Uniformly Functions

belongs to the class $TS_{p,\lambda}(f,g;\varphi,\beta)$, where

$$\varphi \leq p - \frac{(p+\lambda)(p-\alpha)^2}{(p+\lambda)(p+1+\beta-\alpha)^2b_{p+1}-2(p-\alpha)^2}. \quad (148)$$

The result is sharp for functions $f_{\mu}(z)(\mu = 1, 2)$ defined by (115).

**Proof.** By using Theorem 16, we obtain

$$\sum_{k=p+1}^{\infty} \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} \right\}^2 a_{k,1} \leq 1, \quad (149)$$

and

$$\sum_{k=2}^{\infty} \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} \right\}^2 a_{k,2} \leq 1. \quad (150)$$

It follows from (149) and (151) that

$$\sum_{k=2}^{\infty} \frac{1}{2} \left\{ \frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k}{(p-\alpha)} \right\}^2 (a_{k,1}^2 + a_{k,2}^2) \leq 1. \quad (153)$$

Therefore, we need to find the largest $\varphi$ such that

$$\frac{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\varphi+p\beta)]b_k}{(p-\varphi)} \leq 1 \quad (154)$$

that is

$$\varphi \leq p - \frac{2(1+\beta)(k-p)(p-\alpha)^2}{\left(\frac{p+\lambda(k-p)}{p}\right)[k(1+\beta)-(\alpha+p\beta)]b_k - 2(p-\alpha)^2}. \quad (156)$$
Let

\[ H(k) = p - \frac{2(1 + \beta)(k - p)(p - \alpha)^2}{(p + \lambda k - p)(k(1 + \beta) - (\alpha + p\beta))^2 b_k - 2(p - \alpha)^2}, \]  

(157)

then \( H(k) \) is an increasing function of \( k(k \geq p + 1) \). Therefore, we conclude that

\[ \varphi \leq H(p + 1) = p - \frac{2(1 + \beta)(p - \alpha)^2}{(p + \lambda)(p + 1 + \beta - \alpha)^2 b_{p+1} - 2(p - \alpha)^2}, \]

(158)

and hence the proof of Theorem 18 is completed.

**Remark 2**

(i) Putting \( g(z) = z^p + \sum_{k=1}^{\infty} C(k, n)(1 - \gamma k)^n z^k \), where \( C(k, n) \) given by (16) by replacing \( k \) by \( k - p \), \( \lambda = 0 \) and \( \lambda = 1 \), respectively, in Theorems 16, 17 and ??, respectively, we obtain modified Hadamard products for the classes \( S_p(\alpha, \beta, \lambda) \) and \( C_p(\alpha, \beta, \lambda) \), respectively, defined in the introduction.

(ii) Putting \( \lambda = 0 \) and \( b_k = \Omega_k \), where \( \Omega_k \) is given by (11) and \( 0 \leq \alpha < p, \beta \geq 0, \alpha_i \in \mathbb{C}(i = 1, 2, \ldots, l), \beta_j \in \mathbb{C}\{1, -2, \ldots\}(j = 1, 2, \ldots, m) \) in Theorems 16, 17 and ??, respectively we modified the results obtained by Marouf [16, Theorems 4, 3 and 5, respectively].

(iii) Putting \( \lambda = 0 \) and \( b_k = \phi_k(n, \lambda, \delta, p) \), where \( \phi_k \) is given by (13) and \( n \in \mathbb{N}_0, \lambda \geq 0, \delta \geq 0, -p \leq \alpha < p, \beta \geq 0 \) in Theorems 16, 17 and ??, respectively, we obtain the results obtained by Salim et al. [20, Theorems 3, 2 and 4, respectively].

(iv) Putting \( \lambda = 0 \) and \( b_k = \Phi(k) \), where \( \Phi(k) \) is given by (17) and \( -\infty < \mu < 1, -\infty < \gamma < 1, \eta \in \mathbb{R}^+, p \in \mathbb{N}, -p \leq \alpha < p, \beta \geq 0, a \in \mathbb{R}, c \in \mathbb{R} \setminus \mathbb{Z}, z \in \mathbb{U} \) in Theorems 16, 17 and ??, respectively, we obtain the results obtained by Khairnar and More [15, Theorems 4.2, 4.1 and 5.1, respectively, with \( n = 1 \)].

**Remark 3** Specializing the parameters \( p, \lambda, \alpha, \beta \) and function \( g(z) \) in our results, we obtain new results associated to the subclasses \( TS_{p, \lambda}(n, \alpha, \beta) \) and \( TS_{p, \lambda, n, \alpha, \beta}(n, \alpha, \beta) \) defined in the introduction.

8 Open problem

The authors suggest to obtain the same properties of the class consisting of functions \( f(z), g(z) \in T(p) \) and satisfying the subordinating condition:

\[ \frac{(1 - \gamma + \frac{z}{p})z(f * g)'(z) + \frac{z}{p}z^2(f * g)''(z)}{(1 - \gamma)(f * g)(z) + \frac{z}{p}z(f * g)'(z)} \]  

(159)
\[-\left|\frac{(1 - \gamma + \frac{2}{p})z(f * g)'(z) + \frac{2}{p}z^2(f * g)''(z)}{(1 - \gamma)(f * g)(z) + \frac{2}{p}z(f * g)'(z)} - p\right| < p \frac{1 + Az}{1 + Bz} (z \in U). \tag{160}\]

where \(\prec\) denotes the subordinate.

References


