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Complexity of Separately Harmonic Function and Application to Arithmetical Entire Functions

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Abstract

The aim of this paper is to give some results on the growth of an entire separately harmonic function using its complexity. We give also an estimation on Taylor series expansion coefficients and some results on arithmetical entire functions.

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1 Introduction

Nguyen Thanh Van studied separately harmonic functions [Ng3] by weakening Lelong's assumptions and by introducing Leja's polynomial condition.we take two domains $D_1 \subset \mathbb{R}^n$ and $D_2 \subset \mathbb{R}^m$ and a subset $E_1 \subset D_1$.we assume that $h : D_1 \times D_2 \to \mathbb{R}, and$ than $h(x, .)$ is harmonic in D_2 for every $x \in E_1$ and that $h(., y)$ is harmonic in D_1 for every $y \in E_2$.we also assume that the harmonic functions are locally unique in at a certain point $a \in D_1$ and that E_1verify Leja's condition at the same point a ;to finally prove that it exists a compact $K_1 \subset D_1$which contains a and E_1 in a way that $\mathbb{R}^n \setminus K_1$ be related .So, h is harmonic in $D_1 \times D_2$[Si-Ng] .In this paper we give some results on the growth of an entire separately harmonic function using its complexity.We give also some results on arithmetical entire functions.

2 Notations and Definitions

Notation 1 *For a multi-index* $\alpha = (\alpha_1...\alpha_N) \in \mathbb{N}^N$ *and* $Z = (z_1, ..., z_N) \in \mathbb{C}^N$*,with:*

$$\begin{aligned} D_Z^{\alpha} &= \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} z_1 ... \partial^{\alpha_N} z_N} = \left(\frac{\partial}{\partial z_1}\right)^{\alpha_1} .... \left(\frac{\partial}{\partial z_N}\right)^{\alpha_N}, \\ |\alpha| &= \alpha_{1+......+}\alpha_N \quad \text{and} \quad \alpha! = \alpha_1!\, .\alpha_2!....\alpha_N!. \end{aligned}$$

And for any $j = 1, 2..., N$, *write*

$$\begin{aligned} \Delta_j &= \left\{ \left(x_{2j-1}, x_{2j}\right) \in \mathbb{R}^2 \text{ such that } \left|x_{2j-1} + i x_{2j}\right| < 1 \right\}. \\ r.\Delta_j &= \left\{ \left(x_{2j-1}, x_{2j}\right) \in \mathbb{R}^2 \text{ such that } \left|x_{2j-1} + i x_{2j}\right| < r \right\}. \\ \Delta &= \Delta_1 \times ..... \times \Delta_N \subset \mathbb{R}^{2N} \end{aligned}$$

$$\begin{aligned} \widehat{\Delta}_j &= \left\{\left(z_{2j-1}, z_{2j}\right) \in \mathbb{C}^2 \text{ such that } \left|z_{2j-1} + i z_{2j}\right| < 1 \text{ and } \left|z_{2j-1} - i z_{2j}\right| < 1\right\}. \\ r.\widehat{\Delta}_j &= \left\{\left(z_{2j-1}, z_{2j}\right) \in \mathbb{C}^2 \text{ such that } \left|z_{2j-1} + i z_{2j}\right| < r \text{ and } \left|z_{2j-1} - i z_{2j}\right| < r\right\}. \\ \widehat{\Delta} &= \widehat{\Delta}_1 \times \widehat{\Delta}_2 \times ... \times \widehat{\Delta}_N \subset \mathbb{C}^{2N} \end{aligned}$$

Definition 1 *Order and type of an entire function (see [Ro] for more detail).*

Let $\chi : [0\,, +\infty[\longrightarrow [0\,, +\infty[\cup \{+\infty\}$ *be an increasing function. We define the growth order* $\rho(\chi)$ *of* χ *by the formula :*

$$\rho(\chi) = \limsup_{r\to\infty} \frac{\log \chi(r)}{\log r}$$

If χ *is of finite order,* $(0 < \rho(\chi) < \infty)$ *the growth type* $\sigma(\chi)$ *is defined by :*

$$\sigma(\chi) = \limsup_{r\to\infty} \frac{\chi(r)}{r^{\rho}}$$

For any entire function f *on* $\mathbb{C}^N$, *the* N*- growth of* f *is defined by the growth of the function* $\chi(r) = \log^+ M_N(f, r)$ *where* N *is a norm on* $\mathbb{C}^N$ *and* $M_N(f, r) = \sup_{N(z)\le r} |f(z)|$. *For any complete bounded domain* Δ *of center 0 in* $\mathbb{C}^N$, *the* $\Delta-$ *growth of* f *is defined by the growth of the function* $\chi(r) = \log^+ M_\Delta(f, r)$ *where* $M_\Delta(f, r) = \sup_{z\in r.\Delta} |f(z)|$.

Remark that this definition can be extended to entire hanmonic functions on $\mathbb{R}^N$.

Definition 2 *A function U defined on a neighborhood D of $\mathbb{C}^N$ is said separately harmonic function if and only if*

$$\frac{\partial^2 U}{\partial z_j \partial \overline{z_j}}(Z) = 0 \quad , \qquad Z = (z_1, ..., z_N) \in D, \quad j = 1, ..., N$$

Definition 3 *An entire holomorphic function $f \ : \mathbb{C}^N \longrightarrow \mathbb{C}$ (resp. an entire separately harmonic function $U \ : \mathbb{R}^{2N} \longrightarrow \mathbb{R}$) is arithmetical if $f(\mathbb{N}^N) \subset \mathbb{Z}$ (resp. $U(\mathbb{N}^{2N}) \subset \mathbb{Z}$)*

3 Main Results

Theorem 1 *Let $U : \mathbb{R}^{2N} \longrightarrow \mathbb{R}$ be an entire separately harmonic function of order ρ $(0 < \rho < \infty)$ and $\triangle$-type σ $(0 < \sigma < \infty)$. Then there exists a unique entire function $\widehat{U}$ on $\mathbb{C}^{2N}$ of order ρ and $\widehat{\Delta}-$ type σ such that $\widehat{U} = U$ on $\mathbb{R}^{2N}$.*

Proof. Let $U(x_1, ..., x_{2N}) \ : \ \mathbb{R}^{2N} \approx \mathbb{C}^N \longrightarrow \mathbb{R}$ be a real entire separately harmonic function.U can be represented by the following Poisson integral formula :$\forall r > 0$

$$U(x_1, ..., x_{2N}) = \frac{1}{(2\pi)^N} \int_0^{2\pi} ... \int_0^{2\pi} \prod_{j=1}^{N} \frac{r^2 - \left(x_{2j-1}^2 + x_{2j}^2\right)}{\left|re^{i\theta_j} - (x_{2j-1} + ix_{2j})\right|^2} U\left(re^{i\theta_1}, ..., re^{i\theta_N}\right) d\theta_1 ... d\theta_N$$

$(x_1....x_{2N}) \in r.\Delta, \ (x_{2j-1}, x_{2j}) \in r.\Delta_j \ , \ j = 1....N.$

The complexified *of* U is the function $\widehat{U} \ : \ \mathbb{C}^{2N} \longrightarrow \mathbb{C}$ defined by : $\forall$ $r > 0$,

$$\widehat{U}(z_1, ..., z_{2N}) = \frac{1}{(2\pi)^N} \int_0^{2\pi} ... \int_0^{2\pi} \prod_{j=1}^{N} \frac{r^2 - \left(z_{2j-1}^2 + z_{2j}^2\right)}{\left[re^{i\theta_j} - \left(z_{2j-1} + iz_{2j}\right)\right]\left[re^{i\theta_j} - \left(z_{2j-1} - iz_{2j}\right)\right]} U(re^{i\theta_1}, ..., re^{i\theta_N}) d\theta_1 ... d\theta_N \quad (3.1)$$

$(z_1, ..., z_{2N}) \in r.\widehat{\Delta}, \ (z_{2j-1} \ , \ z_{2j}) \in r.\widehat{\Delta}_j \ , \ j = 1, ..., N.$

$\widehat{U}$ is holomorphic on $\mathbb{C}^{2N}$ and equal to U on $\mathbb{R}^{2N}$. The unicity follows from the fact that $\mathbb{R}^{2N} = Reel\left(\mathbb{C}^{2N}\right)$ is a unicity set for holomorphic entire functions on $\mathbb{C}^{2N}$.

For any $(z_{2j-1}\ ,\ \ z_{2j})\in r'.\widehat{\Delta}_j$, $j=1,...N$ we have :

$$\begin{aligned}\left|r^2-\left(z_{2j-1}^2+z_{2j}^2\right)\right| &\leq r^2+\left|z_{2j-1}^2+z_{2j}^2\right| \\ &\leq r^2+r'^2\leq 2r^2.\end{aligned}$$

Then $\left|re^{i\theta_j}-(z_{2j-1}\pm iz_{2j})\right|\geqslant r-|z_{2j-1}\pm iz_{2j}|\geqslant r-r'$, for all $(z_{2j-1}\ ,\ \ z_{2j})\in r'.\widehat{\Delta}_j \quad j=1....N$.

The formula (3.1) and the above inequality gives :$\forall r>r'>0$,

$$\left|\widehat{U}\ \ (z_1,...,z_{2N})\right| \leq \frac{1}{(2\pi)^N}\int_0^{2\pi}...\int_0^{2\pi}\prod_{j=1}^{N}\frac{\left|r^2-\left(z_{2j-1}^2+z_{2j}^2\right)\right|}{\left|re^{i\theta_j}-(z_{2j-1}+iz_{2j})\right|\ \left|re^{i\theta_j}-(z_{2j-1}-iz_{2j})\right|}\left|U(re^{i\theta_1},...,re^{i\theta_N})\right|d\theta_1...d\theta_N$$

$$\left|\widehat{U}\ \ (z_1,...,z_{2N})\right| \leq \frac{2^N r^{2N}}{(r-r')^{2N}}\sup_{z\in r.\Delta}|U(z)| \qquad \forall\ (z_1,...,z_{2N})\in r'.\widehat{\Delta}\ .$$

$$\sup_{(z_1,...,z_{2N})\in r'.\widehat{\Delta}}\left|\widehat{U}\ \ (z_1,...,z_{2N})\right|\leq\frac{2^N r^{2N}}{(r-r')^{2N}}M_\Delta(U,r)$$

$$M_{\widehat{\Delta}}\left(\widehat{U},r'\right)\leq 2^N\left(1-\frac{r'}{r}\right)^{-2N}M_\Delta(U,r)$$

Consequentely, $\forall r>r'>0$:

$$M_\Delta(U,r')\leq M_{\widehat{\Delta}}\left(\widehat{U},r'\right)\leq 2^N\left(1-\frac{r'}{r}\right)^{-2N}M_\Delta(U,r)\,. \tag{3.2}$$

The growth of $\widehat{U}$ follows from inequality (3.2) ∎

Proposition 1 *Let U be an entire separately harmonic function such that :*

$$M_\Delta(U,r)\leqslant C\exp(\sigma.r)\ \ \forall\ r>0.$$

If $U(x)=\sum\limits_{\alpha\in\mathbb{N}^{2N}}a_\alpha x^\alpha$ *is its Taylor expansion at the origin then*

$$|a_\alpha|\leqslant C.A(N)\ \ (2e)^{|\alpha|}\ \ |\alpha|^{2N}\ \ \sigma^{|\alpha|}\qquad\forall\alpha\in\mathbb{N}^{2N},$$

where the constant $A(N)$ depends only of N.

Proof. Let $\widehat{U}(z) = \sum_{\alpha\in\mathbb{N}^{2N}} a_\alpha z^\alpha$ the expansion series of the complexified of U, for all $r > \acute{r} > 0$, $r'.\Delta$ contains the polydisc $D_{r'} = \left\{ z \in \mathbb{C}^{2N} : |z_j| < \frac{r'}{2} \right\}$, then the Cauchy formula applied to $\widehat{U}$, gives :

$$|a_\alpha| \leqslant \frac{\| \widehat{U} \|_{D_{r'}}}{\left(\frac{r'}{2}\right)^{|\alpha|}} = \frac{2^{|\alpha|}}{r'^{|\alpha|}} \| \widehat{U} \|_{D_{r'}} \leqslant \frac{2^{|\alpha|}}{\acute{r}^{\alpha}} M_{\widehat{\Delta}}\left(\widehat{U}, r'\right). \tag{3.3}$$

The inequality (3.2) and (3.3) gives:

$$|a_\alpha| \leqslant 2^{|\alpha|+N} \left(1 - \frac{r'}{r}\right)^{-2N} r'^{-|\alpha|} M_\Delta(U, r) \qquad \forall r > r' > 0. \tag{3.4}$$

Since the function defined by $g(t) = \left(1 - \frac{t}{r}\right) t^{-|\alpha|},\ t \in]0, r[$ reach its maximum at the point $t = \frac{|\alpha|}{2N + |\alpha|} r \in]0, r[$, we have for all $r > 0$:

$$\begin{aligned} |a_\alpha| &\leqslant 2^{|\alpha|+N} \left(\frac{2N + |\alpha|}{2N}\right)^{2N} |\alpha|^{-|\alpha|} (2N + |\alpha|)^{|\alpha|} r^{-|\alpha|} M_\Delta(U, r) \\ &\leqslant 2^{|\alpha|+N} (2N + |\alpha|)^{2N+|\alpha|} (2N)^{-2N} . |\alpha|^{-|\alpha|} r^{-|\alpha|} M_\Delta(U, r) \\ &\leqslant \frac{2^N}{(2N)^{2N}} \left(1 + \frac{2N}{|\alpha|}\right)^{2N+|\alpha|} 2^{|\alpha|} |\alpha|^{2N} r^{-|\alpha|} M_\Delta(U, r). \end{aligned}$$

The function :

$$|\alpha| \mapsto \frac{2^N}{(2N)^{2N}} \left(1 + \frac{2N}{|\alpha|}\right)^{2N+|\alpha|}$$

is bounded by a constant $A(N)$ depending only on N, so :

$$|a_\alpha| \leqslant A(N) . 2^{|\alpha|} |\alpha|^{2N} r^{-|\alpha|} M_\Delta(U, r) \qquad \forall \alpha \in \mathbb{N}^{2N},\ r > 0$$

$$|a_\alpha| \leqslant A(N) . 2^{|\alpha|} |\alpha|^{2N} r^{-|\alpha|} C \exp(\sigma.r) \qquad \forall \alpha \in \mathbb{N}^{2N},\ r > 0 \tag{3.5}$$

Using inequality (3.5) and the fact that $\inf\left(r^{-|\alpha|} \exp(\sigma.r)\right) = \left(\frac{e\sigma}{|\alpha|}\right)^{|\alpha|}$,

$$|a_\alpha| \leqslant C.A(N) . 2^{|\alpha|} |\alpha|^{2N} \left(\frac{e\sigma}{|\alpha|}\right)^{|\alpha|}$$

the proposition is conclued. ∎

Theorem 2 *Let U be an entire separately harmonic function on $\mathbb{R}^{2N} \approx \mathbb{C}^N$, such that :*
$M_\Delta(U,r) \leqslant C.\exp(\sigma.r)$ *then :*

$$| D^\alpha U(0) | \leqslant C.M(N).|\alpha|^{\frac{1}{2}}\sigma^{|\alpha|},\ \forall\alpha \in \mathbb{N}^{2N} \tag{3.6}$$

where the constant $M(N)$ depends only of N.

Proof. U can be represented by the Poisson integral formula :

$$U(x) = \frac{1}{(2\pi)^N}\int_0^{2\pi}.....\int_0^{2\pi} P_r\left(x, re^{i\theta}\right) U\left(re^{i\theta}\right) d\theta$$

$x \in r.\Delta$ $\left((x_{2j-1}, x_{2j}) \in r.\Delta_j \text{ for } j = 1...N\right), r > 0$, where

$$P_r\left(x\, ,\, re^{i\theta}\right) = \prod_{j=1}^{N} P_r^j\left(x_{2j-1}\, , x_{2j}\, ,\ re^{i\theta_j}\right)$$

and

$$\begin{aligned} P_r^j\left(x_{2j-1}, x_{2j}\ ,\ re^{i\theta j}\right) &= \frac{r^2 - \left(x_{2j-1}^2 + x_{2j}^2\right)}{\left|re^{i\theta j} - (x_{2j-1} + ix_{2j})\right|^2} \\ re^{i\theta} &= \left(re^{i\theta_1}, re^{i\theta_2}, ..., re^{i\theta_N}\right) \ \textit{and}\ \ d\theta = d\theta_1.d\theta_2...d\theta_{N.} \end{aligned}$$

For any j the kernel P_r^j is harmonic on an open neighbourhood including the closed disk $\overline{r.\Delta_j} \subset \mathbb{R}^2 \approx \mathbb{C}$, then P_r^j and all its partial derivatives whith respect to x_{2j-1} et x_{2j} are continous in $r\Delta_j \times (\partial(r.\Delta_j))$,since for all $\alpha \in \mathbb{N}^{2N}$

:

$$D_x^\alpha P_r\left(x, re^{i\theta}\right) = \prod_{j=1}^{N} D_{\left(x_{2j-1},x_{2j}\right)}^{\left(\alpha_{2j-1},\alpha_{2j}\right)} P_r^j\left(x_{2j-1}, x_{2j}, re^{i\theta j}\right),$$

P_r and all its partial derivatives whith respect to x_{2j-1} et x_{2j} are continous in $(r.\Delta) \times (\partial(r.\Delta))$, where $\partial(r.\Delta) = (\partial(r.\Delta_1)) \times (\partial(r.\Delta_2)) \times ..... \times (\partial(r.\Delta_N))$, it follows that :

$$D_x^\alpha U(0) = \frac{1}{(2\pi)^N}\int_0^{2\pi}......\int_0^{2\pi} D_x^\alpha P_r\left(0, re^{i\theta}\right) U\left(re^{i\theta}\right) d\theta, \forall\alpha \in \mathbb{N}^{2N}.$$

Thus,to estimate $|D_x^\alpha U(0)|$ it suffices to estimate

$$\left|D_x^\alpha P_r\left(0, re^{i\theta}\right)\right| = \prod_{j=1}^{N}\left|D_{x_{2j-1},x_{2j}}^{(\alpha_{2j-1},\alpha_{2j})}\ P_r^j\left(0, re^{i\theta_j}\right)\right|.$$

This will be do by expressing $P_r^j\left(x_{2j-1}, x_{2j}\ ,\ re^{i\theta j}\right)$ as a series of harmonic polynomials.

Let

$$K_r\left((x,y),re^{i\theta}\right) = \frac{r^2-(x^2+y^2)}{\mid re^{i\theta}-(x+iy)\mid^2}$$
$$= \frac{r^2-(x^2+y^2)}{r^2+(x^2+y^2)-2r\left(x\cos\theta+y\sin\theta\right)}$$

$$K_r\left((x,y),re^{i\theta}\right) = \frac{1-\dfrac{\mid\gamma\mid^2}{r^2}}{1+\dfrac{\mid\gamma\mid^2}{r^2}-2\dfrac{\mid\gamma\mid}{r}t}$$

$$K_r\left((x,y),re^{i\theta}\right) = \sum_{k=0}^{\infty}N(k,2)\frac{\mid\gamma\mid^k}{r^k}L_{k,re^{i\theta}}(t) \tag{3.7}$$

Where $t = \dfrac{x\cos\theta+y\sin\theta}{|\gamma|}$, $|\gamma| = (x^2+y^2)^{\frac{1}{2}}$, $N(k,n) \leq Ak^{n-2}$, A depending only on n, and $L_{k,re^{i\theta}}$ is the Legendre polynomial of degre k. Write

$$T_{k\ ,re^{i\theta}}(\gamma) = |\gamma|^k\ \ L_{k,\,re^{i\theta}}\ (t)$$

$T_{k\ ,re^{i\theta}}$ an homogeneous harmonic polynomial of degre k such that :

i) $T_{k,re^{i\theta}}\left(re^{i\theta}\right) = \left|re^{i\theta}\right|^k = r^k$

ii) $D^\beta T_{k\,,re^{i\theta}}(0) = 0 \quad \forall\beta,\ |\beta| \neq k.$

Hence, it is enough to show that :

$$D_\gamma^\beta K_r\left(\gamma,re^{i\theta}\right) = \sum_{k=0}^{\infty}\frac{N(k,2)}{r^k}D_\gamma^\beta T_{k\ ,\ \ re^{i\theta}}(\gamma)$$

For all fixed $re^{i\theta}$ the function $K_r\left(.\ ,\ re^{i\theta}\right)$is harmonic on an neighbourhood of the closed disc $\overline{D}\,(0,r)$, and therefore real-analytic in the open disk $D\,(0,r)$. So its Taylor series at 0 converges uniformly on an open neighbourhood of the origine to $K_r\left(.\ ,\ re^{i\theta}\right)$. By rearranging the terms of same degree in the Taylor expansion we obtain series of homogeneous polynomials, converging to $K_r\left(.\ ,\ re^{i\theta}\right)$ on some neighbourhood of the origin. By unicity of Taylor expansion we deduce that this series coincide whith the series (3.7) and by derivation we obtain

$$D_\gamma^\beta K_r\left(\gamma\ ,\ re^{i\theta}\right) = \sum_{k=0}^{\infty}N(k,2)r^kD_\gamma^\beta\left(|\gamma|^k\quad L_{k\ ,re^{i\theta}}(t)\right)$$
$$= \sum_{k=0}^{\infty}N(k,2)r^kD_\gamma^\beta\left(T_{k\ ,re^{i\theta}}(\gamma)\right),$$

then :

$$D_\gamma^\beta K_r\left(0,re^{i\theta}\right) = \frac{N(\mid\beta\mid,2)}{r^{|\beta|}}D_\gamma^\beta T_{|\beta|\ ,\ re^{i\theta}}(0)$$

and

$$\left|D_{\gamma}^{\beta}K_r\left(0,re^{io}\right)\right| = |N(|\ \beta\ |,2)|\ \ r^{-|\beta|}\left|D_{\gamma}^{\beta}T_{|\beta|,re^{i\theta}}\left(0\right)\right|.$$

To estimate$\left|D_{\gamma}^{\beta}T_{|\beta|,re^{io}}\left(0\right)\right|$ we use the following lemma due to $\ddot{U}Kuran$ [Ku](see also[Ar2])

Lemma 1 *[Ku]: Let $\mathcal{H}_{m,n}$ the vector space of all homogeneous polynomials of degree m in $\mathbb{R}^n$, $\|\ \|$ denote the norme on $\mathcal{H}_{m,n}$ introduced by Brelot and Choquet [Br-Ch] and defined by the formula*

$$\|\ H\ \| = \left(\frac{1}{S_n}\int_{S(1)}H^2 d\sigma\right)^{\frac{1}{2}},\ H\in H_{m,n}$$

where $S(1)$ is the unit sphere of center 0 of $\mathbb{R}^n$ and S_n denote its mesure .

If $H\in H_{m,n}$ and $|\beta|=m$,then $\left|D^{\beta}H\right|\leqslant m!\left(\dim H_{m,n}\right)^{\frac{1}{2}}.\ \|\ H\ \|$.
The above lemma gives :

$$\left|D^{\beta}T_{k\,,\,re^{i\theta}}\left(0\right)\right|\leqslant|\beta|!\left(\dim H_{|\beta|,2}\right)^{\frac{1}{2}}\ \|\ T_{|\beta|\,,\,re^{i\theta}}\ \|$$

since $\|\ T_{|\beta|\,,\,re^{i\theta}}\ \|^2 = \left(\dim H_{|\beta|,2}\right)^{-1}$ *then :* $|\ D^{\beta}T_{|\beta|,re^{i\theta}}\left(0\right)\ |\leqslant|\beta|!,$

$$so\quad\left|D^{\beta}K_r\left(0,re^{i\theta}\right)\right|\leqslant A\,|\beta|!\ r^{-|\beta|},\ \ \beta\in\mathbb{N}^2.$$

Consequently :

$$\left|D^{\left(\alpha_{2j-1},\alpha_{2j}\right)}P_r^j\left(0,re^{i\theta j}\right)\right|\leqslant A\left(\alpha_{2j-1}+\alpha_{2j}\right)!r^{-\left(\alpha_{2j-1}+\alpha_{2j}\right)}\quad\forall j=1...N$$

$$\left|D^{\left(\alpha_{2j-1},\alpha_{2j}\right)}P_r^j\left(0,re^{i\theta j}\right)\right|\leqslant 2A\left(\alpha_{2j-1}+\alpha_{2j}\right)!r^{-\left(\alpha_{2j-1}+\alpha_{2j}\right)}\quad\forall j=1...N$$

$$\left|D^{\alpha}P_r\left(0,re^{i\theta}\right)\right|\leqslant 2^N A^N\ r^{-|\alpha|}\prod_{j=1}^{N}\left(\alpha_{2j-1}+\alpha_{2j}\right)!$$

But $\prod_{j=1}^{N}\left(\alpha_{2j-1}+\alpha_{2j}\right)!\leqslant|\alpha|!$ *so :*

$$\left|D^{\alpha}P_r\left(0,re^{i\theta}\right)\right|\leqslant 2^N\ \ A^N\,|\alpha|!\ \ r^{-|\alpha|}\qquad\forall\alpha\in\mathbb{N}^{2N}$$

and finally :

$$
\begin{aligned}
\mid D^{\alpha}U(0)\mid &\leqslant 2^{N}\ A^{N}\,|\alpha|!\ r^{-|\alpha|}\,M_{\Delta}\,(u,r) && \forall\alpha\in\mathbb{N}^{2N}, r>0\\
&\leqslant C.2^{N}\ |\alpha|!\ r^{-|\alpha|}\exp\,(\sigma.r) && \forall\alpha\in\mathbb{N}^{2N}, r>0\\
&\leqslant C.2^{N}\ |\alpha|!\ (e\sigma)^{|\alpha|}\,|\alpha|^{-|\alpha|} && \forall\alpha\in\mathbb{N}^{2N}\\
&\leqslant C.2^{N}\ |\alpha|^{\frac{1}{2}}.e^{|\alpha|}\ |\alpha|^{-|\alpha|-\frac{1}{2}}\ |\alpha|!\ \sigma^{|\alpha|} && \forall\alpha\in\mathbb{N}^{2N}.
\end{aligned}
$$

Using Stirling formula we get :

$$|D^{\alpha}U\,(0)| < C\ M\,(N)\mid\alpha\mid^{\frac{1}{2}}\sigma^{|\alpha|}\qquad\forall\alpha\in\mathbb{N}^{2N}.$$

■

Corollary 1 *Under the hypothesis of theorm(2), if* $U\,(x)=\sum_{\alpha\in\mathbb{N}^{2N}}a_{\alpha}x^{\alpha}$ *is the Taylor expansion at the origine of* U *then :*

$$|a_{\alpha}|\leqslant C\ M\,(N)\ \frac{|\alpha|^{\frac{1}{2}}}{\alpha!}\sigma^{|\alpha|},\ \forall\alpha\in\mathbb{N}^{2N}.$$

Proof. *Its follows from* $a_{\alpha}=\dfrac{D^{\alpha}U\,(0)}{\alpha!}.$
■

Corollary 2 *let* U *be an entire separately harmonic function such that* $M_{\Delta}\,(U,r)\leqslant C\exp\,(\sigma r)$ *,* $r>0$ *. If* $\sigma<1$ *and all* $D^{\alpha}U\,(0)$ *are integers then* U *is polynomial.*

Proof. Its follows that $\mid a_{\alpha}\mid=\dfrac{|D^{\alpha}u\,(0)|}{\alpha!}$ goes to 0 when $|\alpha|\to\infty$, which implies that all a_{α} vanish for large rank because $D^{\alpha}U\,(0)$ are integers, then U is polynomial ■

Remark 1 *In [Ar2] , Armitage etablish an inequality of type (3.6) in a more case but in our particular situation we obtain more precise results.*

Theorem 3 *let* U *be a entire separately harmonic function such that* $M_{\Delta}\,(U,r)\leqslant C\exp\,(\sigma r)$*,* $\forall r>0$ *then* $\forall\varepsilon>0\ \exists C\,(\varepsilon)>0$ *such that the complexified* $\widehat{U}$ *satisfies :*

$$\left|\widehat{U}\,(z_{1},z_{2,\ldots,}z_{2N})\right|\leqslant C\,(\varepsilon)\exp\left[(\sigma+\varepsilon)\,(|z_{1}|+|z_{2}|+\ldots..+|z_{2N}|)\right]\quad\forall\ (z_{1,\ldots,}z_{2N})\in\mathbb{C}^{2N}.$$

Proof. Let U an entire separately harmonic function on $R^{2N} \simeq \mathbb{C}^N$, and $U(x) = \sum_{\alpha \in \mathbb{N}^{2N}} a_\alpha x^\alpha$ its Taylor expansion at the origin.

Let $\widehat{U}(z) = \sum_{\alpha \in \mathbb{N}^{2N}} a_\alpha z^\alpha$ the complexified of U. From the corollary (1) there exist a constant $M(N)$ depending only of N such that :

$$|a_\alpha| \leqslant C\ M(N)\ |\ \alpha\ |^{\frac{1}{2}} \frac{\sigma^{|\alpha|}}{\alpha!},\ \forall \alpha \in \mathbb{N}^{2N}.$$

then :

$$\begin{aligned}
|\ \widehat{U}(z_1, \ldots . z_{2N})\ | &\leqslant \sum_{\alpha \in \mathbb{N}^{2N}} C.M(N)\ \frac{|\alpha|^{\frac{1}{2}}}{\alpha!}\ \sigma^{|\alpha|}\ \ |z|^\alpha \\
&\leqslant C.M(N) \sum_{k \geq 0} \sum_{|\alpha| = k} \frac{|\alpha|^{\frac{1}{2}}}{\alpha!}\ \sigma^{|\alpha|}\ \ |z|^\alpha \\
&\leqslant C.M(N) \sum_{k \geq 0} \left[\sum_{|\alpha| = k} |\alpha|^{\frac{1}{2}} . \left(\frac{\sigma}{\sigma + \varepsilon} \right)^{|\alpha|} . \frac{(\sigma+\varepsilon)^{|\alpha|}}{|\alpha|!} . \frac{|\alpha|!}{\alpha!}\ \ |z|^\alpha \right] \\
&\leqslant C.M(N) . \sum_{k \geq 0} \left[\sum_{|\alpha| = k} . k^{\frac{1}{2}} \left(\frac{\sigma}{\sigma + \varepsilon} \right)^k \frac{(\sigma+\varepsilon)^k}{k!} \frac{|\ \alpha\ |!}{\alpha!}\ \ |z|^\alpha \right] \\
&\leqslant \sum_{k \geq 0} C.M(N) . k^{\frac{1}{2}} \left(\frac{\sigma}{\sigma + \varepsilon} \right)^k . \frac{(\sigma+\varepsilon)^k}{k!} \left[\sum_{|\alpha| = k} \frac{|\alpha|!}{\alpha!}\ \ |z|^\alpha \right]
\end{aligned}$$

The fonction $g(k) = C.M(N)\ k^{\frac{1}{2}} \left(\frac{\sigma}{\sigma + \varepsilon} \right)^k$ being bounded, there exist $C(\varepsilon)$ such that :

$$\left| \widehat{U}(z_1, z_2, \ldots, z_{2N}) \right| \leqslant C(\varepsilon) \sum_{k \geq 0} \frac{(\sigma+\varepsilon)^k}{k!} \left(\sum_{|\alpha| = k} \frac{|\alpha|!}{\alpha!}\ \ |z|^\alpha \right)$$

It 's easy to show that : $(|z_1| + \ldots + |z_{2N}|)^k = \sum_{|\alpha| = k} \frac{|\alpha|!}{\alpha!} |z|^\alpha$, and then :

$$\left| \widehat{U}(z_1, z_2, \ldots, z_{2N}) \right| \leqslant C(\varepsilon) \exp\left[(\sigma + \varepsilon) (|z_1| + \ldots + |z_{2N}|) \right] \quad , \forall\ (z_1, \ldots, z_{2N}) \in \mathbb{C}^{2N}$$

In fact this inequality relies the $\Delta-$type of U and the type of $\widehat{U}$ whith respect to a norm N of $\mathbb{C}^{2N}$. ■

4 Arithmetic Entire Function

In this section we give a some results on arithmetical entire functions.

Theorem 4 *Let U an entire separately harmonic functions on $\mathbb{R}^{2N}$ such that:*

$$\exists\ A\ >0\ \ \text{such that}\ \ M_{\Delta}(U,r)\leqslant A\exp(\sigma\, r)\quad \forall\, r>0. \tag{4.1}$$

If $\sigma<\log 2$, then U is a polynom.
Proof. *Let U an entire separately harmonic function on $R^{2N}\simeq\mathbb{C}^N$,such that:*

$$\exists\ A\ >0\ \ ,\ \ M_{\Delta}(U,r)\leqslant A\exp(\sigma\, r)\quad \forall\, r>0. \tag{4.2}$$

and $\sigma<\log 2$, so U develops according

$$U(x)=\sum_{\alpha\in\mathbb{N}^{2N}}\frac{D^{\alpha}U(0)}{\alpha!}x^{\alpha}$$

Development converge uniformly on any compact of $\mathbb{C}^N$.

From the theorem 2 there exist a constant $M(N)$ depending only of N such that:

$$\left|\frac{D^{\alpha}U(0)}{\alpha!}\right|\ \leqslant\ C\ M(N)\mid\alpha\mid^{\frac{1}{2}}\frac{\sigma^{|\alpha|}}{\alpha!},\ \forall\alpha\in\mathbb{N}^{2N}.$$

$$If\ \sigma\ <\ \log 2,\ then\ |a_{\alpha}|<C\ M(N)\mid\alpha\mid^{\frac{1}{2}}\frac{\log 2^{|\alpha|}}{\alpha!}$$

$$so\ \ |a_{\alpha}|\ \longrightarrow\ 0\ when\ \mid\alpha\mid\longrightarrow+\infty, which\ implies\ that\ all\ a_{\alpha}vanish\ for\ large\ rank$$

then U is a polynom .

Corollary 3 *Let U be an entire separately harmonic function on $\mathbb{R}^{2N}$ satisfies inequality (4.1) with $\sigma<\log 2$.*
i) If U is arithmetical, then U is a polynom.
ii) If $U\left(\mathbb{N}^{2N}\right)=0$ then $U\equiv 0$ on $\mathbb{R}^{2N}$
Proof. *From theorem(4) U is a polynom.* ∎

Proposition 2 *Let U be an entire separately harmonic function on $\mathbb{R}^{2N}$ satisfies inequality (4.1) .*

If $U(\mathbb{N}_m)\subset\mathbb{Z}$ and $\sigma<\log 2$, then there exists a polynomial P with m real variables such as:

$U(x_1,...,x_m,0_{....},0)=P(x_{1,....,}x_m)\qquad (x_1,...,x_m)\in\mathbb{R}^m.$

or $\mathbb{N}_m=\left\{(x_1,...,x_{2N})\in\mathbb{N}^{\,2N}\quad:\quad x_{m+1}=x_{m+2}=....=x_{2N}=0\right\}$

∎

Proof. Let ϕ be an arithmetical entire function defined by :$(x_{1,....,}x_m) \longrightarrow U(x_{1,...,}x_m, 0_{....,}0)$,and $\sigma < \log 2$,by corollary 3 the function ϕ is a polynom.

Then there exists a polynomial P with m real variables such as:

$$U(x_{1,...,}x_m, 0_{....,}0) = P(x_{1,....,}x_m) \qquad (x_{1,...,}x_m) \in \mathbb{R}^m.$$

■

5 Open Problem

Let V be an entire separately harmonic function on $\mathbb{R}^{2N}$ satisfies inequality (4.1).If $V(\mathbb{R}_m) \subset \mathbb{Z}$ and $\sigma < \log 2$.

Is there a polynomial P with m real variables such as:
$V(x_{1,...,}x_m, 0_{....,}0) = P(x_{1,....,}x_m) \qquad (x_{1,...,}x_m) \in \mathbb{R}^m$.
or $\mathbb{R}_m = \left\{(x_1, ..., x_{2N}) \in \mathbb{R}^{2N} \quad : \quad x_{m+1} = x_{m+2} = ... = x_{2N} = 0\right\}$

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