Int. J. Open Problems Compt. Math., Vol. 9, No. 1, March 2016
ISSN 1998-6262; Copyright ©ICSRS Publication, 2016
www.i-csrs.org

On the Exact Number of Zeros of Certain Even Degree Polynomails: A Sharkovsky Theorem Approach

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Abstract

In this paper we give some rigorous conditions to determine the exact number of zeros of certain even degree polynomails. The analysis is based on the so called *Sharkovsky theorem* that gives a complete description of possible sets of periods for continuous mappings defined on an interval. Two open problems are given based on some rigorous analysis focused to certain type of real polynomial functions with several conditions on their coefficients.

Keywords: Polynomials, counting zeros, periodic three point, Sharkovsky theorem.

AMS Mathematics Subject Classification (2010): 30A10.

1 Introduction

The problem of finding or characterizing the zeros of polynomial functions has a long history. The obtained results varied from theory, algorithms and numerical simulations. Historically, this study began with the fundamental theorem of algebra proved by Gauss. The most known results in this direction is the fact that a real polynomial of degree n has at most n real zeros. There is also the so called *Descartes rule of signs* concerning the number of positive zeros. Some generalizations of Descartes rule are know such as the Budan-Fourier theorem that gives an upper bound for the number of zeros of a polynomial. Also, the Sturm's theorem that gives a method for determining the exact number of zeros in an interval [9, chapter 6] and [6, chapter 2]. Recent results uses the classical Eneström-Kakeya theorem to restricts the location of the zeros based on a condition imposed on the coefficients of the polynomial under invistigation. See [7] and references therain.

In this paper, we will use a dynamical system result concerning periodic points of a continuous function. The result is called *Sharkovsky theorem* [1-2-3-4-5] that gives a complete description of possible sets of periods for continuous mappings defined on an interval. The interval need not be closed or bounded. The main idea used here is the notion of *topological conjugation* between continuous mappings. In this case we construct infinitely many polynomials with any desired number of real zeros. Generally, the use of the Sharkovsky theorem is known in the field of dynamical mappings (the study of iterations of mappings), but as far as we know, this result is never used to investigate the nature and the number of zeros of ceratin polynomial functions.

2 Sharkovsky theorem and counting zeros of certain polynomials

Let us consider the function

$$f\left(x\right) = ax^2 + bx + c, a \neq 0 \tag{1}$$

Then we get

$$\left\{ \begin{array}{c} f^2\left(x\right) = a^3x^4 + 2a^2bx^3 + a\left(b + 2ac + b^2\right)x^2 + b\left(b + 2ac\right)x + c\left(b + ac + 1\right) \\ f^3\left(x\right) = \xi_1 x^8 + \xi_2 x^7 + \xi_3 x^6 + \xi_4 x^5 + \xi_5 x^4 + \xi_6 x^3 + \xi_7 x^2 + \xi_8 x + \xi_9 \\ ... \\ f^k\left(x\right) = f\left(f^{k-1}\left(x\right)\right) = a\left(f^{k-1}\left(x\right)\right)^2 + bf^{k-1}\left(x\right) + c, k = 2, 3, ... \end{array} \right. \tag{2}$$

where

$$\left\{ \begin{array}{c} \xi_1 = a^7 \\ \xi_2 = 4a^6b \\ \xi_3 = 4ca^6 + 6a^5b^2 + 2a^5b \\ \xi_4 = 12ca^5b + 4a^4b^3 + 6a^4b^2 \\ \xi_5 = a^3\left(6a^2c^2 + 12ab^2c + 6abc + 2ac + b^4 + 6b^3 + b^2 + b\right) \\ \xi_6 = 2a^2b\left(6a^2c^2 + 2ab^2c + 6abc + 2ac + b^3 + b^2 + b\right) \\ \xi_7 = a\left(4a^2c^2 + 4a^3c^3 + b^2 + b^3 + b^4 + 6a^2b^2c^2 + 4ab^2c + 6ab^3c + 6a^2bc^2 + 4abc\right) \\ \xi_8 = b\left(4a^3c^3 + 6a^2bc^2 + 4a^2c^2 + 2ab^2c + 4abc + b^2\right) \\ \xi_9 = c\left(a^3c^3 + 2a^2bc^2 + 2a^2c^2 + ab^2c + 3abc + ac + b^2 + b + 1\right) \end{array} \right. \tag{3}$$

In (2), f^k represents the composition of f with itself k times. It is a 2^k-degree polynomial function verifying the functional equation described in the last part of (2).

In this paper, we are interested mainlely on the number of real zeros of the following equation:

$$f^k(x) - x = 0 \tag{4}$$

From (2) we remark that the expression of f^k become very complicated with the increasing of the value of the integer k. So, it is impossible to solves (4) directely or know the number of its real zeros. To resolve this problem we use the Sharkovsky theorem described as follow: Define a periodic point x of f as the real zero of the equation (4) and k is called the *period* of x. If $k = 1$, then x is called *fixed point*. A *Sharkovsky ordering* of the set of natural numbers $\mathbb{N}$ is given by:

$$3 \rhd 5 \rhd 7... \rhd 2.3 \rhd 2.5 \rhd 2.7 \rhd ... \rhd 2^2.3 \rhd 2^2.5... \rhd 2^3 \rhd 2^2 \rhd 2 \rhd 1. \tag{5}$$

This is a total ordering of the set $\mathbb{N}$. Hence, the Sharkovsky theorem can be stated as follow:

Theorem 1 *If m is a period for f and $m \rhd l$ in the ordering (5), then l is also a period for f.*

All proofs of the Sharkovsky theorem 1 are elementary. For example the intermediate value theorem is the deepest method. From the ordering (5), we remark that if f has a periodic point of period 3, then it has periodic points of all periods. This result is also proved by Li and York in [10] confirming that the existence of a 3-periodic orbit can be considered as a route to chaos. See [11-12-13-14-15] for more details on chaos theory. In particular, Li and Yorke proved the following theorem:

Theorem 2 *Let J be an interval and let $\varphi : J \to J$ be continuous. Assume there is a point $u \in J$ for which the points $v = \varphi(u), w = \varphi^2(u), d = \varphi^3(u)$, satisfy*

$$\begin{cases} d \leq u < v < w \\ \qquad or \\ d \geq u > v > w \end{cases} \tag{6}$$

then for every $k = 1, 2, ...$ there is a periodic point in J having period k, i.e., the equation $\varphi^k(x) - x = 0$ has exactely k real and distinct zeros.

If φ has a periodic orbit $\{z_i\}_{0 \leq i \leq 2}$ with period 3, then equation (4) has exactely k real and distinct zeros for every $k = 1, 2, ...$

By using (6) for the polynomial function f given by (1) we get the following theorem:

Theorem 3 *Assume there is a point $u \in J$ such that*

$$\begin{cases} f^3(u) \leq u < f(u) < f^2(u) \\ or \\ f^3(u) \geq u > f(u) > f^2(u) \end{cases} \tag{7}$$

where $f(u), f^2(u)$ and $f^3(u)$ are defined in (2) by setting $x = u$, then for every $k = 1, 2, ...$ the polynomial equation (4) has exactely k real and distinct zeros.

Solving a double inequality of the form (7) is quite difficulte or impossible. Then we must use the conditions for the periodic orbit $\{z_i\}_{0 \leq i \leq 2}$ as follow:

$$\begin{cases} z_{i+1} = f(z_i), i = 0, 1, 2 \\ z_3 = f(z_2) = z_0 \\ z_i = f^3(z_i), i = 0, 1, 2 \\ z_i \neq f^j(z_i), i, j = 0, 1, 2. \end{cases} \tag{8}$$

The last conditions of (8) means that the resulting periodic orbit $\{z_i\}_{0 \leq i \leq 2}$ is not a fixed point, not a periodic point of period 2 of the function f.

The fixed points of f are the real zeros of the equation $ax^2 + (b-1)x + c = 0$ denoted by $u_1 = -\frac{b+\sqrt{-2b-4ac+b^2+1}-1}{2a}$ and $u_2 = \frac{-b+\sqrt{-2b-4ac+b^2+1}+1}{2a}$ if $-2b-4ac+b^2+1 > 0$ and $u_1 = u_2 = -\frac{b-1}{2a}$ if $-2b-4ac+b^2+1 = 0$. A periodic point of period 2 of the function f is the real solution of the equation $f^2(x) - x = 0$ which is not a fixed point, i.e., $a^3x^4 + 2a^2bx^3 + (a(b^2+2ac)+ab)x^2 + (b^2+2acb-1)x + (c+ac^2+bc) = 0$. This is a quartic equation which can have at most 4 zeros deonted by $\{w_j\}_{1 \leq j \leq 4}$ with the conditions $u_1, u_2 \notin \{w_j\}_{1 \leq j \leq 4}$.

Hence, it is easy to prove the following result:

Theorem 4 *Assume that there exists a set of 3 points $\{z_i\}_{0 \leq i \leq 2}$ such that*

$$\begin{cases} z_{i+1} = az_i^2 + bz_i + c, i = 0, 1, 2 \\ z_0 = az_2^2 + bz_2 + c = z_3 \\ \xi_1 z_i^8 + \xi_2 z_i^7 + \xi_3 z_i^6 + \xi_4 z_i^5 + \xi_5 z_i^4 + \xi_6 z_i^3 + \xi_7 z_i^2 + (\xi_8 - 1) z_i + \xi_9 = 0, i = 0, 1, 2 \\ z_0, z_1, z_2 \notin \left\{ \{w_j\}_{1 \leq j \leq 4}, u_1, u_2 \right\} \end{cases} \tag{9}$$

Then for all integer $k \geq 1$ there exists a set of k points $\left\{u_i^{(k)}\right\}_{0 \leq i \leq k-1}$ such that:

$$\begin{cases} u_{i+1}^{(k)} = a\left(u_i^{(k)}\right)^2 + bu_i^{(k)} + c, i = 0, ..., k-1 \\ u_0^{(k)} = a\left(u_{k-1}^{(k)}\right)^2 + b\left(u_{k-1}^{(k)}\right) + c = u_k^{(k)} \\ u_i^{(k)} = f^k\left(u_i^{(k)}\right), i = 0, ..., k-1 \\ u_i^{(k)} \neq f^j\left(u_i^{(k)}\right), i = 0, ..., k-1, j = 1, ..k-1. \end{cases} \tag{10}$$

that is, the polynomial equation (4) has exactely k distinct real zeros for all integer $k \geq 1$.

Note that the set of conditions (9) is not empty since it was proved analytically that the special case of the logistic map $g(x) = rx(1-x), 0 \leq r \leq 4$ has a periodic point of period 3 for $r \simeq 3.83187405...$ [8], i.e., the equation $g^k(x) - x = 0$ has exactely k distinct real zeros for all integer $k \geq 1$.

From this analytical result we can show that there exists infinitely many functions of the form (1) such that the equations (4) has exactely k distinct real zeros:

Theorem 5 *For all $0 < r \leq 4$, there exists infinitely many functions of the form (1) such that the equation (4) has exactely k distinct real zeros for all integer $k \geq 1$.*

Indeed, we say that the functions $f(x) = ax^2+bx+c$ and $g(x) = rx(1-x)$ are affinely conjugate if there exists an *affine transformation* $h(x) = px + q$ $(p \neq 0)$ such that:

$$(g \circ h)(x) = (h \circ f)(x), \tag{11}$$

for all $x \in \mathbb{R}$. This implies that the *dynamical mappings* defined by $f(x)$ and $g(x)$ have identical topological properties. In particular, they have the same number of fixed and periodic points. Hence, for certain values of a, b and c, the equation (4) has exactely k distinct real zeros for all integer $k \geq 1$. Indeed, equation (11) is equivalent to:

$$\left(-rp^2 - ap\right)x^2 + \left(-r\left(p\left(q-1\right)+pq\right) - bp\right)x - \left(q + cp + qr\left(q-1\right)\right) = 0 \tag{12}$$

for all $x \in \mathbb{R}$. This implies that all the coefficients in (12) are zero, that is, if $r \neq 0$, then $p = -\frac{a}{r}$, $q = \frac{r-b}{2r}$ and $c = \frac{b^2-2b-r^2+2r}{4a}$ since $a \neq 0$. Finally, there exists infinitely many functions of the form $f(x) = ax^2 + bx + \frac{b^2-2b-r^2+2r}{4a}$ such that the equation (4) has exactely k distinct real zeros for all integer $k \geq 1$.

3 Counting zeros of some polynomials of degree 2^k

For $k \geq 1$, let us consider an arbitrary polynomial function of degree 2^k of the form:

$$p_{2^k}(x) = a_{2^k}x^{2^k} + a_{2^k-1}x^{2^k-1} + ... + a_1x + a_0 = \sum_{j=0}^{2^k} a_jx^j, a_{2^k} \neq 0 \tag{13}$$

and consider $w_{2^k}(x) = f^k(x) - x$ as a polynomial function of the form:

$$w_{2^k}(x) = \delta_{2^k} x^{2^k} + \delta_{2^k-1} x^{2^k-1} + ... + (\delta_1 - 1)x + \delta_0 = \sum_{j=0}^{2^k} \delta_j x^j, \delta_{2^k} \neq 0 \quad (14)$$

which has exactely k real zeros (generally, $w_{2^k}(x)$ can be considered as any polynomial function with k real zeros). We wante to find some conditions on the coefficients $a_0, a_2, ...a_{2^k}$ in which the number of real zeros of $p_{2^k}(x)$ is exactely the integer $k \geq 1$. To do this, we use the realtion conjugation relation:

$$(p_{2^k} \circ h)(x) = (h \circ w_{2^k})(x) \quad (15)$$

with $h(x) = px + q$ $(p \neq 0)$ for all $x \in \mathbb{R}$ and we get the equation:

$$a_{2^k}(px+q)^{2^k} + ... + a_1(px+q) + a_0 - \left(p\delta_{2^k}x^{2^k} + ... + p(\delta_1 - 1)x + p\delta_0 + q\right) = 0 \quad (16)$$

By using the *Newton's binomial theorem* we get:

$$\begin{cases} (px+q)^{2^k} = q^{2^k} + p^{2^k}x^{2^k} + 2^k p^{2^k-1} q x^{2^k-1} + \sum_{m=1}^{2^k-2} \frac{2^k!}{m!\left(2^k-m\right)!} p^m x^m q^{2^k-m} \\ (px+q)^{2^k-1} = q^{2^k-1} + p^{2^k-1}x^{2^k-1} + \sum_{m=1}^{2^k-2} \frac{\left(2^k-1\right)!}{m!\left(2^k-1-m\right)!} p^m x^m q^{2^k-1-m} \\ ..... \\ (px+q)^{2^k-d} = q^{2^k-d} + p^{2^k-d}x^{2^k-d} + \sum_{m=1}^{2^k-d-1} \frac{\left(2^k-d\right)!}{m!\left(2^k-d-m\right)!} p^m x^m q^{2^k-d-m}, d = 0, ..., 2^k \end{cases} \quad (17)$$

Thus, the coefficient of the polynomial in the right side of (16) are given by:

$$\begin{cases} x^{2^k} : a_{2^k} p^{2^k} - p\delta_{2^k} \\ x^{2^k-1} : a_{2^k} 2^k p^{2^k-1} q + a_{2^k-1} p^{2^k-1} - p\delta_{2^k-1} \\ .... \\ x^0 : \sum_{m=0}^{2^k} a_m q^m - p\delta_0 - q \end{cases} \quad (18)$$

Letting these coefficient equal to zero (because (15) holds for all $x \in \mathbb{R}$) we can find the expressions (with possible conditions) for the coefficients $(a_i)_{0 \leq i \leq 2^k}$ by solving the system of equations:

$$\left\{\begin{array}{c} a_{2^k}p^{2^k} - p\delta_{2^k} = 0 \\ a_{2^k}2^k p^{2^k-1}q + a_{2^k-1}p^{2^k-1} - p\delta_{2^k-1} = 0 \\ \dots. \\ \sum_{m=0}^{2^k} a_m q^m - p\delta_0 - q = 0 \end{array}\right. \tag{19}$$

If the system of equations (19) has one real solution with respect to p and q and no contradictions in the expressions of $(a_i)_{0\le i\le 2^k}$ and their possible conditions, then it is easy to prove the following theorem:

Theorem 6 *For any polynomial function of degree 2^k with k distinct real zeros, there exist another non trivial polynomial function of degree 2^k with k distinct real zeros.*

If (19) has no real solutions or it has several real solutions with respect to p and q then no polynomial function of degree 2^k with k distinct real zeros can exists by using this method.

As some examples, let us consider the cases $k = 1$ and $k = 2$: For $k = 1$, we get $p_2(x) = a_2x^2 + a_1x + a_0$ $(a_2 \neq 0), w_2(x) = \delta_2x^2 + \delta_1x + \delta_0$ $(\delta_2 \neq 0, \delta_1^2 - 4\delta_2\delta_0 = 0)$, $p = \frac{\delta_2}{a_2}, q = \frac{\delta_1 - a_1}{2a_2}$ and $a_0 = \frac{2\delta_1 - 2a_1 - \delta_1^2 + a_1^2 + 4\delta_0\delta_2}{4a_2}$. For $k = 2$, we get $p_4(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$ $(a_4 \neq 0)$, $w_4(x) = \delta_4x^4 + \delta_3x^3 + \delta_2x^2 + \delta_1x + \delta_0$ $(\delta_4 \neq 0)$, $p = \sqrt[3]{\frac{\delta_4}{a_4}}$, $q = \frac{\delta_3 - p^2a_3}{4p^2a_4}$, $a_0 = q + p\delta_0 - qa_1 - q^2a_2 - q^3a_3 - q^4a_4, a_1 = \delta_1 - 2qa_2 - 3q^2a_3 - 4q^3a_4$ and $a_2 = -\frac{1}{p}\left(-\delta_2 + 3pqa_3 + 6pq^2a_4\right)$.

4 Open problems

In this section, we propose the following open problems:

Problem 1 Is it possible that the above analysis given for f defined by (1) can be extended to arbitrary polynomials of the form $p(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ $(n \geq 2)$ and checking the conditions for a periodic point of period three? If so, then any equation of the form $p^k(x) - x = 0$ has exactely k zeros. The choice of f in (1) is the minimum case where periodic three point can be found since chaos cannot appear in linear mapping.

Problem 2 Is a similar approach possible for the case of odd degree polynomails?

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