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Ascending Series Solution to Airy's Inhomogeneous Boundary Value Problem

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Abstract

A two-point boundary value problem involving the non-homogeneous differential equation of Sir George Biddell Airy with a right-hand-side that is a function of the independent variable, is considered. Solution is expressed in terms of two recently introduced integral functions that are expressible in terms of ascending series. Evaluation of these functions gives rise to an open problem that is stated in this work.

Keywords: *Airy's inhomogeneous equation, variable forcing function, ascending series representation, Nield-Kuznetsov function*

1 Introduction

Airy's differential equation continues to receive considerable attention in the literature due to its mathematical implications and physical applications, [6]. Mathematically, finding and expressing solutions to Airy's equation provides us with further insights into series convergence and the introduction of innovative series techniques. In addition, advances in the knowledge of behaviour of this equation and its solution can provide us with the ability to reduce mathematical and physical problems to Airy's equation. Recently, the use of Airy's functions in the study of flow over porous layers, [5], resulted in the introduction of a new function that was further analyzed by Hamdan and Kamel [3] to discover its salient features and the differential equations that this function satisfies. From an applications point of view, Airy's equation arises in various applications of mathematical physics, including fluid flow and the study of optics. [1,4].

Although analysis of Airy's equation has been directed in the main part to the homogeneous Airy's differential equation and the inhomogeneous equation with a constant forcing function, recent analysis has branched into the inhomogeneous equation with a variable forcing function. In fact, in a recent article, [2], an initial value problem was considered in which solution to the variable forcing function equation was obtained in terms of two recently introduced integral functions, whose approximations were obtained using asymptotic series. These same integral functions arise in two-point boundary value problem associated with Airy's differential equation with a variable right-hand-side, as will be illustrated in the current work. The goal here is to analyze the two-point boundary value problem, find the general solution and express it in ascending series form, and deduce a solution satisfying the boundary conditions. The method used, that is ascending series approximations of the resulting integral functions, is also valid for initial boundary value problem. We end this work with a statement of an open problem associated with series approximations.

2 Problem Formulations

We wish to solve the following inhomogeneous Airy's ordinary differential equation

(1) $$y'' - xy = f(x)$$

With two-point boundary conditions expressed as

(2) $$y(a) = \alpha$$

(3) $$y(b) = \beta$$

where $\alpha, \beta \in \Re$ and $x \in [a,b]$.

Equation (1) is the well-known Airy's differential equation that has its roots in the 19th Century work of Airy, and continues to receive considerable attention in the literature due to its applications in mathematical physics (*c.f.* [1,6] and the references therein).

General solution to (1) has been discussed in detail in [3], and takes the form

$$y_g = c_1 A_i(x) + c_2 B_i(x) + \pi\{K_i(x) - f(x)N_i(x)\} \tag{4}$$

In which c_1 and c_1 are arbitrary constants, and $A_i(x)$ and $B_i(x)$ are the well-known Airy's functions of the first and second kind, respectively, defined by, [6]:

$$A_i(x) = \frac{1}{\pi}\int_0^\infty \cos(xt + \frac{1}{3}t^3)dt \tag{5}$$

$$B_i(x) = \frac{1}{\pi}\int_0^\infty [\sin(xt + \frac{1}{3}t^3) + \exp(xt - \frac{1}{3}t^3)]dt \tag{6}$$

and have the following non-zero wronskian, [1]

$$w(A_i(x), B_i(x)) = A_i(x)B_i'(x) - B_i(x)A_i'(x) = \frac{1}{\pi}. \tag{7}$$

The function $N_i(x)$ is the recently introduced Nield-Kuznetsov function, [5], defined in terms of Airy's functions as

$$N_i(x) = A_i(x)\int_0^x B_i(t)dt - B_i(x)\int_0^x A_i(t)dt \tag{8}$$

and has the following integral representation

$$N_i(x) = \frac{1}{3\pi}\int_0^\infty [2\sin(xt + \frac{1}{3}t^3) - \exp(xt - \frac{1}{3}t^3)]dt. \tag{9}$$

The function, $K_i(x)$, introduced in [3], is defined by either of the following forms:

$$K_i(x) = A_i(x)\int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} f'(t)dt - B_i(x)\int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} f'(t)dt \tag{10}$$

$$K_i(x) = f(x)N_i(x) - \left\{A_i(x)\int_0^x f(t)B_i(t)dt - B_i(x)\int_0^x f(t)A_i(t)dt\right\} \tag{11}$$

Now, upon using boundary conditions (2) and (3) in (4), we can obtain the following values of the arbitrary constants and render the boundary value problem solved:

$$c_1 = \frac{\pi[f(b)N_i(b) - K_i(b) + \frac{\beta}{\pi}]B_i(a) - \pi[f(a)N_i(a) - K_i(a) + \frac{\alpha}{\pi}]B_i(b)}{[A_i(b)B_i(a) - A_i(a)B_i(b)]} \tag{12}$$

$$c_2 = \frac{\pi[f(b)N_i(b) - K_i(b) + \frac{\beta}{\pi}]A_i(a) - \pi[f(a)N_i(a) - K_i(a) + \frac{\alpha}{\pi}]A_i(b)}{[A_i(a)B_i(b) - A_i(b)B_i(a)]} \tag{13}$$

It is clear from the above discussion that finding the arbitrary constants, and evaluating the solution over the interval *[a,b]*, necessitates evaluating $A_i(x)$, $B_i(x)$, $N_i(x)$ and $K_i(x)$ for $a \le x \le b$. While this is a challenging task, we will attempt to approximate these integral functions on the given interval using series approximations.

3 Ascending Series Approximations

Airy's functions, $A_i(x)$ and $B_i(x)$, their derivatives and integrals can be approximated using infinite series. Two popular series approximations are the asymptotic series and ascending series methods that have been reported in the literature (*cf.* [1,6] and the references therein). These same series can be used to derive series expressions for $N_i(x)$ and $K_i(x)$. Asymptotic series approximations have been discussed in a previous article, [2], in connection with initial value problems. We will in the current work discuss ascending series approximations for the boundary value problem at hand.

Ascending series representations for *Ni(x), Ki(x)* and their derivatives and integrals are developed using the ascending series representations of $A_i(x)$ and $B_i(x)$, their derivatives and integrals, reported in [6].

Letting

$$a_1 = A_i(0) \approx 0.3550280538878172 \tag{14}$$

$$a_2 = -A_i'(0) \approx 0.2588194037928067 \tag{15}$$

(16) $$F_1(x) = \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}$$

(17) $$F_2(x) = \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!}$$

(18) $$F_1'(x) = \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!}$$

(19) $$F_2'(x) = \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}$$

where $(b)_k$ is the Pochhammer symbol, defined as, [6]:

(20) $$(b)_0 = 1$$

(21) $$(b)_k = \frac{\Gamma(b+k)}{\Gamma(b)} = b(b+1)(b+2)...(b+k-1); k > 0$$

the following representations can then be obtained, [12]:

(22) $$A_i(x) = a_1 F_1'(x) - a_2 F_2'(x)$$

(23) $$B_i(x) = \sqrt{3}[a_1 F_1'(x) + a_2 F_2'(x)]$$

(24) $$\int_0^x A_i(t)dt = a_1 F_1(x) - a_2 F_2(x)$$

(25) $$\int_0^x B_i(t)dt = \sqrt{3}a_1 F_1(x) + \sqrt{3}a_2 F_2(x)\,.$$

Using (16)-(21) in (22)-(25) we obtain the following ascending series representations of Airy's functions and their integrals:

(26) $$A_i(x) = a_1 \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} - a_2 \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}$$

(27) $$B_i(x) = \sqrt{3}a_1 \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} + \sqrt{3}a_2 \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}$$

(28) $$\int_0^x A_i(t)dt = a_1 \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} - a_2 \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!}$$

(29) $$\int_0^x B_i(t)dt = \sqrt{3}a_1 \sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} + \sqrt{3}a_2 \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!}\,.$$

In order to obtain an ascending series representation for $N_i(x)$, we substitute (22)-(25) in (8) to get either

$$N_i(x) = 2\sqrt{3}a_1 a_2 [F_1'(x)F_2(x) - F_2'(x)F_1(x)] \tag{30}$$

or

$$N_i(x) = A_i(x)[\sqrt{3}a_1 F_1(x) + \sqrt{3}a_2 F_2(x)] - B_i(x)[a_1 F_1(x) - a_2 F_2(x)]. \tag{31}$$

Equations (30) and (31) have the following equivalent summation expressions, respectively

$$N_i(x) = 2\sqrt{3}a_1 a_2 \left[\begin{array}{l} \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!} \right\} \left\{ \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \right\} \\ - \left\{ \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \right\} \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} \right\} \end{array} \right] \tag{32}$$

$$\begin{aligned} N_i(x) = A_i(x) &\left\{ \sqrt{3}a_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} + \sqrt{3}a_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \right\} \\ - B_i(x) &\left\{ a_1 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!} - a_2 \sum_{k=0}^{\infty} \left(\frac{2}{3}\right)_k \frac{3^k x^{3k+2}}{(3k+2)!} \right\}. \end{aligned} \tag{33}$$

In order to obtain an ascending series representation for $K_i(x)$, we express the integrals on RHS of (10) in terms (22) to (25) as follows:

$$\int_0^x \left\{ \int_0^t A_i(\tau)d\tau \right\} f'(t)dt = \int_0^x \{ a_1 f'(t)F_1(t) - a_2 f'(t)F_2(t) \} dt \tag{34}$$

and

$$\int_0^x \left\{ \int_0^t B_i(\tau)d\tau \right\} f'(t)dt = \int_0^x \{ \sqrt{3}a_1 f'(t)F_1(t) + \sqrt{3}a_2 f'(t)F_2(t) \} dt. \tag{35}$$

Upon integrating by parts and using (18) and (19), we obtain

$$\int_0^x f'(t)F_1(t)dt = f(x)F_1(x) - \int_0^x f(t) \left\{ \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)_k \frac{3^k t^{3k}}{(3k)!} \right\} dt \tag{36}$$

and

$$\int_0^x f'(t)F_2(t)dt = f(x)F_2(x) - \int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt. \tag{37}$$

Using (36) and (37) in (34) and (35) we obtain, respectively

$$\begin{aligned}\int_0^x \left\{\int_0^t A_i(\tau)d\tau\right\} f'(t)dt &= a_1\left[f(x)F_1(x) - \int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k t^{3k}}{(3k)!}\right\}dt\right] \\ &- a_2\left[f(x)F_2(x) - \int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt\right]\end{aligned} \tag{38}$$

$$\begin{aligned}\int_0^x \left\{\int_0^t B_i(\tau)d\tau\right\} f'(t)dt &= \sqrt{3}a_1\left[f(x)F_1(x) - \int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k t^{3k}}{(3k)!}\right\}dt\right] \\ &+ \sqrt{3}a_2\left[f(x)F_2(x) - \int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt\right].\end{aligned} \tag{39}$$

Equation (10) then takes the following form with the help of (38) and (39):

$$\left\{\begin{aligned}&K_i(x) = A_i(x)f(x)[\sqrt{3}a_1F_1(x) + \sqrt{3}a_2F_2(x)] \\ &+ B_i(x)f(x)[a_2F_2(x) - a_1F_1(x)] \\ &- A_i(x)\left[\sqrt{3}a_1\int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k t^{3k}}{(3k)!}\right\}dt + \sqrt{3}a_2\int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt\right] \\ &+ B_i(x)\left[a_1\int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k t^{3k}}{(3k)!}\right\}dt - a_2\int_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt\right]\end{aligned}\right. \tag{40}$$

which simplifies to the following final form of $K_i(x)$:

$$(41)\qquad \begin{cases} K_i(x) = f(x)N_i(x) - \left[\sqrt{3}a_2 A_i(x) + a_2 B_i(x)\right] * \\ \int\limits_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt \\ -\left[\sqrt{3}a_1 A_i(x) - a_1 B_i(x)\right]\int\limits_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k t^{3k}}{(3k)!}\right\}dt \end{cases}.$$

Ascending series form of the general solution (4) is obtained by substituting (26), (27), (33) and (41) in equation (4). After some simplification we obtain:

$$(42)\qquad \begin{cases} y_g = \left(a_2[\sqrt{3}c_2 - c_1] + 2\sqrt{3}\pi a_1 a_2 \int\limits_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k t^{3k}}{(3k)!}\right\}dt\right) * \\ \left[\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}\right] \\ +\left(a_1[\sqrt{3}c_2 + c_1] - 2\sqrt{3}\pi a_1 a_2 \int\limits_0^x f(t)\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k t^{3k+1}}{(3k+1)!}\right\}dt\right) * \\ \left[\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!}\right] \end{cases}$$

Values of the arbitrary constants are obtained from (12) and (13) once the values of $A_i(x), B_i(x), N_i(x), K_i(x)$ at $x = a$ and $x = b$ are obtained from (26), (27), (33) and (41). A concrete example is implemented in the following section to illustrate the procedure.

4 An Illustrative Example

Suppose we are required to solve equation (1) with $f(x) = \sqrt{x}$ with the boundary conditions $y(0) = 0$ and $y(1) = 1$. Substituting $f(x) = \sqrt{x}$ in (41) and (42), and integrating, the function $K_i(x)$ and the general solution take the forms, respectively:

$$(43)\quad \begin{cases} K_i(x) = \sqrt{x}N_i(x) - \left[\sqrt{3}a_2 A_i(x) + a_2 B_i(x)\right]* \\ \left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+5/2}}{(3k+1)!(3k+5/2)}\right\} \\ -\left[\sqrt{3}a_1 A_i(x) - a_1 B_i(x)\right]\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k+3/2}}{(3k)!(3k+3/2)}\right\} \end{cases}$$

$$(44)\quad \begin{cases} y_g = \left(a_2[\sqrt{3}c_2 - c_1] + 2\sqrt{3}\pi a_1 a_2\left\{\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k+3/2}}{(3k)!(3k+3/2)}\right\}\right)* \\ \left[\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+1}}{(3k+1)!}\right] \\ +\left(a_1[\sqrt{3}c_2 + c_1] - 2\sqrt{3}\pi a_1 a_2\left\{\sum_{k=0}^{\infty}\left(\frac{2}{3}\right)_k \frac{3^k x^{3k+5/2}}{(3k+1)!(3k+5/2)}\right\}\right)* \\ \left[\sum_{k=0}^{\infty}\left(\frac{1}{3}\right)_k \frac{3^k x^{3k}}{(3k)!}\right] \end{cases}.$$

In order to satisfy the given boundary conditions, we find the arbitrary constants, c_1 and c_2, given by equations (12) and (13). This requires calculating $A_i(x), B_i(x), N_i(x), K_i(x)$ and *f(x)* at $x = 0$ and $x = 1$. Definitions (8) and (11) give us the values $N_i(0) = 0$ and $K_i(0) = 0$. The literature, [1,6], reports the values $A_i(0) = 0.3550280538878172$ and $B_i(0) = \sqrt{3}A_i(0) = 0.6149266274460007$. We thus need to calculate $A_i(1), B_i(1), N_i(1), K_i(1)$. These are computed from equations (26), (27), (33) and (43) where we have used the first five terms of the series and retained 16 significant digits in the calculations. It is noted that the first omitted term (sixth term) in each of the series (16)-(19) is less than 10^{-8} at $x = 1$. This is considered as a measure of accuracy of the computed solutions.

The following values are thus obtained:

$A_i(1) = 0.1352924154742438$

$B_i(1) = 1.207423592970825$

$N_i(1) = -0.1672560920565251$

$K_i(1) = -0.0791473538868198$

and the values of arbitrary constants, computed using (12) and (13), are:

$$c_1 = -1.287255812781822$$
$$c_2 = 0.7431974900254956\,.$$

Upon using these values of arbitrary constants in (44), equation (44) represents the complete solution to the boundary value problem.

4 Conclusion

In this work, we have considered Airy's non-homogeneous equation with a right-hand-side that is a function of the independent variable, subject to boundary conditions. Solution is cast in terms of four integral functions (the two Airy's functions and two recently introduced integral functions). All functions are expressed in terms of ascending series in order to facilitate accurate computations.

5 Open Problem

Accurate evaluation of the integral functions $N_i(x)$ and $K_i(x)$ represents a challenge at two levels: 1) Both functions are defined in terms of the functions $A_i(x)$ and $B_i(x)$, thus embedding a dependence of $N_i(x)$ and $K_i(x)$ on Airy's functions. 2) Evaluation of $N_i(x)$ and $K_i(x)$ relies heavily on our ability to accurately evaluate Airy's functions, and choosing the most efficient representation.

The above two points warrant further investigation into the possibility of finding different or better representations of the integral functions in terms of elementary functions, or in terms of functions with less elaborate computational requirements.

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