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An Efficient Algorithm for Generalized Fisher's Equation by the Variational Homotopy Perturbation Method

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Abstract

In this paper, we apply the variational homotopy perturbation method to resolve the generalized Fisher's equation. The numerical solutions are obtained as series solutions and compared with the exact solution. The result showed that this algorithm based on the VHPM method is very effective and can be applied to other problems.

Keywords: *Generalized Fisher's equation, Variational homotopy perturbation method, Nonlinear equations, Soliton solution.*
AMS Subject Classifications: 35C08, 35C10, 35Q51, 65N10.

1 Introduction

There is no secret to the researcher in the field of nonlinear partial differential equations, that the solution of this class of equations is not easy. So we find that many researchers have done and are still great efforts to find methods to solve this type of equations. These efforts resulted in the consolidation of this research field in many methods, we find among them the variational iteration method (VIM) that established by He ([4], [13]) and the homotopy perturbation method that established also by He in 1998 ([7], [11], [9], [12]). But recently, we find that Muhammad Aslam Noor and Syed Tauseef Mohyud-Din [14] have provided a new method based on the combined between these two methods and became known by variational homotopy perturbation method, in

short written as VHPM. This method is applied by many researchers to solve various linear and nonlinear problems (see [16], [15], [17], [19]).
Our concern in this work is to apply the VHPM for solving the generalized Fisher equation of the form

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1 - u^{\beta}), \tag{1}$$

The simplest form of Eq. (1) is

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1 - u), \tag{2}$$

and it was proposed by Fisher [1] as a model for the propagation of a mutant gene. Fisher's equation is widely used in physical processes and do not have a precise analytical solution till 1979 where a particular solution of Eq. (2) considered in [6], was found by Ablowitz and Zepetella [2]. The exact solution of (2) is given by

$$u(x,t) = \left(\frac{1}{1 + e^{\sqrt{\frac{\alpha}{6}}x - \frac{5}{6}\alpha t}} \right)^2, \tag{3}$$

and the exact solution of

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1 - u^{\beta}) \tag{4}$$

is given by

$$u(x,t) = \left(\frac{1}{2} + \frac{1}{2}\mathrm{Tanh}\left[\frac{b}{2} - \frac{\beta\left(x - \frac{t(4+\beta)}{\sqrt{4+2\beta}}\right)}{2\sqrt{4+2\beta}} \right] \right)^{\frac{2}{\beta}}, \tag{5}$$

where $b \in (-\infty, +\infty)$.

2 Variational Homotopy Perturbation Method

Muhammad Aslam Noor and Syed Tauseef Mohyud-Din [14] gives the idea of the basis of this method, they consider the following general differential equation

$$Lu + Nu = g(x), \tag{6}$$

where N, L are nonlinear and linear operator respectively, $g(x)$ is an inhomogeneous term. According to the variational iteration method, we can construct a correct functional

$$\begin{aligned} u_{n+1} &= u_n + \int_0^t \lambda(\tau) L\left(u_n(x,\tau)\right) d\tau \\ &+ \int_0^t \lambda(\tau) N\left(\tilde{u}_n(x,\tau)\right) d\tau - \int_0^t \lambda(\tau) g(x,\tau) d\tau, \end{aligned} \tag{7}$$

where λ is a general Lagrange multiplier, which can be identified optimally via VIM. The subscripts n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation. That is, $\delta\tilde{u}_n(t) = 0$ and (7) is called a correct functional. Now, we apply the homotopy perturbation method

$$\sum_{i=0}^{\infty} p^i u_i = u_0 + p\int_0^t \lambda(\tau)\left(\sum_{i=0}^{\infty} p^i L(u_n) + \sum_{i=0}^{\infty} p^i N(u_i)\right) d\tau - p\int_0^t \lambda(\tau) g(x,\tau) d\tau, \tag{8}$$

which is the variational homotopy perturbation method and is formulated by the coupling of VIM and Adomian's polynomials. A comparison of like powers of p gives solutions of various orders.

3 VHPM for the generalized Fisher's equation

we consider the following generalized Fisher's equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \alpha u(1-u^\beta) \\ u(x,0) = f(x) \end{cases} . \tag{9}$$

According to the VIM method, the correction variational functional of equation (9) given by

$$u_{k+1} = u_k + \int_0^t \lambda(\tau)\left(\frac{\partial u_k}{\partial \tau} - \frac{\partial^2 u_k}{\partial x^2}\right) - \int_0^t \lambda(\tau)\left(\alpha u_k(1-(u_k)^\beta)\right) d\tau. \tag{10}$$

The Lagrange multiplier is calculated optimally via variational theory and it yields the stationary conditions $\left\{\begin{array}{c} \lambda' = 0 \\ \lambda + 1 = 0 \end{array}\right\}$. Hence, the general Lagrange multiplier $\lambda = -1$.

Substituting this value $\lambda = -1$ into (10), we have the iteration formula

$$u_{k+1} = u_k - \int_0^t \left(\frac{\partial u_k}{\partial \tau} - \frac{\partial^2 u_k}{\partial x^2}\right) d\tau - \int_0^t \left(\alpha u_k(1-(u_k)^\beta)\right) d\tau. \tag{11}$$

While applying the variational homotopy perturbation method, one obtains

$$\begin{aligned} u_0 + pu_1 + p^2u_2 + ... &= f(x) + p\int_0^t \frac{\partial^2}{\partial^2 x}\left(\sum_{i=0}^{\infty} p^i u_i(x,\tau)\right) d\tau \\ &+ p\int_0^t \alpha\left(\sum_{i=0}^{\infty} p^i u_i(x,\tau))\right)\left(1-(\sum_{i=0}^{\infty} p^i u_i(x,\tau))^{\beta}\right) d\tau. \end{aligned} \tag{12}$$

Comparing the coefficients of like powers of p, one obtains

$$\begin{aligned} &p^0 : \ u_0(x,t) = f(x), \\ &p^1 : u_1(x,t) = \textstyle\int_0^t \left\{\alpha u_0 + \frac{\partial^2 u_0}{\partial x^2} - \alpha(u_0)^{\beta+1}\right\} d\tau, \\ &p^2 : u_2(x,t) = \textstyle\int_0^t \left(\alpha u_1 + \frac{\partial^2 u_1}{\partial x^2}\right) d\tau - \int_0^t \alpha(\beta+1)u_1(u_0)^{\beta} d\tau, \\ &p^3 : u_3(x,t) = \textstyle\int_0^t \left(\alpha u_2 + \frac{\partial^2 u_2}{\partial x^2}\right) - \int_0^t \alpha(\beta+1)u_2(u_0)^{\beta} d\tau - \int_0^t \alpha\beta(\frac{\beta+1}{2})(u_1)^2(u_0)^{\beta-1} d\tau, \\ &\vdots \end{aligned} \tag{13}$$

So, we obtain the components which constitute $u(x,t)$ and thus, we will have

$$u(x,t) = u_0 + u_1 + u_2 + u_3 + \cdots \tag{14}$$

For later numerical computation, we let the expression

$$\varphi_n(x,t) = \sum_{i=0}^{n} u_i(x,t), \tag{15}$$

to denote the n-term approximation to $u(x,t)$.

4 Numerical results

In this section, we will solve the Fisher's equation according to the following two cases: first case for $\alpha = 6$, $\beta = 1$, second case for $\alpha = 1$, $\beta = 6$.

Example 4.1

First, we consider the equation (9) with $\alpha = 6$, $\beta = 1$

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 6u(1-u), \tag{16}$$

with the initial condition

$$u(x,0) = \frac{1}{(1+e^x)^2}. \tag{17}$$

The exact solution of (16)–(17) is

$$u(x,t) = \frac{1}{(1+e^{(x-5t)})^2}. \tag{18}$$

By using the equations (13), we have

$$\begin{array}{l} p^0:\ u_0(x,t)=\frac{1}{(1+e^x)^2},\\ p^1: u_1(x,t)=\int_0^t\left\{6u_0+\frac{\partial^2 u_0}{\partial x^2}-6(u_0)^2\right\}d\tau,\\ p^2: u_2(x,t)=\int_0^t\left\{6u_1+\frac{\partial^2 u_1}{\partial x^2}-12u_1u_0\right\}d\tau,\\ p^3: u_3(x,t)=\int_0^t\left\{6u_2+\frac{\partial^2 u_2}{\partial x^2}-12u_2u_0-6(u_1)^2\right\}d\tau,\\ \vdots \end{array} \tag{19}$$

This yields to

$$\begin{array}{c} u_0(x,t)=\frac{1}{(1+e^x)^2},\\ u_1(x,t)=\frac{10e^x}{(1+e^x)^3}t,\\ u_2(x,t)=\frac{25e^x(2e^x-1)}{(1+e^x)^4}t^2.\\ u_3(x,t)=\frac{125e^x(4e^{2x}-7e^x+1)}{3(1+e^x)^5}t^3,\\ \vdots \end{array} \tag{20}$$

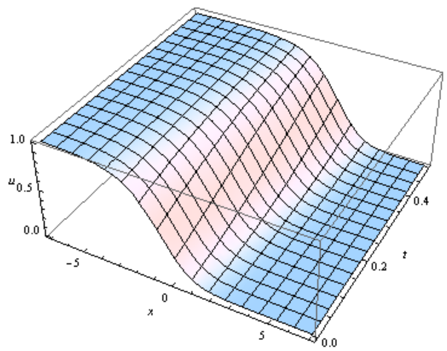
So according to (15), we obtain the 3-term approximation $\varphi_3(x,t)$ to $u(x,t)$, which constitutes the third order approximation of $u(x,t)$, as

$$\begin{aligned} \varphi_3(x,t) &= \sum_{i=0}^{3} u_i(x,t)=\frac{1}{(1+e^x)^2}+\frac{10e^x}{(1+e^x)^3}t \\ &+ \frac{25e^x(2e^x-1)}{(1+e^x)^4}t^2+\frac{125e^x(4e^{2x}-7e^x+1)}{3(1+e^x)^5}t^3. \end{aligned} \tag{21}$$

Now, an expansion of the exact solution (18) in Taylor series over $t=0$ to order 3 gives

$$\begin{aligned} u(x,t) &= \frac{1}{(1+e^x)^2}+\frac{10e^x}{(1+e^x)^3}t+\frac{25e^x(2e^x-1)}{(1+e^x)^4}t^2 \\ &+ \frac{125e^x(4e^{2x}-7e^x+1)}{3(1+e^x)^5}t^3+O[t]^4, \end{aligned} \tag{22}$$

which is exactly the same as obtained by the VHPM in (21). So we can see that this method is more effective and gives approximation solution in the form of a development in Taylor series of the exact solution.



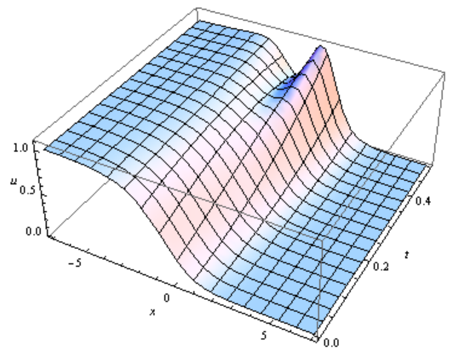


Figure 1: (Left): Exact solution (18) of Eqs. (16)-(17); (Right): Series approximation solution of Eqs. (16)-(17) by VHPM method with four terms.

Example 4.2

Second, we consider the equation (9) with $\alpha = 1$ and $\beta = 6$ defined as

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = u(1 - u^6), \tag{23}$$

with the initial condition

$$u(x, 0) = \frac{1}{\left(1 + e^{\frac{3}{2}x}\right)^{1/3}}. \tag{24}$$

The exact solution of (23)–(24) is given by

$$u(x, t) = \left(\frac{1}{2} - \frac{1}{2}\mathrm{Tanh}\left[\frac{3}{4}\left(-\frac{5t}{2} + x\right)\right]\right)^{\frac{1}{3}}, \tag{25}$$

or in the same way

$$u(x, t) = \left(\frac{1}{1 + e^{\frac{3}{4}(-5t+2x)}}\right)^{1/3}. \tag{26}$$

By using the equations (13), we have

$$\begin{aligned}
&p^0 : \ u_0(x,t) = \tfrac{1}{(1+e^{3x/2})^{1/3}},\\
&p^1 : \ u_1(x,t) = \textstyle\int_0^t \left\{u_0 + \tfrac{\partial^2 u_0}{\partial x^2} - (u_0)^7\right\} d\tau,\\
&p^2 : \ u_2(x,t) = \textstyle\int _0^t \left\{u_1 + \tfrac{\partial^2 u_1}{\partial x^2} - 7u_1(u_0)^6\right\} d\tau,\\
&p^3 : \ u_3(x,t) = \textstyle\int_0^t \left\{u_2 + \tfrac{\partial^2 u_2}{\partial x^2} - 7u_2(u_0)^6 - 21(u_1)^2(u_0)^5\right\} d\tau,\\
&\vdots
\end{aligned} \tag{27}$$

This yields to

$$
\begin{aligned}
u_0\left(x,t\right) &= \tfrac{1}{(1+e^{\frac{3x}{2}})^{1/3}},\\
u_1\left(x,t\right) &= \tfrac{5}{4}e^{\frac{3x}{2}}\left(\tfrac{1}{1+e^{3x/2}}\right)^{4/3}t,\\
u_2\left(x,t\right) &= \tfrac{25}{32}e^{\frac{3x}{2}}\left(-3+e^{\frac{3x}{2}}\right)\left(\tfrac{1}{1+e^{\frac{3x}{2}}}\right)^{7/3}t^2,\\
u_3\left(x,t\right) &= \frac{125e^{\frac{3x}{2}}\left(\frac{1}{1+e^{3x/2}}\right)^{\frac{10}{3}}\left(9-18e^{\frac{3x}{2}}+e^{3x}\right)}{384}t^3,\\
&\vdots
\end{aligned}
\tag{28}
$$

So according to (15), we obtain the 3-term approximation $\varphi_3(x,t)$ to $u(x,t)$, which constitutes the third order approximation of $u(x,t)$, as

$$
\begin{aligned}
\varphi_3(x,t) &= \sum_{i=0}^{3}u_i(x,t) \qquad (29)\\
&= \frac{1}{(1+e^{\frac{3x}{2}})^{1/3}}+\frac{5}{4}te^{\frac{3x}{2}}\left(\frac{1}{1+e^{\frac{3x}{2}}}\right)^{4/3}+\frac{25}{32}e^{\frac{3x}{2}}\left(-3+e^{\frac{3x}{2}}\right)\left(\frac{1}{1+e^{\frac{3x}{2}}}\right)^{7/3}t^2\\
&+ \frac{125e^{\frac{3x}{2}}\left(\frac{1}{1+e^{3x/2}}\right)^{10/3}\left(9-18e^{\frac{3x}{2}}+e^{3x}\right)t^3}{384}.
\end{aligned}
$$

An expansion of the exact solution (26) in Taylor series over $t=0$ to order 3 gives

$$
\begin{aligned}
U_3(x,t)) &= \left(\frac{1}{1+e^{\frac{3x}{2}}}\right)^{1/3}+\frac{5}{4}e^{\frac{3x}{2}}\left(\frac{1}{1+e^{\frac{3x}{2}}}\right)^{4/3}t\\
&= \frac{25}{32}e^{\frac{3x}{2}}\left(-3+e^{\frac{3x}{2}}\right)\left(\frac{1}{1+e^{3x/2}}\right)^{7/3}t^2\\
&+ \frac{125e^{\frac{3x}{2}}\left(\frac{1}{1+e^{\frac{3x}{2}}}\right)^{\frac{10}{3}}\left(9-18e^{\frac{3x}{2}}+e^{3x}\right)t^3}{384}+O[t]^4,
\end{aligned}
\tag{30}
$$

which is exactly the same as obtained by the VHPM in (29). So we can see that this method is more effective and gives approximation solution in the form of a development in Taylor series of the exact solution.

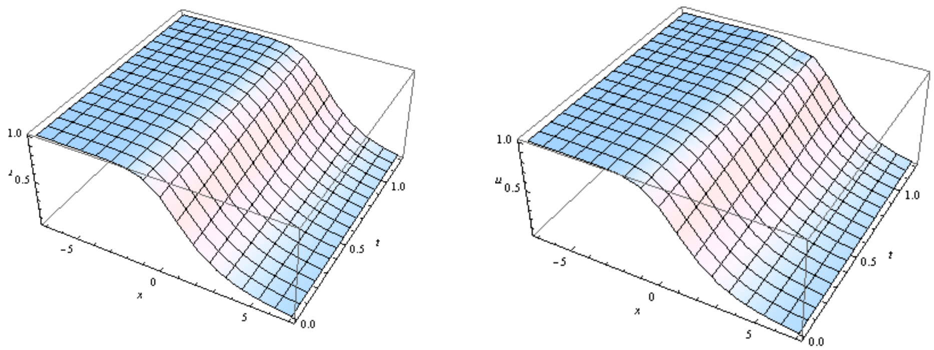


Figure 2: (Left): Exact solution (26) of Eqs. (23)–(24); (Right): Series approximation solution of Eqs. (23)–(24) by VHPM method with four terms.

5 Conclusion

In this work, the variational homotopy perturbation method (VHPM) is considered for solving the generalized Fisher's equation. The numerical results obtained with our algorithm for the different values of α and β showed that the VHPM is a powerful and reliable method for finding the approximate analytical solutions of the generalized Fisher's equation.

6 Open Problem

Several methods have been applied to find the solution of Fisher's equation as: ADM, VIM, HPM, ... So, is it possible to applied one of these to solve the fractional generalized fisher's equation: ${}^{c}D_t^{\lambda}u = {}^{c}D_x^{\gamma}u + \alpha u(1-u^{\beta})$, $\lambda \in]0,1]$ and $\gamma \in]1,2]$, where ${}^{c}D^{\lambda}$ and ${}^{c}D^{\gamma}$, are the Caputo fractional derivative of order λ, γ respectively ?

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