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Application of Homotopy Perturbation Transform Method to Resolve Lane - Emden and Emden - Fowler Equations

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Abstract

In this study, we will apply homotopy perturbation transform method (HPTM) to solve Lane-Emden and Emden-Fowler equations and we will also make a comparison between the result obtained by this method and the exact solutions of the same equations presented in other research. The results showed that HPTM has a high efficiency and effectiveness in solving these equations and can be applied to other nonlinear partial differential equations to obtain approximate or exat solutions if they exist.

Keywords: *Homotopy perturbation method, Laplace transform, Emden–Fowler equation, Lane–Emden equation, Analytical solution.*

1 Introduction

Since the linear and nonlinear differential equations have emerged have been made and are still making big efforts by researchers to find methods to solve this class of equations. These efforts resulted in the consolidation of this research field in many methods, we find among them the homotopy perturbation method (HPM). This method was established in 1998 by He ([1]-[5]) and applied to various linear and nonlinear problems (see ([6]-[16]). The method has the advantage of dealing directly with the problem. That is, the equations are solved without transforming them also avoids linearization, discretization or any unrealistic assumption and provides an efficient numerical solution. In dealing with nonlinear equations the nonlinearity terms is replaced by a series. Then it is an easy algorithm for computing the solution. As a result, it yields a very rapidly convergent series solution, and usually a few iterations lead to very accurate approximation of the exact solution [10]. In recent years, many researchers have paid attention to study the solutions of linear and nonlinear equations by using various methods combined with Laplace transform [18]. Among these are the variational iteration method coupled with Laplace transform method [17] and the homotopy perturbation method with Laplace transform (see [18]-[25]). It should be noted here that the homotopy perturbation transform method (HPTM) is a combination of Laplace transform method, homotopy perturbation method (HPM) and He's polynomials. The objective of this paper is to directly apply the homotopy perturbation transform method (HPTM) proposed by Khan and Wu [18] to consider the rational approximation solution of the Lane–Emden and Emden–Fowler equations.

2 Homotopy perturbation transform method

Khan and Wu [18] gives the idea of the basis of this method, where they considered a general nonlinear non-homogeneous partial differential equation with initial conditions of the form

$$Du(x,t) + Ru(x,t) + Nu(x,t) = g(x,t), \tag{1}$$

with the initial conditions

$$u(x,0) = h(x), u_t(x,0) = f(x), \tag{2}$$

where D is the second order linear differential operator $D = \frac{\partial^2}{\partial t^2}$, N represent the general nonlinear differential operator, R is the linear differential operator of less order D and $g(x,t)$ is the source term. Taking the Laplace transform (denoted throughout this paper by L) on both sides of (1)

$$L[Du(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[g(x,t)]. \tag{3}$$

Using the differentiation property of the Laplace transform, we have

$$L[u(x,t)] = \frac{1}{s}h(x) + \frac{1}{s^2}f(x) + \frac{1}{s^2}L[g(x,t)] - \frac{1}{s^2}L[Ru(x,t)] - \frac{1}{s^2}L[Nu(x,t)]. \tag{4}$$

Operating with the Laplace inverse on both sides of (4) gives

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s^2}L[Ru(x,t) + Nu(x,t)]\right], \tag{5}$$

where $G(x,t)$ represents the term arising from the source term and the prescribed initial conditions. Now, we apply the homotopy perturbation method

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{6}$$

and the nonlinear term can be decomposed as

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{7}$$

for some He's polynomials H_n that are given by

$$H_n(u_0, ..., u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, n = 0, 1, 2, ... \tag{8}$$

Substituting (6) and (7) in (5), we get

$$\sum_{n=0}^{\infty} p^n u_n = G(x,t) - p\left(L^{-1}\left[\frac{1}{s^2}L\left[R\sum_{n=0}^{\infty} p^n u_n + \sum_{n=0}^{\infty} p^n H_n(u)\right]\right]\right), \tag{9}$$

which is the coupling of the Laplace transform and the homotopy perturbation method using He's polynomials. Comparing the coefficient of like powers of p, the following approximations are obtained

$$\begin{aligned} p^0 &: u_0(x,t) = G(x,t),\\ p^1 &: u_1(x,t) = -L^{-1}\left[\tfrac{1}{s^2}L\left[Ru_0(x,t) + H_0(u)\right]\right],\\ p^2 &: u_2(x,t) = -L^{-1}\left[\tfrac{1}{s^2}L\left[Ru_1(x,t) + H_1(u)\right]\right],\\ p^3 &: u_3(x,t) = -L^{-1}\left[\tfrac{1}{s^2}L\left[Ru_2(x,t) + H_2(u)\right]\right],\\ &\vdots \end{aligned} \tag{10}$$

3 Applications

In this section, we apply the homotopy perturbation transform method (HPTM) for solving Lane–Emden and Emden–Fowler equations of index m, where m in $[0,5]$.

3.1 Lane–Emden equation of index m

The Lane–Emden equation of index m is given by

$$u''(x) + \frac{2}{x}u'(x) + u^m = 0, \quad u(0) = 1, \quad u'(0) = 0. \tag{11}$$

It should be noted here that the exact solutions exist only for $m = 0$;1 and 5.

To overcome the difficulty encountered by the singularity at $x = 0$, we use the transformation

$$v(x) = xu(x), \tag{12}$$

so that

$$\begin{aligned} v' &= xu' + u, \\ v'' &= xu'' + 2u'. \end{aligned} \tag{13}$$

Substituting (12) and (13) into (11) gives

$$v'' + x^{1-m}v^m = 0, \qquad m = 0, 1, 2, ..., \tag{14}$$

with the initial conditions

$$v(x) = 0, \quad v'(x) = 1. \tag{15}$$

Using the initial conditions (15) and the formula (9), we get

$$\sum_{n=0}^{\infty} p^n v_n(x) = x - p\left(L^{-1}\left[\frac{1}{s^2}L\left[x^{1-m}\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right), \tag{16}$$

where the first few components of He's polynomials [27], are given by

$$\begin{gathered} H_0(v) = v_0^m, \\ H_1(v) = mu_1 v_0^{m-1}, \\ H_2(v) = mu_2 v_0^{m-1} + \tfrac{(m-1)m}{2} v_1^2 v_0^{m-2}, \\ H_3(v) = mu_0^{m-1} v_3 + m(m-1)v_0^{m-2} v_1 v_2 + \tfrac{m(m-1)(m-2)}{6} v_0^{m-3} v_1^3. \\ \vdots \end{gathered} \tag{17}$$

Comparing the coefficients of like powers of p in the formula (16), we have

$$\begin{gathered} p^0 : v_0(x) = x, \\ p^1 : v_1(x) = -L^{-1}\left[\tfrac{1}{s^2}L\left[x^{1-m}H_0(v)\right]\right], \\ p^2 : v_2(x) = -L^{-1}\left[\tfrac{1}{s^2}L\left[x^{1-m}H_1(v)\right]\right], \\ p^3 : v_3(x) = -L^{-1}\left[\tfrac{1}{s^2}L\left[x^{1-m}H_2(v)\right]\right], \\ p^4 : v_4(x) = -L^{-1}\left[\tfrac{1}{s^2}L\left[x^{1-m}H_3(v)\right]\right], \\ \vdots \end{gathered} \tag{18}$$

Using He's polynomials (17) and the iteration formulas (18), we obtain

$$\begin{aligned} v_0(x) &= x, \\ v_1(x) &= -\tfrac{x^3}{3!}, \\ v_2(x) &= m\tfrac{x^5}{5!}, \\ v_3(x) &= -\tfrac{m(8m-5)}{3\cdot 7!}x^7, \\ v_4(x) &= \tfrac{m(122m^2-183m+70)}{9\cdot 9!}x^9, \\ &\vdots \end{aligned} \tag{19}$$

Recall, from (12) that $u(x) = \frac{v(x)}{x}$. The first four terms of the decomposition series solution for (11) is given as

$$u(x) = 1 - \frac{x^2}{3!} + m\frac{x^4}{5!} - \frac{m(8m-5)}{3\cdot 7!}x^6 + \frac{m(122m^2-183m+70)}{9\cdot 9!}x^8 + \cdots \tag{20}$$

As stated before, the exact solutions exist only for three cases, namely:
Case 1: For $m = 0$, the exact solution is given by

$$u(x) = 1 - \frac{x^2}{3!}, \tag{21}$$

Case 2: For $m = 1$, the solution $u(x)$ in series form is given by

$$u(x) = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \frac{x^8}{9!} + \cdots, \tag{22}$$

therefore the exact solution is given in the form

$$u(x) = \frac{\sin x}{x}. \tag{23}$$

Case 3: For $m = 5$, the solution $u(x)$ in series form is given by

$$u(x) = 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + \cdots, \tag{24}$$

and thus, we get the exact solution of the equation (11)

$$u(x) = \left(1 + \frac{x^2}{3}\right)^{-\frac{1}{2}}, \tag{25}$$

which is the exact solutions to the Lane–Emden equation as presented in [26].

3.2 Emden–Fowler equation of index m

The Emden–Fowler equation of index m is given by

$$u''(x) + \frac{2}{x}u'(x) + ax^n u^m = 0, \quad u(0) = 1, \quad u'(0) = 0, \tag{26}$$

It should be noted here also that the exact solutions exist only for $m = 0;1$ and 5.

The only value that cause us problems in solving this equation is $x = 0$, then we exclude this value and we solve the equation. To overcome this difficulty, we use the transformations (12) and and (13) into (26) gives

$$v'' + ax^{1+n-m}v^m = 0, \qquad m = 0, 1, 2, ..., \tag{27}$$

with the initial conditions

$$v(x) = 0, \quad v'(x) = 1. \tag{28}$$

In a similar way as above, the utilisation of formula (9) and the initial conditions (28) gives

$$\sum_{n=0}^{\infty} p^n v_n(x) = x - p\left(L^{-1}\left[\frac{1}{s^2}L\left[ax^{1+n-m}\sum_{n=0}^{\infty} p^n H_n(v)\right]\right]\right). \tag{29}$$

Comparing the coefficient of like powers of p, we have

$$\begin{aligned} p^0 &: v_0(x) = x, \\ p^1 &: v_1(x) = -L^{-1}\left[\tfrac{1}{s^2}L\left[ax^{1+n-m}H_0(v)\right]\right], \\ p^2 &: v_2(x) = -L^{-1}\left[\tfrac{1}{s^2}L\left[ax^{1+n-m}H_1(v)\right]\right], \\ p^3 &: v_3(x) = -L^{-1}\left[\tfrac{1}{s^2}L\left[ax^{1+n-m}H_2(v)\right]\right], \\ &\vdots \end{aligned} \tag{30}$$

Using He's polynomials (17) and the iteration formulas (30), we obtain

$$\begin{aligned} v_0(x) &= x, \\ v_1(x) &= -\tfrac{a}{(n+2)(n+3)}x^{n+3}, \\ v_2(x) &= \tfrac{a^2 m}{2(2n+5)(n+3)(n+2)^2}x^{2n+5}, \\ v_3(x) &= -\tfrac{a^3 m(8m+3mn-2n-5)}{6(2n+5)(3n+7)(n+3)^2(n+2)^3}x^{3n+7}, \\ &\vdots \end{aligned} \tag{31}$$

Recall, from (12) that $u(x) = \frac{v(x)}{x}$. The first four terms of the decomposition series solution for (26) is given as

$$u(x) = 1 - \frac{a}{(n+2)(n+3)}x^{n+2} + \frac{a^2 m}{2(2n+5)(n+3)(n+2)^2}x^{2n+4} - \frac{a^3 m(8m+3mn-2n-5)}{6(2n+5)(3n+7)(n+3)^2(n+2)^3}x^{3n+6} + \cdots \tag{32}$$

From the series solution (33), we conclude that $n \neq -3, -2, -\frac{5}{2}, -\frac{7}{3}, -\frac{9}{4}, \cdots$ Similarly as above, the exact solutions exist only for three cases, namely:
Case 1: For $m = 0$ and $n = 0$, the exact solution is given by

$$u(x) = 1 - \frac{a}{6}x^2. \tag{33}$$

Case 2: For $n = 0$ and $m = 1$, the solution $u(x)$ in series form is given by

$$u(x) = 1 - \frac{(\sqrt{a}x)^2}{3!} + \frac{(\sqrt{a}x)^4}{5!} - \frac{(\sqrt{a}x)^6}{7!} + \cdots, \tag{34}$$

therefore the exact solution is given in the form

$$u(x) = \frac{\sin\sqrt{a}x}{\sqrt{a}x}. \tag{35}$$

Case 3: For $n = 0$ and $m = 5$, the solution $u(x)$ in series form is given by

$$u(x) = 1 - a\frac{x^2}{6} + a^2\frac{x^4}{24} - a^3\frac{5x^6}{432} + \cdots, \tag{36}$$

and thus, we get the exact solution of the equation (26) as follows

$$u(x) = \left(1 + a\frac{x^2}{3}\right)^{-\frac{1}{2}}, \tag{37}$$

which is the exact solutions to the Emden–Fowler equation as presented in [26].

4 Conclusion

After the direct application of homotopy perturbation transform method and from the results obtained, we can say that this method is easy to implement and effective, as it allows us to know the exact solution after calculate the first three terms only. As a result, the conclusion that comes through this work is that HPTM can be applied to other nonlinear partial differential equation, due to the efficiency and flexibility in the application to get the possible results.

5 Open Problem

We can resolve the same problem by using the Sumudu transform method coupled with HPM and also for Lane-Emden and Emden-Fowler equations of fractional order.

References

[1] J. H. He, Homotopy perturbation technique. *Comput. Meth. Appl. Mech. Eng*, 178 (3), (1999), pp. 257–262.

[2] J. H. He, Application of homotopy perturbation method to nonlinear wave equations. *Chaos Solitons Fractals*, 26 (3), (2005), pp. 695–700.

[3] J. H. He, A coupling method of homotopy technique and perturbation technique for nonlinear problems. *Int. J. Nonlinear Mech*, 35 (3), (2000), pp. 37–43.

[4] J. H. He, Some asymptotic methods for strongly nonlinear equations. *Int. J. Modern Phys. B*, 20 (10), (2006), pp. 1141–1199.

[5] J. H. He, A new perturbation technique which is also valid for large parameters. *J. Sound Vib*, 229, (2000), pp. 1257–1263.

[6] J. Biazar, H. Ghazvini, Exact solutions for non-linear Schrdinger equations by He's homotopy perturbation method. *Phys. Lett. A*, 366, (2007) pp. 79–84.

[7] A. M. A. El-Sayed, A. Elsaid, D. Hammad, A Reliable Treatment of Homotopy Perturbation Method for Solving the Nonlinear Klein-Gordon Equation of Arbitrary (Fractional) Orders. *Hindawi Pub. Corp. J. Appl. Math. A. ID 581481*, 7, (2012), 13p.

[8] H. El Qarnia, Application of homotopy perturbation method to non-homogeneous parabolic partial and non linear differential equations. *W. J. Mod. Sim*, 5(3), (2009), pp. 225-231.

[9] B. Ganjavi, H. Mohammadi, D. D. Ganji, A. Barari, Homotopy Perturbation Method and Variational Iteration Method for Solving Zakharov-Kuznetsov Equation. *American J. Appl. Sci*, 5 (7), (2008), pp. 811-817.

[10] A. Golmankhaneh, A. Golmankhaneh, D. Baleanu, Homotopy perturbation method for solving a system of Schrdinger-Korteweg-de Vries equations. *Romanian Reports Phys*, 63(3), (2011), pp. 609-623.

[11] P. K. Guptaa, M. Singh, Homotopy perturbation method for fractional Fornberg–Whitham equation. *Comp. Math. Appl*, 61, (2011), pp. 250-254.

[12] E. Hesameddini, The Use of Variational Iteration Method and Homotopy Perturbation Method for Painlev Equation I. *Appl. Math. Sci*, 38 (3), (2009), pp. 1861-1871.

[13] N. A. Khan, M. Ayaz, L. Jin, A. Yildirim, On approximate solutions for the time-fractional reaction-diffusion equation of Fisher type. *Int. J. Phy. Sci*, 6 (10), (2011), pp. 2483-2496.

[14] Y. Liu, Z. Li, Y. Zhang, Homotopy perturbation method to fractional biological population equation. *Frac. Diff. Cal*, 1 (1), (2011), pp. 117-124.

[15] S. Momani, Z. Odibat, Homotopy perturbation method for nonlinear partial differential equations of fractional order. *Phy. Lett. A 365*, (2007), pp. 345-350.

[16] M. Matinfar, M. Ghanbari, Homotopy Perturbation Method for the Generalized Fisher's Equation. *J. Appl. Math., Islamic Azad Uni. Lahijan*, 7 (4), (2011), pp. 39-44.

[17] K. Afshan, M. Syed Tauseef, Coupling of laplace transform and correction functional for wave equations. *W. J. Mod. Simul*, 9 (3), (2013), pp. 173-180.

[18] Y. Khan, Q. Wu, Homotopy perturbation transform method for nonlinear equations using He's polynomials. *Comput. Math. Appl*, 61 (8), (2011), pp. 1963-1967.

[19] J. Singh, D. Kumar, S. Kumar, New treatment of fractional Fornberg–Whitham equation via Laplace transform. *Ain Shams Eng. J. 4*, (2013), pp. 557-562.

[20] M. Madani, M. Fathizadeh, Y. Khan, A. Yildirim, On the coupling of the homotopy perturbation method and Laplace transformation. *Math. Comput. Mod*, 53 (9), (2011), pp. 1937-1945.

[21] V. G. Gupta, S. Gupta, Homotopy Perturbation Transform Method for Solving Initial Boundary Value Problems of Variable Coefficients. *Int. J. Nonlinear Sci*, 12 (3), (2011), pp. 270-277.

[22] S. Kumara, A. Yildirim, Y. Khan, L. Weid, A fractional model of the diffusion equation and its analytical solution using Laplace transform. *Scientia Iranica B*, 19 (4), (2012), pp. 1117-1123.

[23] J. Singh, D. Kumar, S. Rathore, Application of Homotopy Perturbation Transform Method for Solving Linear and Nonlinear Klein-Gordon Equations. *J. Information and Computing Sci*, 7 (2), (2012), pp. 131-139.

[24] J. Singh, D. Kumar, An appliquation of homotopy perturbation transform method to fractional heat and wave-lake equations. *J. Fra. Cal. Appl*, 4 (2), (2013), pp. 290 - 302.

[25] S. Gupta, J. Singh, D. Kumar, Applications of Homotopy Perturbation Transform Method for Solving Time-Dependent Functional Differential Equations. *Int. J. Nonlinear Sci*, 16 (1), (2013), pp. 37-49.

[26] A. M. Wazwaz, The variational iteration method for analytic treatment for linear and nonlinear ODEs. *Appl. Math. Comp*, 212, (2009), pp. 120–134.

[27] Y. Zhu, Q. Chang, S. Wu, A new algorithm for calculating Adomian polynomials. *Appl. Math. Comp*, 169, (2005), pp. 402–416.

[28] D. ZIANE, K. BELGHABA and M. HAMDI CHERIF, Fractional homotopy perturbation transform method for solving the time-fractional KdV, K(2,2) and Burgers equations. *Int. J. Open Problems Compt. Math.,* Vol. 8, No. 2, (2015), pp. 63–75.