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On Mixed Transitivity of Arens Products

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Abstract

Our matter is to investigate conditions of transitivity of Arens products in non regular Banach algebras. To this end we introduce the notion of links, i.e. some special elements of the second dual of a Banach algebra that allow a transitive passage between different Arens products. There will be two classes of links, both closed two sided ideals of the second dual space. We shall characterize these classes, determining some of their properties and giving a few examples.

Keywords: *Means and invariance of means. Amenable, solvable and paradoxical groups. Arens products. Bounded approximately unital Banach algebras. Mixed identities.*

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1 Introduction

Throughout this article $\mathcal{U}$ will be a complex Banach algebra. Its bidual space $\mathcal{U}^{**}$ will be considered with the usual first and second Arens products $\Box$ and $\Diamond$ respectively (cf. [4], p. 250; [9], §1.4, p. 46; [1]). As it is well known these products extend the product of $\mathcal{U}$ to the second conjugate space $\mathcal{U}^{**}$ of $\mathcal{U}$. Further, $\mathcal{U}^{**}$ becomes an associative Banach algebra in both cases and the underlying $\mathcal{U}$ is called regular (or Arens regular) when the two Arens products coincide. The elements of $\mathcal{U}$ will be denoted as $a, b, \ldots$; the elements of $\mathcal{U}^*$ will be denoted as $a^*, b^*, \ldots$; etc.. For a^*, a^{**}, b^{**} let

$$L_{a^*}(a^{**}) = a^* a^{**} \text{ and } \mathcal{L}_{a^{**}}(b^{**}) = \mathcal{R}_{b^{**}}(a^{**}) = a^{**}\Box b^{**} - a^{**}\Diamond b^{**}.$$

Clearly $L_{a^*} \in \mathcal{B}(\mathcal{U}^{**},\mathcal{U}^*)$ and $\mathcal{L}_{a^{**}}, \mathcal{R}_{b^{**}} \in \mathcal{B}(\mathcal{U}^{**})$. If X is any normed space we shall denote χ_X to the natural immersion $X \hookrightarrow X^{**}$.

Section 2 contains our principal results. In Proposition 2.1 we shall relate some mixed transitive relationships between Arens products and the algebraic behauviour of the above linear ope-rators $\mathcal{L}_{a^{**}}$ and $\mathcal{R}_{b^{**}}$´s. The concepts of *links* will be introduced in Definition 2.2. Thereafter, in Th. 2.3 we shall consider the links as ideals and the corresponding properties of clases of links of closed subalgebras or quotient algebras. In particular, it will be characterized the Arens regularity of *bounded approximately unital Banach algebras* in terms of links. In Proposition 2.4 we shall characterize different classes of links in connection with other linear operators or continuity properties. In Proposition 2.6 it will be shown that the second dual space of a group algebra of a discrete infinite abelian group is always greater than some classes of links. In Example 2 we shall evaluate some links in the context of a non regular measure Banach algebra on a locally compact - non compact space on the positive integers. Finally, in Section 3 we close our study by sharing some derived unsolved problems.

2 The main results

Proposition 2.1 *Let* $a^{**} \in \mathcal{U}^{**}$.

(i) The following assertions are equivalent:

(ia) $\mathcal{L}_{a^{**}} \in \mathcal{B}_r(\mathcal{U}^{**}, \Box)$.

(ib) $a^{**}\Diamond\left[b^{**}\Box c^{**}\right] = \left[a^{**}\Diamond b^{**}\right]\Box c^{**}$ for all $b^{**}, c^{**} \in \mathcal{U}^{**}$.

(ic) $c^{**}\left(a^* a^{**}\right) = \left(c^{**} a^*\right) a^{**}$ if $a^* \in \mathcal{U}^*$ and $c^{**} \in \mathcal{U}^{**}$.

(ii) $\mathcal{R}_{a^{**}} \in \mathcal{B}_l(\mathcal{U}^{**}, \Box)$ if and only if

$$\left[b^{**}\Box c^{**}\right]\Diamond a^{**} = b^{**}\Box\left[c^{**}\Diamond a^{**}\right] \quad \text{for all } b^{**}, c^{**} \in \mathcal{U}^{**}.$$

(iii) $\mathcal{L}_{a^{**}} \in \mathcal{B}_r(\mathcal{U}^{**}, \Diamond)$ if and only if

$$a^{**}\Box\left[b^{**}\Diamond c^{**}\right] = \left[a^{**}\Box b^{**}\right]\Diamond c^{**} \text{ for all } b^{**}, c^{**} \in \mathcal{U}^{**}.$$

(iv) The following assertions are equivalent:

(iva) $\mathcal{R}_{a^{**}} \in \mathcal{B}_l(\mathcal{U}^{**}, \Diamond)$.

(ivb) $\left[b^{**}\Diamond c^{**}\right]\Box a^{**} = b^{**}\Diamond\left[c^{**}\Box a^{**}\right]$ for all $b^{**}, c^{**} \in \mathcal{U}^{**}$.

(ivc) $\left(a^{**} a^*\right) b^{**} = a^{**}\left(a^* b^{**}\right)$ if $a^* \in \mathcal{U}^*$ and $b^{**} \in \mathcal{U}^{**}$.

$(ia \Leftrightarrow ib)$ It is immediate because $\square$ is associative.

$(ib \Rightarrow ic)$ Given $a^* \in \mathcal{U}^*$ and $b^{**}, c^{**} \in \mathcal{U}^{**}$ we have

$$\begin{aligned} \langle c^{**}(a^* a^{**}) - (c^{**} a^*) a^{**}, b^{**} \rangle &= \langle a^* a^{**}, b^{**} \square c^{**} \rangle - \langle c^{**} a^*, a^{**} \lozenge b^{**} \rangle \\ &= \langle a^*, a^{**} \lozenge [b^{**} \square c^{**}] - [a^{**} \lozenge b^{**}] \square c^{**} \rangle \\ &= 0, \end{aligned}$$

and the conclusion follows because b^{**} is arbitrary.

$(ic \Rightarrow ib)$ Analogously, if $b^{**}, c^{**} \in \mathcal{U}^{**}$

$$\begin{aligned} \langle a^*, a^{**} \lozenge [b^{**} \square c^{**}] - [a^{**} \lozenge b^{**}] \square c^{**} \rangle &= \langle c^{**}(a^* a^{**}) - (c^{**} a^*) a^{**}, b^{**} \rangle \\ &= \langle 0_{\mathcal{U}^*}, b^{**} \rangle \\ &= 0 \end{aligned}$$

for all $a^* \in \mathcal{U}^*$.

(ii) It is immediate because $\square$ is associative.

(iii) It is immediate because $\lozenge$ is associative.

(iv) Ídem to (i).

Definition 2.2 *An element $b^{**} \in \mathcal{U}^{**}$ is called a link of $\mathcal{U}$ of type $(\lozenge, \square)$ or $(\square, \lozenge)$ according as*

$$a^{**} \lozenge [b^{**} \square c^{**}] = [a^{**} \lozenge b^{**}] \square c^{**} \text{ or } a^{**} \square [b^{**} \lozenge c^{**}] = [a^{**} \square b^{**}] \lozenge c^{**}$$

for all $a^{**}, c^{**} \in \mathcal{U}^{**}$. We denote $\mathcal{E}_{\mathcal{U}}(\lozenge, \square)$ and $\mathcal{E}_{\mathcal{U}}(\square, \lozenge)$ to the classes of links of $\mathcal{U}$ of type $(\lozenge, \square)$ and $(\square, \lozenge)$ respectively.

Theorem 2.3 **(i)** *$\mathcal{E}_{\mathcal{U}^{op}}(\lozenge_{op}, \square_{op}) = \mathcal{E}_{\mathcal{U}}(\lozenge, \square)$ and $\mathcal{E}_{\mathcal{U}^{op}}(\square_{op}, \lozenge_{op}) = \mathcal{E}_{\mathcal{U}}(\square, \lozenge)$.*

(ii) *$\mathcal{E}_{\mathcal{U}}(\lozenge, \square)$ is a closed ($\lozenge$-left, $\square$-right) ideal.*

(iii) *$\mathcal{E}_{\mathcal{U}}(\square, \lozenge)$ is a closed ($\square$-left, $\lozenge$-right) ideal.*

(iv) *If $\mathcal{U}$ is besides a $*$-algebra, $\mathcal{E}_{\mathcal{U}}(\lozenge, \square)$ and $\mathcal{E}_{\mathcal{U}}(\square, \lozenge)$ are $*$-closed.*

(v) *Let $\mathcal{V}$ be a closed subalgebra of $\mathcal{U}$ and let $\iota : \mathcal{V} \hookrightarrow \mathcal{U}$ be the inclusion map. Then*

$$\iota^{**}(\mathcal{V}) \cap \mathcal{E}_{\mathcal{U}}(\lozenge, \square) \subseteq \iota^{**}[\mathcal{E}_{\mathcal{V}}(\lozenge, \square)] \text{ and } \iota^{**}(\mathcal{V}) \cap \mathcal{E}_{\mathcal{U}}(\square, \lozenge) \subseteq \iota^{**}[\mathcal{E}_{\mathcal{V}}(\square, \lozenge)].$$

(vi) Let I be a closed ideal of $\mathcal{U}$ and let $q : \mathcal{U} \to \mathcal{U}/I$ be the quotient map. Then $q^{**}[\mathcal{E}_{\mathcal{U}}(\lozenge, \square)] \subseteq \mathcal{E}_{\mathcal{U}/I}(\lozenge, \square)$ and $q^{**}[\mathcal{E}_{\mathcal{U}}(\square, \lozenge)] \subseteq \mathcal{E}_{\mathcal{U}/I}(\square, \lozenge)$.

(vii) If $\mathcal{U}^{**}$ has a mixed identity E, $\mathcal{U}$ is Arens regular if and only if E is a link of type $(\Box, \Diamond)$.

(i-iv) and (vii) Are straightforward.

(v) It follows by applying the Hahn-Banach theorem and observing that

$$\iota^{**} \in Hom\left[(\mathcal{V}_{\Box}^{**}, \mathcal{U}_{\Box}^{**})\right] \cap Hom\left[\left(\mathcal{V}_{\Diamond}^{**}, \mathcal{U}_{\Diamond}^{**}\right)\right].$$

(vi) Observe that q^{**} is surjective. For, let $R \in (\mathcal{U}/I)^{**}$. By the Hahn-Banach theorem the rank of q^{*} is closed because it is the annihilator I^{o} of I within $\mathcal{U}^{*}$. Since q is surjective q^{*} is injective. By the open mapping theorem q^{*} becomes a Banach space isomorphism between $(\mathcal{U}/I)^{*}$ and I^{o}. Thus, by the Hahn-Banach theorem there is an extension $r \in \mathcal{U}^{**}$ of the linear form $R \circ (q^{*})^{-1}$ on the rank of q^{*}. Clearly $q^{**}(r) = R$. Now, it suffices to note that $q^{**} \in Hom\,[\mathcal{U}_{\Box}^{**}, (\mathcal{U}/I)_{\Box}^{**}] \cap Hom\,[\mathcal{U}_{\Box}^{**}, (\mathcal{U}/I)_{\Box}^{**}]$.

Proposition 2.4 *Let $b^{**} \in \mathcal{U}^{**}$.*

(i) $b^{**} \in \mathcal{E}_{\mathcal{U}}(\Diamond, \Box)$ if and only if

$$\left(c^{**}a^{*}\right)a^{**} - c^{**}\left(a^{*}a^{**}\right) \in \ker\left(b^{**}\right) \text{ if } a^{*} \in \mathcal{U}^{*} \text{ and } a^{**}, c^{**} \in \mathcal{U}^{**}.$$

(ii) The following assertions are equivalent:

(iia) $b^{**} \in \mathcal{E}_{\mathcal{U}}(\Box, \Diamond)$.

(iib) $b^{**} \cdot_{\Box} L_{a^{*}}^{*}(c^{**}) = \chi_{\mathcal{U}^{*}}\left[\left(b^{**} \Diamond c^{**}\right)a^{*}\right]$ if $a^{*} \in \mathcal{U}^{*}$ and $c^{**} \in \mathcal{U}^{**}$.

(iic) If $a^{*} \in \mathcal{U}^{*}$ the linear operator

$$\Lambda_{a^{*}} : \mathcal{U}^{**} \to \mathcal{U}^{***},\ \Lambda_{a^{*}}(c^{**}) = \chi_{\mathcal{U}^{*}}\left[\left(b^{**} \Diamond c^{**}\right)a^{*}\right]$$

is (w^{*}, w^{*})-continuous.

(i) It is immediate.

$(iia \Rightarrow iib)$ Let $a^{*} \in \mathcal{U}^{*}$ and $a^{**}, c^{**} \in \mathcal{U}^{**}$. Then

$$\begin{aligned}
\left\langle a^{**}, b^{**} \cdot_{\Box} L_{a^{*}}^{*}(c^{**})\right\rangle &= \left\langle a^{*}\left(a^{**} \Box b^{**}\right), c^{**}\right\rangle \\
&= \left\langle a^{*}, \left(a^{**} \Box b^{**}\right) \Diamond c^{**}\right\rangle \\
&= \left\langle a^{*}, a^{**} \Box \left(b^{**} \Diamond c^{**}\right)\right\rangle \\
&= \left\langle \left(b^{**} \Diamond c^{**}\right)a^{*}, a^{**}\right\rangle \\
&= \left\langle a^{**}, \chi_{\mathcal{U}^{*}}\left[\left(b^{**} \Diamond c^{**}\right)a^{*}\right]\right\rangle.
\end{aligned}$$

$(iib \Rightarrow iic)$ Let $\{c_i^{**}\}_{i\in I}$ be a net in $\mathcal{U}^{**}$ so that $w^*\text{-}\lim_{i\in I} c_i^{**} = 0_{\mathcal{U}^{**}}$. If $a^* \in \mathcal{U}^*$ and $a^{**} \in \mathcal{U}^{**}$ we have

$$\begin{aligned} 0 &= \lim_{i\in I} \langle a^* (a^{**}\Box b^{**}), c_i^{**}\rangle \\ &= \lim_{i\in I} \langle a^{**}, b^{**} \cdot_{\Box} L_{a^*}^* (c_i^{**})\rangle \\ &= \lim_{i\in I} \langle a^{**}, \Lambda_{a^*} (c_i^{**})\rangle, \end{aligned}$$

i.e. $w^*\text{-}\lim_{i\in I} \Lambda_{a^*}(c_i^{**}) = 0_{\mathcal{U}^{***}}$.

$(iic \Rightarrow iib)$ If $a \in \mathcal{U}$ and $a^* \in \mathcal{U}^*$ we see that

$$\begin{aligned} b^{**} \cdot_{\Box} L_{a^*}^* (\chi_{\mathcal{U}} (a)) &= b^{**} \cdot_{\Box} [a\chi_{\mathcal{U}^*} (a^*)] \\ &= \chi_{\mathcal{U}^*} [b^{**} (aa^*)] \\ &= \chi_{\mathcal{U}^*} [(b^{**}a) a^*]. \end{aligned}$$

If $c^{**} \in \mathcal{U}^{**}$ let $c^{**} = w^*\text{-}\lim_{j\in J} \chi_{\mathcal{U}} (c_j)$ for some bounded net $\{c_j\}_{j\in J}$ in $\mathcal{U}$. Hence

$$\begin{aligned} \chi_{\mathcal{U}^*} [(b^{**}\Diamond c^{**}) a^*] &= \Lambda_{a^*} (c^{**}) \\ &= w^*\text{-}\lim_{j\in J} \Lambda_{a^*} (\chi_{\mathcal{U}} (c_j)) \\ &= w^*\text{-}\lim_{j\in J} \chi_{\mathcal{U}^*} [(b^{**}c_j) a^*] \\ &= b^{**} \cdot_{\Box} L_{a^*}^* (c^{**}). \end{aligned}$$

$(iib \Rightarrow iia)$ Let $a^* \in \mathcal{U}^*, a^{**}, c^{**} \in \mathcal{U}^{**}$. Then

$$\begin{aligned} \langle a^*, (a^{**}\Box b^{**}) \Diamond c^{**}\rangle &= \langle L_{a^*} (a^{**}\Box b^{**}), c^{**}\rangle \\ &= \langle a^{**}, b^{**} \cdot_{\Box} L_{a^*} (c^{**})\rangle \\ &= \langle a^{**}, \chi_{\mathcal{U}^*} [(b^{**}\Diamond c^{**}) a^*]\rangle \\ &= \langle (b^{**}\Diamond c^{**}) a^*, a^{**}\rangle \\ &= \langle a^*, a^{**}\Box (b^{**}\Diamond c^{**})\rangle. \end{aligned}$$

Corollary 2.5 *$\mathcal{U}^{**} = \mathcal{E}_{\mathcal{U}} (\Diamond, \Box)$ if and only if $(c^{**}a^*) a^{**} = c^{**} (a^*a^{**})$ for all $a^* \in \mathcal{U}^{**}$ and $a^{**}, c^{**} \in \mathcal{U}^{**}$. In this case $\chi_{\mathcal{U}} (\mathcal{U}) \subseteq \mathcal{E}_{\mathcal{U}} (\Box, \Diamond)$.*

The first assertion follows by Prop. 2.4(i). On the other hand, it suffices to observe that

$$\begin{aligned} \langle a^*, (a^{**}b) \Diamond c^{**} - a^{**}\Box (bc^{**})\rangle &= \langle b, (c^{**}a^*) a^{**} - c^{**} (a^*a^{**})\rangle \\ &= 0 \end{aligned}$$

whenever $b \in \mathcal{U}$, $a^* \in \mathcal{U}^*$ and $a^{**}, c^{**} \in \mathcal{U}^{**}$.

Proposition 2.6 *Let $\mathcal{G}$ be an infinite abelian discrete group wich has more than one invariant mean. Then* $\mathcal{E}_{l^1(\mathcal{G})}(\Diamond,\Box) \subsetneq l^1(\mathcal{G})^{**}$.

Before proceeding to the proof let us make three remarks. First, $l^1(\mathcal{G})$ is not regular because the discrete group $\mathcal{G}$ is infinite and abelian (cf. [3]). Further, for completeness it is worth mentioning that $l^1(\mathcal{G})$ is not regular whenever $\mathcal{G}$ is an infinite locally compact Hausdorff group (cf. [10]). Secondly, it is well known that if $\mathcal{G}$ is a locally compact group $L^\infty(\mathcal{G})$ is a Banach $M(\mathcal{G})$-bimodule, where $M(\mathcal{G})$ denotes the Banach algebra of complex, complete, regular Borel measures on $\mathcal{G}$ of finite variation (cf. [6], (20.12)). If $\mathcal{G}$ is abelian, $\mu \in M(\mathcal{G})$ and $F \in L^\infty(\mathcal{G})$ the action of $M(\mathcal{G})$ on $L^\infty(\mathcal{G})$ is defined almost everywhere with respect to the Haar measure of $\mathcal{G}$ as

$$(\mu * F)(g) = (F * \mu)(g) = \int_{\mathcal{G}} F\left(h^{-1}g\right) d\mu(h).$$

In the third place, $\mathfrak{M}(\mathcal{G})$ will denote the set of invariant means on $L^\infty(\mathcal{G})$, i.e. those $m \in L^\infty(\mathcal{G})^*$ so that $\langle \chi_{\mathcal{G}}, m\rangle = \|m\| = 1$ and $\langle \delta_g * F, m\rangle = \langle F, m\rangle$ if $g \in \mathcal{G}$ and $F \in L^\infty(\mathcal{G})$. The existence of invariant means depends intrinsically of the group structure of $\mathcal{G}$. For instance, it would require any of the three equivalent conditions: (i) the group $\mathcal{G}$ is *not paradoxical*; (ii) the existence of a $\mathcal{G}$-invariant finitely additive probability measure on $\mathcal{P}(\mathcal{G})$ (cf. [2]); (iii) the group $\mathcal{G}$ is *amenable* (cf. [7]). An amenable group has only one invariant mean if and only if it is finite (cf. [5], Th. 4). The existence of at least two invariant means of an infinite group $\mathcal{G}$ is guaranteed under any of the following conditions: (a) That $\mathcal{G}$ be a solvable group; (b) That $\mathcal{G}$ be amenable and contains an element of infinite order; (c) That $\mathcal{G}$ be a locally finite group, i.e. every finite subset of $\mathcal{G}$ generate a finite subgroup of $\mathcal{G}$ (cf. [5], Corollary 3).

Let $r, t \in \mathfrak{M}(\mathcal{G})$, $F \in l^\infty(\mathcal{G})$, $g \in \mathcal{G}$ and $e_g^1 = (\delta_{g,h})_{h\in\mathcal{G}}$ in $l^1(\mathcal{G})$, where $\delta_{g,h}$ denotes the Kronecker symbol. Then

$$\left\langle e_g^1, (tF)\,r\right\rangle = \left\langle e_g^1(tF), r\right\rangle. \tag{1}$$

If $k, h \in \mathcal{G}$ we see that

$$\left\langle e_k^1, e_g^1(tF)\right\rangle = \left\langle e_{kg}^1, tF\right\rangle = \left\langle F e_{kg}^1, t\right\rangle \tag{2}$$

and

$$\left\langle e_h^1, F e_{kg}^1\right\rangle = F_{kgh} = \left\langle e_h^1, \delta_{(kg)^{-1}} * F\right\rangle.$$

Consequently, $F e_{kg}^1 = \delta_{(kg)^{-1}} * F$, where $\delta_{(kg)^{-1}}$ is the Dirac measure carried by $\left\{(kg)^{-1}\right\}$. By (2) is

$$\left\langle e_k^1, e_g^1(tF)\right\rangle = \left\langle \delta_{(kg)^{-1}} * F, t\right\rangle = \langle F, t\rangle,$$

i.e. $e_g^1(t\digamma) = \langle \digamma, t\rangle \chi_{\mathcal{G}}$. Now, by (1) is $\langle e_g^1, (t\digamma)\, r\rangle = \langle \digamma, t\rangle$ for all $g \in \mathcal{G}$, i.e. $(t\digamma)\, r = \langle \digamma, t\rangle \chi_{\mathcal{G}}$. Analogously, $t\,(\digamma r) = \langle \digamma, r\rangle \chi_{\mathcal{G}}$. So, if $s \in \mathcal{E}_{l^1(\mathcal{G})}(\Diamond, \Box)$ we obtain

$$0 = \langle \digamma, (r\Diamond s)\,\Box t - r\Diamond\,(s\Box t)\rangle = \langle (t\digamma)\, r - t\,(\digamma r)\,, s\rangle = \langle \digamma, t-r\rangle \langle \chi_{\mathcal{G}}, s\rangle\,.$$

Therefore, if $r \neq t$ in $\mathfrak{M}(\mathcal{G})$ there is $\digamma \in l^\infty(\mathcal{G})$ so that $\langle \digamma, t-r\rangle \neq 0$, i.e.

$$\mathcal{E}_{l^1(\mathcal{G})}(\Diamond, \Box) \subseteq \left\{ s \in l^1(\mathcal{G})^{**} : \langle \chi_{\mathcal{G}}, s\rangle = 0 \right\}.$$

Let $\mathcal{T}$ denote the topology on $\mathbb{N}$ generated by the following sets: $\{2n-1\}$ and $\{4n\}$ with $n \in \mathbb{N}$, $A \triangleq \{4n-2\}_{n\in\mathbb{N}:2\nmid n}$ and $B \triangleq \{4n-2\}_{n\in\mathbb{N}:2|n}$. Then $(\mathbb{N}, \mathcal{T})$ becomes a locally compact non compact space and it is easy to see that A and B do not have proper Borel subsets. So, any $\mu \in M(\mathbb{N}, \mathcal{T})$ determines a unique set $\{\mu_A, \mu_B\} \cup \left\{\mu_{\{n\}}\right\}_{n\in\mathbb{N}:2\nmid n \text{ or } 4|n}$ of complex numbers so that

$$\|\mu\| = |\mu_A| + |\mu_B| + \sum_{n\in\mathbb{N}:2\nmid n \text{ or } 4|n} \left|\mu_{\{n\}}\right|$$

and

$$\mu = \mu_A\delta_A + \mu_B\delta_B + \sum_{n\in\mathbb{N}:2\nmid n \text{ or } 4|n} \mu_{\{n\}}\delta_{\{n\}}, \tag{3}$$

where δ_A, δ_B and $\delta_{\{n\}}$ are the Borel measures carried by A, B and $\{n\}$ respectively. Let $*$ be the commutative multiplication $M(\mathbb{N}, \mathcal{T})$ so that

$$\delta_A * \delta_A = \delta_A,\;\; \delta_A * \delta_B = 0,\;\; \delta_B * \delta_B = \delta_B,$$

$\delta_C * \delta_{\{n\}} = 0$ if $C = A$ or B and

$$\delta_{\{n\}} * \delta_{\{m\}} = \begin{cases} \max\{n,m\} & if \quad n \text{ and } m \text{ are even,} \\ \min\{n,m\} & if \quad n \text{ and } m \text{ are odd,} \\ n & if \quad n \text{ is even and } m \text{ is odd.} \end{cases}$$

This multiplication is associative and the product of positive measures is a po-sitive measure, i.e. it is a general measure multiplication and $M(\mathbb{N}, \mathcal{T})$ becomes a Banach algebra (cf. [8], Prop. 2.1). Let us prove that $M(\mathbb{N}, \mathcal{T})$ is not Arens regular. For, if we set $f = \sum_{n=1}^{\infty}(-1)^n \chi_{\{4n-2\}}$ then $f \in C_0(X)$. Precisely, if D is any open disk in the complex plane let $D_0 = D \cap \{-1, 0, 1\}$. If $D_0 = \varnothing$, $f^{-1}(D) = \varnothing$. If $D_0 = \{-1\}$, $f^{-1}(D) = A$. If $D_0 = \{1\}$, $f^{-1}(D) = B$. If $D_0 = \{0\}$, $f^{-1}(D) = \mathbb{N} - (A \cup B)$. We can conclude that $f^{-1}(D)$ is open at all and that f is continuous. Further, given $\varepsilon > 0$ then $\{|f| \geq \varepsilon\} = A \cup B$ is compact. Now, let $\{n_k\}$ and $\{n_h\}$ be increasing subsequences of $\mathbb{N}$ so that

$$c^{**} = w^*\text{-}\lim_{k\to\infty} \chi_{M(\mathbb{N},\mathcal{T})}\left[\delta_{8n_k-6}\right] \text{ and } a^{**} = w^*\text{-}\lim_{h\to\infty} \chi_{M(\mathbb{N},\mathcal{T})}\left[\delta_{8n_h-2}\right]$$

are well defined elements of $M(\mathbb{N},\mathcal{T})^{**}$. Since $C_0(X) \hookrightarrow M(\mathbb{N},\mathcal{T})^{*}$ we see that

$$\begin{aligned} \langle f, c^{**}\square a^{**}\rangle &= \lim_{k\to\infty}\lim_{h\to\infty}\left\langle f, \delta_{\{8n_k-6\}} * \delta_{\{8m_h-2\}}\right\rangle \\ &= 1 \\ &\neq -1 \\ &= \lim_{h\to\infty}\lim_{k\to\infty}\left\langle f, \delta_{\{8n_k-6\}} * \delta_{\{8m_h-2\}}\right\rangle \\ &= \langle f, c^{**}\Diamond a^{**}\rangle, \end{aligned} \tag{4}$$

i.e. $M(\mathbb{N},\mathcal{T})$ is not Arens regular. Further, $\mathcal{E}_{M(\mathbb{N},\mathcal{T})}(\Diamond,\square) \neq \left\{0_{M(\mathbb{N},\mathcal{T})}\right\}$, for instance $\chi_{M(\mathbb{N},\mathcal{T})}[\delta_A]$ is a non zero $(\Diamond,\square)$-type link of $M(\mathbb{N},\mathcal{T})$. For, let d^{**}, $e^{**} \in M(\mathbb{N},\mathcal{T})^{**}$, say

$$d^{**} = w^{*}\text{-}\lim_{i\in I}\chi_{M(\mathbb{N},\mathcal{T})}\left(\mu^{i}\right) \text{ and } e^{**} = \lim_{j\in J}\chi_{M(\mathbb{N},\mathcal{T})}\left(\upsilon^{j}\right)$$

for some bounded nets $\{\mu^i\}_{i\in I}$ and $\{\upsilon^j\}_{j\in J}$ respectively. If $g \in M(\mathbb{N},\mathcal{T})^{*}$ we obtain

$$\begin{aligned} \langle g, (d^{**}\delta_A)\square e^{**} - d^{**}\Diamond(\delta_A e^{**})\rangle &= \langle \delta_A(e^{**}g), d^{**}\rangle - \langle (gd^{**})\delta_A, e^{**}\rangle \\ &= \lim_{i\in I}\left\langle \mu^i\delta_A, e^{**}g\right\rangle - \lim_{j\in J}\left\langle \delta_A\upsilon^j, gd^{**}\right\rangle \\ &= \lim_{i\in I}\mu^i_A\langle g\delta_A, e^{**}\rangle - \lim_{j\in J}\upsilon^j_A\langle \delta_A g, d^{**}\rangle \\ &= \left(\lim_{i\in I}\mu^i_A\lim_{j\in J}\upsilon^j_A - \lim_{j\in J}\upsilon^j_A\lim_{i\in I}\mu^i_A\right)\langle \delta_A, g\rangle \\ &= 0. \end{aligned}$$

Further, if $\mu \in M(\mathbb{N},\mathcal{T})$ as in (3) it is readily seen that

$$\langle \mu, c^{**}f\rangle = -\langle \mu, fa^{**}\rangle = -\sum_{n\in\mathbb{N}:2\nmid n \text{ or } 4|n}\mu_{\{n\}}.$$

Since $(c^{**}f)a^{**} = c^{**}f$ and $c^{**}(fa^{**}) = fa^{**}$ we conclude that $\mathcal{E}_{M(\mathbb{N},\mathcal{T})}(\Diamond,\square)$ is a proper subset of $M(\mathbb{N},\mathcal{T})^{**}$. If $u \in \mathbb{N}$ is odd we obtain that

$$\begin{aligned} \left\langle f, c^{**}\square\left(\delta_{\{u\}}a^{**}\right)\right\rangle &= \langle f, c^{**}\square a^{**}\rangle \\ &\neq \langle f, c^{**}\Diamond a^{**}\rangle \\ &= \left\langle f, \left(c^{**}\delta_{\{u\}}\right)\Diamond a^{**}\right\rangle, \end{aligned}$$

i.e. $\chi_{M(\mathbb{N},\mathcal{T})}[M(\mathbb{N},\mathcal{T})] \nsubseteq \mathcal{E}_{M(\mathbb{N},\mathcal{T})}(\square,\Diamond)$.

3 Open problems

(i) Let $\mathcal{S}$ be the subspace generated by elements of the form

$$(c^{**}a^{*})\,a^{**} - c^{**}\,(a^{*}a^{**})\,,\ \text{with } a^{*} \in \mathcal{U}^{*} \text{ and } a^{**}, c^{**} \in \mathcal{U}^{**}.$$

By Proposition 2.4(i), for the existence of a non-zero link of type $(\Diamond, \Box)$ it is necessary and sufficient that $\mathcal{S}$ be non dense in $\mathcal{U}^{*}$. If $\mathcal{U}$ is not a radical algebra and $\Delta_{\mathcal{U}}$ is the maximal ideal space of $\mathcal{U}$, it would be interesting to decide the relationships between $\mathcal{S}$ and $\Delta_{\mathcal{U}}$.

Problems 3.1 **(ii)** *By Corollary 2.5 the second dual space consists of links of type* $\mathcal{E}_{\mathcal{U}}(\Diamond, \Box)$ *if and only if* $\mathcal{S} = (0_{\mathcal{U}^{*}})$, *and then every element of the underlying algebra es a link of type* $(\Box, \Diamond)$. *What could be said if the second dual space consists of links of type* $\mathcal{E}_{\mathcal{U}}(\Box, \Diamond)$*?-*

(iii) *Theorem 2.3(vii) characterizes the regularity of bounded approximately unital Banach algebras in terms of links of type* $(\Box, \Diamond)$. *Is it possible a similar result in terms of links of* $(\Diamond, \Box)$*?-*

(iv) *Is there a non regular Banach algebra whose second dual space equal both classes of links?*

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