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M-strongly solid varieties and Q-free clones

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Abstract

A variety of algebras is called strongly solid if and only if every its identity is a strong hyperidentity. The clone of a strongly solid variety is free with respect to itself. M-solid varieties generalize the concept of solidity. In this paper, we describe the clone of an arbitrary M-strongly solid variety.

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1 Introduction

Let $n \in \mathbb{N}$ and $X_n := \{x_1, \ldots, x_n\}$ be an n-elements set. The set X_n is called an *alphabet* and its elements are called *variables*. To construct an n-ary term of type τ, we also need a set $\{f_i | i \in I\}$ of operation symbols, indexed by the set I. The set X_n and $\{f_i | i \in I\}$ have to be disjoint. To every operation symbol f_i we assign a natural number $n_i \geq 1$, call the arity of f_i. The sequence $\tau = (n_i)_{i \in I}$ is called a type.

Definition 1.1 *Let $n \geq 1$. The* n-ary terms *of type τ are defined inductively as follows:*

(i) Every variable $x_i \in X_n$ is an n-ary term *of type τ.*

(ii) If $t_1, \ldots, t_{n_i}$ are n-ary terms of type τ and f_i is an n_i-ary operation symbol, then $f_i(t_1, \ldots, t_{n_i})$ is an n-ary term *of type τ.*

The set $W_\tau(X_n)$ of all n-ary terms of type τ is the smallest set containing $x_1, \ldots, x_n$ that is closed under finite application of (ii). The set of all terms of type τ over the alphabet $X := \{x_1, x_2, \ldots\}$ is defined as $W_\tau(X) := \bigcup\limits_{n=1}^{\infty} W_\tau(X_n)$. By using step (ii) in the definition of n-ary terms of type τ, the term algebra

$$\mathcal{F}_\tau(X) := (W_\tau(X), (f_i)_{i\in I}^{\mathcal{F}_\tau(X)}).$$

of type τ, the so-called *absolutely free algebra*, can be constructed where $W_\tau(X)$ is the a carrier set and for each operation symbol f_i and $t_1, \ldots, t_{n_i} \in W_\tau(X)$,$f_i^{\mathcal{F}_\tau(X)}(t_1, \ldots, t_{n_i}) := f_i(t_1, \ldots, t_{n_i})$.

The concept of a hypersubstitution was introduced by K. Denecke, D. Lau, R. Pöschel and D. Schweigert in 1991 [1]. They used it as the tool to study hyperidentities and solid varieties. Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$.

Definition 1.2 *A* hypersubstitution *of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \longrightarrow W_\tau(X)$ which maps n_i-ary operation symbols to n_i-ary terms. Let $Hyp(\tau)$ be the set of all hypersubstitutions of type τ.*

For every $\sigma \in Hyp(\tau)$ induces a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ as follows, for any $t \in W_\tau(X)$, $\hat{\sigma}[t]$ is inductively defined by

(i) $\hat{\sigma}[t] := t$ if $t \in X$.

(ii) $\hat{\sigma}[f_i(t_1, \ldots, t_{n_i})] := \sigma(f_i)(\hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$.

Using the induced maps $\hat{\sigma}$, a binary operation $\circ_h$ can be defined on the set $Hyp(\tau)$. For any hypersubstitutions $\sigma_1, \sigma_2 \in Hyp(\tau)$, $\sigma_1 \circ_h \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ i.e.

$$\forall i \in I, (\sigma_1 \circ_h \sigma_2)(f_i) = \hat{\sigma}_1[\sigma_2(f_i)].$$

Let σ_{id} be the hypersubstitution which maps each n_i-ary operation symbol f_i to the term $f_i(x_1, \ldots, x_{n_i})$. It turns out that $\underline{Hyp(\tau)} = (Hyp(\tau), \circ_h, \sigma_{id})$ is a monoid where σ_{id} is the identity element.

In 2000, S. Leeratanavalee and K. Denecke [4] generalized the concept of a hypersubstitution to a generalized hypersubstitution. We used it as a tool to study strong hyperidentities and used strong hyperidentities to classify varieties into collections called strong hypervarieties. Varieties which are closed under arbitrary application of generalized hypersubstitutions are called strongly solid.

Definition 1.3 *Let $\tau = (n_i)_{i\in I}$ be a type with the sequence of operation symbols $(f_i)_{i\in I}$. A generalized hypersubstitution of type τ, for simply, a generalized hypersubstitution is a mapping $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ which maps each n_i-ary operation symbol of type τ to a term of this type which does not necessarily preserve the arity.*

We denoted the set of all generalized hypersubstitutions of type τ by $Hyp_G(\tau)$. We begin with a number of definitions and some notations. Firstly, we define inductively the concept of *generalized superposition of terms* $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$.

Definition 1.4 *Let $W_\tau(X)$ be the set of all term of type τ. The operation $S^m : W_\tau(X)^{m+1} \to W_\tau(X)$ is defined inductively as follows:*

(i) If $t = x_j, 1 \leq j \leq m$, then $S^m(x_j, t_1, \ldots, t_m) := t_j$.

(ii) If $t = x_j, m < j \in \mathbb{N}$, then $S^m(x_j, t_1, \ldots, t_m) := x_j$.

(iii) If $t = f_i(s_1, \ldots, s_{n_i})$, then
$S^m(t, t_1, \ldots, t_m) := f_i(S^m(s_1, t_1, \ldots, t_m), \ldots, S^m(s_{n_i}, t_1, \ldots, t_m))$.

To define a binary operation on $Hyp_G(\tau)$, we extend a generalized hypersubstitution σ to a mapping $\hat{\sigma} : W_\tau(X) \to W_\tau(X)$ inductively defined as follows:

(i) $\hat{\sigma}[t] := t$ if $t \in X$.

(ii) $\hat{\sigma}[t] := S^{n_i}(\sigma(f_i), \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_{n_i}])$ if t is a compound term, $f_i(t_1, \ldots, t_{n_i})$.

Then we define a binary operation $\circ_G$ on $Hyp_G(\tau)$ by $\sigma_1 \circ_G \sigma_2 := \hat{\sigma}_1 \circ \sigma_2$ where $\circ$ denotes the usual composition of mappings and $\sigma_1, \sigma_2 \in Hyp_G(\tau)$.

We proved the following propositions.

Proposition 1.5 ([4]) *For arbitrary terms $t, t_1, \ldots, t_n \in W_\tau(X)$ and for arbitrary generalized hypersubstitutions $\sigma, \sigma_1, \sigma_2$ we have*

(i) $S^n(\hat{\sigma}[t], \hat{\sigma}[t_1], \ldots, \hat{\sigma}[t_n]) = \hat{\sigma}[S^n(t, t_1, \ldots, t_n)]$.

(ii) $(\hat{\sigma}_1 \circ \sigma_2)\hat{} = \hat{\sigma}_1 \circ \hat{\sigma}_2$. □

Proposition 1.6 ([4]) *$\underline{Hyp_G(\tau)} = (Hyp_G(\tau), \circ_G, \sigma_{id})$ is a monoid where σ_{id} is the identity element and the set of all hypersubstitutions of type τ forms a submonoid of $\underline{Hyp_G(\tau)}$.* □

Let $\underline{M}$ be a submonoid of $\underline{Hyp_G(\tau)}$ and V be a variety of type τ. The variety V is called *M-strongly solid variety* if

$$\forall s \approx t \in IdV, \forall \sigma \in M(\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV).$$

An identity $s \approx t \in IdV$ is called *M-strong hyperidentity* if

$$\forall \sigma \in M(V \models \hat{\sigma}[s] \approx \hat{\sigma}[t]).$$

If $\underline{M} = \underline{Hyp(\tau)}$, then we speak of *strongly solid variety* and *strong hyperidentity*, respectively.

2 Main results

A *clone* as a set of operations defined on the same set, closed under the superposition and containing all projections can be equipped with algebraic structure which gives a *heterogeneous* algebra

$$\mathcal{C} := ((C^{(n)})_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (e^n_i)_{n\in\mathbb{N}^+, 1\leq i\leq n}) \quad (\mathbb{N}^+ = \{1, 2, 3, ...\})$$

where $C^{(n)}$ is a set of n-ary operations on the set A and $e^n_i, 1 \leq i \leq n$ are the n-ary projections with $e^n_i(a_1, ..., a_n) := a_i$ for all $a_1, ..., a_n \in A$.

Definition 2.1 *Let $C^{(n)}, n \geq 1$ be the set of n-ary operations on the set A. Then the $(n+1)$-ary superposition operation $S^n : (C^{(n)})^{n+1} \to C^{(n)}$ is defined by $S^n(f, g_1, ..., g_n)(a_1, ..., a_n) := f[g_1(a_1, ..., a_n), ..., g_n(a_1, ..., a_n)]$ for all $a_1, ..., a_n \in A$. This can be generalized to an operation $S^n_m : C^{(n)} \times (C^{(m)})^n \to C^{(m)}$ defined by*

$$S^n_m(f, g_1, ..., g_n) := f[g_1, ..., g_n]$$

and $f[g_1, ..., g_n](a_1, ..., a_m) = f(g_1(a_1, ..., a_m), ..., g_n(a_1, ..., a_m))$ for all $a_1, ..., a_m \in A$.

Let $\mathcal{A} = (A; (f^{\mathcal{A}}_i)_{i\in I})$ be an algebra of a type τ. Let $O^{(n)}_A$ be the set of all n-ary operations $f^A : A^n \to A$ and put $\mathcal{O}_A := \bigcup_{n=1}^{\infty} O^{(n)}_A$. We set $F^A := \{f^A_i | i \in I\}$ and $F^{A(n)} := F^A \cap O^{(n)}_A$. Let $\mathcal{O}_A$ be the heterogeneous clone where the carrier set are the sets $O^{(n)}_A$ for every $n \in \mathbb{N}^+$. Then the clone $\mathcal{T}(\mathcal{A})$ of all term operations of $\mathcal{A}$ is the subclone of $\mathcal{O}_A$ generated by $(F^{A(n)})_{n\in\mathbb{N}^+}$: $\mathcal{T}(\mathcal{A}) := \langle (F^{A(n)})_{n\in\mathbb{N}^+}\rangle_{\mathcal{O}_A}$. The carrier sets of $\mathcal{T}(\mathcal{A})$ are the sets $T^{(n)}(\mathcal{A})$ of all n-ary term operations of $\mathcal{A}$. For $\mathcal{A} = \mathcal{F}_\tau(X)$(the absolutely free algebra of type τ for a short written as $\mathcal{F}_\tau$) instead of $\mathcal{T}(\mathcal{A})$, we will write clone(τ) and if $\mathcal{F}_V(X)$ is the free algebra with respect to V, we write clone(V) instead of $\mathcal{T}(\mathcal{F}_V(X))$.

It is well-known and easy to check that this algebra satisfies the clone axioms

(C1) $\widetilde{S}^p_m(X_0, \widetilde{S}^n_m(Y_1, X_1, ..., X_n), ..., \widetilde{S}^n_m(Y_p, X_1, ..., X_n)) \approx$
$\widetilde{S}^n_m(\widetilde{S}^p_m(X_0, Y_1, ..., Y_p), X_1, ..., X_n)$,

(C2) $\widetilde{S}^n_m(\lambda_i, X_1, ..., X_n) \approx X_i, 1 \leq i \leq n$,

(C3) $\widetilde{S}^n_m(X_1, \lambda_1, ..., \lambda_n) \approx X_1$,

where $\widetilde{S}^p_m$ and $\widetilde{S}^n_m$ are operation symbols corresponding to the operations S^p_m and S^n_m of *clone*(τ) where $\lambda_1, ..., \lambda_n$ are nullary operation symbols and $X_1, ..., X_n, Y_1, ..., Y_p$ are variables.

Definition 2.2 *Let $\mathcal{C} := ((C^{(n)})_{n\in\mathbb{N}^+}; (S^n_m)_{m,n\in\mathbb{N}^+}, (e^n_i)_{n\in\mathbb{N}^+,1\leq i\leq n})$ be a clone , and let $(X_n)_{n\in\mathbb{N}^+}, X_n \subseteq C^{(n)}$ be a generating system of the clone $\mathcal{C}$. Then a system $\varphi = (\varphi_n)_{n\in\mathbb{N}^+}$ of a mappings $\varphi_n : X_n \to C^{(n)}$ with $\varphi_n(e^n_i) = e^n_i, n \in \mathbb{N}^+$ for a projections is called a clone substitution. By $Subst_{\langle(X_n)_{n\in\mathbb{N}^+}\rangle}$, we denote the set of all clone substitutions.*

Definition 2.3 *A set $I := (I_n)_{n\in\mathbb{N}^+}, I_n \subseteq C^{(n)}$ for every $n \in \mathbb{N}^+$ is said to be independent with respect to a family Q of a mappings $\psi = (\psi_n)_{n\in\mathbb{N}^+}$ such that $\psi_n : I_n \to C^{(n)}$,(Q-independent) if every ψ can be extended to a homomorphism $\overline{\psi}$ of the subclone $\langle I\rangle_C$ of $\mathcal{C}$ generated by I into $\mathcal{C}$ i.e., $\overline{\psi} : \langle I\rangle_C \to \mathcal{C}$.*

Definition 2.4 *Let $\mathcal{C}$ be a clone and let $Q \subseteq Subst_{\langle(X_n)_{n\in\mathbb{N}^+}\rangle}$ where $(X_n)_{n\in\mathbb{N}^+}$ is a generating system of $\mathcal{C}$. Then Q is called Q-free with respect to itself if $(X_n)_{n\in\mathbb{N}^+}$ is Q- independent.*

If $Q = Subst_{\langle(X_n)_{n\in\mathbb{N}^+}\rangle}$ we have the usual concept of freeness with respect to itself. The extension $\widehat{\varphi}$ of elements $\varphi \in Subst_{\langle(X_n)_{n\in\mathbb{N}^+}\rangle}$ to arbitrary elements of $(C^{(n)})_{n\in\mathbb{N}^+}$ are defined in the usual inductive way. If $\varphi_1, \varphi_2 \in Subst_{\langle(X_n)_{n\in\mathbb{N}^+}\rangle}$ we define a product $\varphi_1 \circ_s \varphi_2$ of substitutions by $\widehat{\varphi}_1 \circ \varphi_2$. This is again a substitution from $Subst_{\langle(X_n)_{n\in\mathbb{N}^+}\rangle}$. Since this product is associative and since the identity φ_{id} belongs to $Subst_{\langle(X_n)_{n\in\mathbb{N}^+}\rangle}$, we obtain a monoid.

Proposition 2.5 *(i) There is a bijection between the set $Hyp_G(\tau)$ of all generalized hypersubstitutions of type τ, and the set $Subst_{\langle(F^{\mathcal{F}_\tau(n)})_{n\in\mathbb{N}^+}\rangle}$ of all clone substitutions of clone(τ).*
(ii) For every variety V of type τ every generalized hypersubstitution of type τ defines a clone homomorphism clone$(\tau) \to$ clone(V).

Proof. *(i) Let $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ be a generalized hypersubstitution of type τ. We set $F := \{f_i | i \in I\}$ and let $F^{(n)}$ be the set of all n-ary operation symbols from F. Then we define a family $\sigma := (\sigma_n)_{n \in \mathbb{N}^+}$ of a mapping such that $\sigma_n : F^{(n)} \to W_\tau(X)$. Since clone$(\tau)$ is generated by $(F^{\mathcal{F}_\tau(n)})_{n \in \mathbb{N}^+}$, for every $\sigma := (\sigma_{n_i})_{n_i \in \mathbb{N}^+}$ we define a family $\varphi := (\varphi_{n_i})_{n_i \in \mathbb{N}^+}$ of a mappings $\varphi_{n_i} : F^{\mathcal{F}_\tau(n_i)} \to clone^{(n_i)}(\tau)$ by $\varphi_{n_i}(f_i^{\mathcal{F}_\tau(X)}) = \sigma_{n_i}(f_i^{\mathcal{F}_\tau(X)})$ where $clone^{(n_i)}(\tau)$ is the n_ith carrier set of $clone^{(n)}(\tau)$. Since $\sigma_{n_i}(f_i)^{\mathcal{F}_\tau(X)}$ is the term operation of $\mathcal{F}_\tau(X)$ induced by the term $\sigma(f_i)$, we get $\sigma_{n_i}(f_i)^{\mathcal{F}_\tau(X)}$ is an element of $clone^{(n_i)}(\tau)$. Remember that any n-ary term of $W_\tau(X)$ induced an n-ary element of $clone^{(n)}(\tau)$ in the following inductive way :*

(i) if $x_i \in X_n$, then $x_i^{\mathcal{F}_\tau(X)} := e_i^{n,\mathcal{F}} \in clone^{(n)}(\tau)$,

(ii) if $x_j \in X \setminus X_n$, then $x_j^{\mathcal{F}_\tau(X)} := x_j^{n,\mathcal{F}} \in clone^{(n)}(\tau)$ where $x_j^{n,\mathcal{F}}$ is the term operation of $\mathcal{F}_\tau(X)$ induced by the term x_j,

(iii) if $f_i(t_1, ..., t_{n_i})$ is a composed term and if $t_i^{\mathcal{F}_\tau(X)}, i = 1, ..., n_i$ are the n-ary term operations induced by t_i, then we define

$$[f_i(t_1, ..., t_{n_i})]^{\mathcal{F}_\tau(X)} = S_n^{n_i}(f_i^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_{n_i}^{\mathcal{F}_\tau(X)}) \in \text{clone}^{(n)}(\tau).$$

Therefore $\varphi : F^{\mathcal{F}_\tau(X)} \to \text{clone}(\tau)$ is a clone substitution. By definition above, σ defines φ uniquely.

Conversely, we assume that $\varphi : F^{\mathcal{F}_\tau(X)} \to \text{clone}(\tau)$ is a clone substitution. Then for each f_i we choose a term $\sigma(f_i) \in W_\tau(X)$ such that $\sigma(f_i)^{\mathcal{F}_\tau(X)} = \varphi(f_i)^{\mathcal{F}_\tau(X)}$. It is clear that $\sigma : \{f_i | i \in I\} \to W_\tau(X)$ is a generalized hypersubstitution and the image of this generalized hypersubstitution is the clone substitution φ. Then we have φ defines σ uniquely.

(ii) We show that the mapping $\widehat{\varphi} : \text{clone}(\tau) \to \text{clone}(V)$ defined by $t^{\mathcal{F}_\tau(X)} \mapsto \widehat{\sigma}[t]^{\mathcal{F}_\tau(X)}$ is an homomorphism $\widehat{\varphi}$ of clone(τ). Because of the bijection $t \mapsto t^{\mathcal{F}_\tau(X)}$ for every $t \in W_\tau(X)$ mentioned above the mapping φ is well-defined. Since $e_i^{n,\mathcal{F}_\tau(X)} = t^{\mathcal{F}_\tau(X)}$ for $t = x_i \in W_\tau(X_n)$, we have $\varphi(e_i^{n,\mathcal{F}_\tau(X)}) = \varphi(x_i^{\mathcal{F}_\tau(X)}) = \widehat{\sigma}(x_i)^{\mathcal{F}_V(X)} = x_i^{\mathcal{F}_V(X)} = e_i^{\mathcal{F}_V(n)}$. This projections are mapped to projections. Finally, we prove

$$\widehat{\varphi}(S_m^n(t^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)})) = S_m^n(\widehat{\varphi}(t^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)}))$$

by induction on the complexity of term t by the axioms (C1) and (C2). Let $t \in W_\tau(X_n)$ and $t_1, ..., t_n, s_1, ..., s_n \in W_\tau(X_m)$.

If $t = x_i \in X_n$, then $\widehat{\varphi}(S_m^n(t^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$

$$= \widehat{\varphi}(S_m^n(x_i^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$$

$$= \widehat{\varphi}(S_m^n(e_i^{n,\mathcal{F}}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$$

$$= \widehat{\varphi}(t_i^{\mathcal{F}_\tau(X)}) = S_m^n(\widehat{\varphi}(x_i^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)})).$$

If $t = x_j \in X \setminus X_n$ then $\widehat{\varphi}(S_m^n(t^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$

$= \widehat{\varphi}(S^n_m(x_j^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$
$= \widehat{\varphi}(x_j^{\mathcal{F}_\tau(X)})$
$= S^n_m(\widehat{\varphi}(x_j^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)}))$.
If $t = f_i(s_1, ..., s_n)$ *and assume that the formula is satisfied for* $s_1, ..., s_n$ *then*
$\widehat{\varphi}(S^n_m(t^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$
$= \widehat{\varphi}(S^n_m(f_i(s_1, ..., s_n)^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$
$= \widehat{\varphi}(S^n_m(S^{n_i}_m(f_i^{\mathcal{F}_\tau(X)}, s_1^{\mathcal{F}_\tau(X)}, ..., s_{n_i}^{\mathcal{F}_\tau(X)}), t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}))$
$= \widehat{\varphi}(S^{n_i}_m(f_i^{\mathcal{F}_\tau(X)}, S^n_m(s_1^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)}), ..., S^n_m(s_{n_i}^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)}, ..., t_n^{\mathcal{F}_\tau(X)})))$
$= S^{n_i}_m((\varphi(f_i^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(S^n_m(s_1^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)})), ..., widehat\varphi(S^n_m(s_1^{\mathcal{F}_\tau(X)}, t_1^{\mathcal{F}_\tau(X)})))$
$= S^{n_i}_m((\varphi(f_i^{\mathcal{F}_\tau(X)}), S^n_m(\widehat{\varphi}(s_1^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)})), ..., \widehat{\varphi}(s_{n_i}^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}),$
$..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)})))$
$= S^n_m(S^{n_i}_m(\varphi(f_i^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(s_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(s_{n_i}^{\mathcal{F}_\tau(X)})), \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)}))$
$= S^n_m(S^{n_i}_m(\sigma(f_i^{\mathcal{F}_\tau(X)}), \widehat{\sigma}[s_1]^{\mathcal{F}_\tau(X)}, ..., \widehat{\sigma}[s_{n_i}]^{\mathcal{F}_\tau(X)}), \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)}))$
$= S^n_m(\widehat{\sigma}[t]^{\mathcal{F}_\tau(X)}, \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)}))$
$= S^n_m(\widehat{\varphi}[t]^{\mathcal{F}_\tau(X)}, \widehat{\varphi}(t_1^{\mathcal{F}_\tau(X)}), ..., \widehat{\varphi}(t_n^{\mathcal{F}_\tau(X)}))$.

□

Note that Proposition 2.5 (i) expresses the well-known fact that generalized hypersubstitution of type τ and clone substitutions of clone(τ) are essentially the same thing if the generating family of clone(τ) consist of the basic operations of the free algebra $\mathcal{F}_\tau(X)$. The reason for that is the natural bijection between terms of type τ and the term operations of the absolutely free algebra $\mathcal{F}_\tau(X)$ on countably many generators.
Since clone(V) is the quotient algebra clone(τ)/IdV, where IdV has to be regarded as a heterogeneous full invariant congruence on clone(τ), there is a natural homomorphism

$$nat_V : \text{clone}(\tau) \to \text{clone}(V).$$

The homomorphism from Proposition 2.5 is composition of the extensions of clone substitutions corresponding to generalized hypersubstitutions and nat_V. As a consequence of Proposition 2.5, we have:

Corollary 2.6 *The monoid* $(Hyp_G(\tau); \circ_G, \sigma_{id})$ *is isomorphic to the monoid* $(Subst_{\langle (F^{\mathcal{F}_\tau(n)})_{n\in\mathbb{N}^+}\rangle}; \circ_s, \varphi_{id})$ *where* $\circ_s$ *is defined by* $\varphi_1 \circ_s \varphi_2 := \widehat{\varphi}_1 \circ_s \varphi_2$ *and where* φ_{id} *is the identical clone substitution of clone*(τ).
Proof. *By Proposition 2.5, we have a bijection between* $Hyp_G(\tau)$ *and* $Subst_{\langle (F^{\mathcal{F}_\tau(n)})_{n\in\mathbb{N}^+}\rangle}$. *Then we have* $\varphi_{id}(f_i^{\mathcal{F}_\tau(X)}) = f_i^{\mathcal{F}_\tau(X)} = f_i(x_1, ..., x_{n_i})^{\mathcal{F}_\tau(X)} = \sigma_{id}(f_i)^{\mathcal{F}_\tau(X)}$ *and if* $\sigma_1, \sigma_2 \in Hyp_G(\tau)$ *then* $(\sigma_1 \circ_G \sigma_2)(f_i)^{\mathcal{F}_\tau(X)} = \widehat{\sigma}_1[\sigma_2(f_i)]^{\mathcal{F}_\tau(X)} = \widehat{\varphi}_1(\sigma_2(f_i)^{\mathcal{F}_\tau(X)}) = \widehat{\varphi}_1(\varphi_2(f_i)^{\mathcal{F}_\tau(X)}) = (\varphi_1 \circ_s \varphi_2)(f_i)^{\mathcal{F}_\tau(X)}$. □

If $M \subseteq Hyp_G(\tau)$ is a submonoid of the monoid of all generalized hypersubstitutions of type τ, then by Proposition 2.5 there is a subset $Q \subseteq Subst_{\text{clone}(\tau)}$ corresponding to M. Now we asking whether a similar proposition is true for clone(V) if V is an M-strongly solid variety of type τ.

Lemma 2.7 *Let V be an M-strongly solid variety of type τ and let clone(V) be the clone of all term operations of the V-free algebra $\mathcal{F}_V(X)$. Then to M it corresponds a set of clone substitutions of clone(V).*

Proof. *Let $\{f_i^{\mathcal{F}_V(X)} | i \in I\}$ be a generating system of clone(V). For any $\sigma \in M$ we define a mapping*

$$\varphi_V^\sigma : \{f_i^{\mathcal{F}_V(X)} | i \in I\} \to \text{clone}(V)$$

by $\varphi_V^\sigma(f_i^{\mathcal{F}_V(X)}) = \sigma(f_i)^{\mathcal{F}_V(X)}$ where $\sigma(f_i)^{\mathcal{F}_V(X)}$ is the term induced by $\sigma(f_i)$ on the V-free algebra $\mathcal{F}_V(X)$. We show that φ_V^σ is well-defined, assume that $f_i^{\mathcal{F}_V(X)} = f_j^{\mathcal{F}_V(X)}$, then $f_i(x_1, ..., x_{n_i}) \approx f_j(x_1, ..., x_{n_i}) \in IdV$ where IdV denotes the set of all identities satisfies in V. Since V is M-strongly solid for every $\sigma \in M$, we have $\hat{\sigma}[f_i(x_1, ..., x_{n_i})] \approx \hat{\sigma}[f_j(x_1, ..., x_{n_i})] \in IdV$, and by definition of the extension $\hat{\sigma}$ we get

$$\sigma(f_i)^{\mathcal{F}_V(X)}(x_1^{\mathcal{F}_V(X)}, ..., x_{n_i}^{\mathcal{F}_V(X)}) = \sigma(f_j)^{\mathcal{F}_V(X)}(x_1^{\mathcal{F}_V(X)}, ..., x_{n_j}^{\mathcal{F}_V(X)})$$

and thus

$$S_n^{n_i}(\sigma(f_i)^{\mathcal{F}_V(X)}, e_1^{n_i,\mathcal{F}_V(X)}, ..., e_{n_i}^{n_i,\mathcal{F}_V(X)}) = S_n^{n_j}(\sigma(f_i)^{\mathcal{F}_V(X)}, e_1^{n_i,\mathcal{F}_V(X)}, ..., e_{n_j}^{n_i,\mathcal{F}_V(X)}).$$

By axiom (C3), it follows $\sigma(f_i)^{\mathcal{F}_V(X)} = \sigma(f_j)^{\mathcal{F}_V(X)}$ and by the definition of φ_V^σ, we have $\varphi_V^\sigma(f_i^{\mathcal{F}_V(X)}) = \varphi_V^\sigma(f_j^{\mathcal{F}_V(X)})$. Since $\sigma(f_i)^{\mathcal{F}_V(X)}$ is an n_i-ary operation from clone(V), the mapping φ_V^σ can be regarded as a family $\varphi_V^\sigma = ((\varphi_V^\sigma)_n)_{n \in \mathbb{N}^+}$. For projections in $\{f_i^{\mathcal{F}_V(X)} | i \in I\}$ we have $\varphi_V^\sigma(e_i^{n_i,\mathcal{F}_V(X)}) = \sigma(e_i^{n_i})^{\mathcal{F}_V(X)} = \hat{\sigma}(e_i^{n_i}(x_1, ..., x_{n_i}))^{\mathcal{F}_V(X)} = \hat{\sigma}(x_i)^{\mathcal{F}_V(X)} = x_i^{\mathcal{F}_V(X)} = e_i^{n_i,\mathcal{F}_V(X)}$. This show that φ_V^σ is a clone substitution of clone(V). If conversely φ_V^σ is a clone substitution of clone(V), then it defines a generalized hypersubstitution σ with $\sigma(f_i)^{\mathcal{F}_V(X)} = \varphi_V^\sigma(f_i^{\mathcal{F}_V(X)})$ for every $i \in I$. □

To prove that $\varphi_V^\sigma$is well-defined, we needed that if two operation symbols induce the same term operations of $\mathcal{F}_V(X)$, then their images under a generalized hypersubstitution σ also these properties. We define:

Definition 2.8 *A generalized hypersubstitution σ of type τ is called generalized meaningful for the variety V of type τ from $f_i^{\mathcal{F}_V(X)} = f_j^{\mathcal{F}_V(X)}$, it follows that $\sigma(f_i)^{\mathcal{F}_V(X)} = \sigma(f_j)^{\mathcal{F}_V(X)}$.*

Now let Q_M be the set of clone substitutions of clone(V) corresponding to the submonoid M of $Hyp_G(\tau)$ by Proposition 2.5(i). Then we obtain the following characterization of M-strongly solidity:

Theorem 2.9 *For a submonoid $M \subseteq Hyp_G(\tau)$ the variety V of type τ is M-strongly solid if and only if each $\sigma \in M$ is generalized meaningful for V, and for $Q_M = \{\varphi_V | \sigma \in M\}$ the algebra clone(V) is Q_M-free with respect to itself with Q_M-basis $F^{\mathcal{F}_V(X)}$.*

Proof. *Assume that V is M-strongly solid. By Lemma 2.7, every $\sigma \in M$ is generalized meaningful for V. Let $\varphi : \{f_i^{\mathcal{F}_V(X)} | i \in I\} \to$ clone(V) be an element in Q_M (the set of all clone substitutions of clone(V) corresponding by Lemma 2.7 to M). By definition of φ, there is a generalized hypersubstitution $\sigma \in M$ such that for every $i \in I$ we have $\varphi(f_i^{\mathcal{F}_V(X)}) = \sigma(f_i^{\mathcal{F}_V(X)})$. We show that φ can be extended to a clone endomorphism of clone(V). Clearly, $\{f_i^{\mathcal{F}_V(X)} | i \in I\}$ is a generating system of clone(V). The mapping clone(τ) $\to$clone(V) : $t \mapsto t^{\mathcal{F}_V(X)}$ is obviously a surjective homomorphism with the kernel IdV. For any $\sigma \in M$, $\sigma[IdV]$ is the kernel of the homomorphism clone(τ) $\to$clone(V) : $t \mapsto \hat{\sigma}[t]^{\mathcal{F}_V(X)}$ considered in Proposition 2.5(ii). Since V is M-strongly solid, every identity of V is an M-strong hyperidentity, that mean $IdV \subseteq \sigma[IdV]$. By the general homomorphism theorem, there exists an homomorphism clone(V) $\to$ clone(V) : $t^{\mathcal{F}_V(X)} \mapsto \hat{\sigma}[t]^{\mathcal{F}_V(X)}$ and this homomorphism extends φ. So clone(V) is Q_M-free with respect to itself and $(F^{\mathcal{F}_V(X)(n)})_{n \in \mathbb{N}^+}$ is a Q_M-free independent generating system.*

Conversely, we assume that clone(V) is Q_M-free freely generated by the Q_M-free independent set $(F^{\mathcal{F}_V(X)(n)})_{n \in \mathbb{N}^+}$. That means, every $\varphi \in Q_M$ can be extended to a clone endomorphism of clone(V). Since every $\sigma \in M$ is generalized meaningful for V from $f_i^{\mathcal{F}_V(X)} = f_j^{\mathcal{F}_V(X)}$, we obtain $\varphi(f_i^{\mathcal{F}_V(X)}) = \sigma(f_i^{\mathcal{F}_V(X)}) = \sigma(f_j^{\mathcal{F}_V(X)}) = \varphi(f_j^{\mathcal{F}_V(X)})$. If $s \approx t \in IdV$, then $s^{\mathcal{F}_V(X)} = t^{\mathcal{F}_V(X)}$ and applying the extension of φ we get $\hat{\varphi}(s^{\mathcal{F}_V(X)}) = \hat{\varphi}(t^{\mathcal{F}_V(X)})$ and thus $\hat{\sigma}[s] \approx \hat{\sigma}[t] \in IdV$. This true for any $\sigma \in M$ and $s \approx t$ is an M-strong hyperidentity. $\square$

3 Open Problem

Let $Reg_G(\tau)$ be the set of all regular generalized hypersubstitutions of type τ. The theory can be used to describe by $Reg_G(\tau)$-strongly solid variety.

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