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Fractional homotopy perturbation transform method for solving the time-fractional KdV, K(2,2) and Burgers equations

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Abstract

In this paper, the fractional homotopy perturbation transform method (FHPTM) is employed to obtain approximate analytical solutions of the time-fractional KdV, K(2,2) and Burgers equations. The FHPTM can easily be applied to many problems and is capable of reducing the size of computational work. The fractional derivative is described in the Caputo sense. The results show that the FHPTM is an appropriate method for solving nonlinear fractional derivative equation.

Keywords: *Fractional homotopy perturbation transform method, KdV equation, K(2,2) equation, Burgers equation, analytical solution.*

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1 Introduction

The investigation of the exact solutions to nonlinear equations play an important role in the study of nonlinear physical phenomena. To date, various nonlinear equations were presented, which described, for example, the motion of the isolated waves, and in many fields such as hydrodynamic, plasma physics, nonlinear optic,... etc. In most cases it is difficult to solve nonlinear problems, especially [7]. To overcome such problems, the homotopy perturbation method (HPM) was established in 1998 by He ([1]-[5]) and applied to various linear and nonlinear problems including Al-Saif and Abood [7], Biazar and Ghazvini [8], El-Sayed et al. [9],... etc. The method has the advantage of dealing directly with the problem. That is, the equations are solved without transforming them also avoids linearization, discretization or any unrealistic assumption and provides an efficient numerical solution [12]. In dealing with nonlinear equations the nonlinearity terms is replaced by a series. Then it is an easy algorithm for computing the solution. As a result, it yields a very rapidly convergent series solution, and usually a few iterations lead to very accurate approximation of the exact solution. Then it is an easy algorithm for computing the solution [12].

The aim of this paper is to directly apply fractional homotopy perturbation transform method (FHPTM) [24] to consider the rational approximation solution of the time-fractional KdV, K(2,2) and Burgers equations of this form:

$$D_{*t}^{\alpha}u - 3(u^2)_x + u_{xxx} = 0, 0 < \alpha \le 1, \tag{1}$$

$$D_{*t}^{\alpha}u + (u^2)_x + (u^2)_{xxx} = 0, 0 < \alpha \le 1, \tag{2}$$

$$D_{*t}^{\alpha}u + \frac{1}{2}(u^2)_x - u_{xx} = 0, 0 < \alpha \le 1, \tag{3}$$

When $\alpha = 1$, the fractional equations reduces to the KdV, K(2,2) and Burgers equations of the form:

$$u_t - 3(u^2)_x + u_{xxx} = 0, \tag{4}$$

$$u_t + (u^2)_x + (u^2)_{xxx} = 0, \tag{5}$$

$$u_t + \frac{1}{2}(u^2)_x - u_{xx} = 0, \tag{6}$$

Eq. (4) is the pioneering equation that gives rise to solitary wave solutions. Solitons: waves with infinite support are generated as a result of the balance between the nonlinear convection $(u^n)_x$ and the linear dispersion u_{xxx} in these equations. Solitons are localized waves that propagate without change of their shape and velocity properties and stable against mutual collisions [25].

The $K(n,n)$ equation [25]

$$u_t + (u^n)_x + (u^n)_{xxx} = 0, \tag{7}$$

is the pioneering equation for compactons. In solitary waves theory, compactons are defined as solitons with finite wavelengths or solitons free of exponential tails. Compactons are generated as a result of the delicate interaction between nonlinear convection $(u^n)_x$ with the genuine nonlinear dispersion $(u^n)_{xxx}$ in (7).

The Burgers equation appears in fluid mechanics. This equation incorporates both convection and diffusion in fluid dynamics, and is used to describe the structure of shock waves [25].

2 Basic definitions

There are several definitions of a fractional derivative of order $\alpha \geq 0$ (see [19]-[23]). The most commonly used definitions are the Riemann--Liouville and Caputo. We give some basic definitions and properties of the fractional calculus theory which are used further in this paper .

Definition 2.1 A real function $f(t), t>0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p>\mu$, such that $f(t)=t^p f_1(t)$, where $f_1(t) \in C([0,\infty))$, and it is said to be in the space C_μ^m, $m \in \mathbb{N}$ if $f^{(m)} \in C_\mu$.

Definition 2.2 The left sided Riemann--Liouville fractional integral of order $\alpha \geq 0$ of a function $f \in C_\mu$, $\mu \geq -1$, is defined as:

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \alpha>0, t>0 \\ f(t), \alpha=0, \end{cases} \tag{8}$$

where $\Gamma(.)$ is the well-known Gamma function.

Definition 2.3 The fractional derivative of $f \in C_{-1}^m$ in the Caputo's sense is defined as:

$$D_*^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} f^{(m)}(\tau) d\tau, m-1<\alpha<m, \\ \frac{d^m}{dt^m} f(t), \alpha=m, \end{cases} \tag{9}$$

where $m \in \mathbb{N}^*$.

Remark 2.1 According to the formula (9), we can obtain:

$D_*^\alpha C = 0;$ where C is a constant

and

$$D_*^{\alpha} t^{\beta} = \begin{cases} \dfrac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} t^{\beta-\alpha}, \beta > \alpha - 1, \\ 0, \alpha = m, \beta \le \alpha - 1. \end{cases} \tag{10}$$

Remark 2.2 In this paper, we consider the time-fractional derivative in the Caputo's sense. When $\alpha \in \mathbb{R}^+$, the time-fractional derivative is defined as:

$$D_{*t}^{\alpha} u(x,t) = \frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}}$$
$$= \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m}(\tau) d\tau, 0 \le m-1 < \alpha < m, \\ \dfrac{\partial^m u(x,t)}{\partial t^m}, \alpha = m. \end{cases} \tag{11}$$

Definition 2.4 The Laplace transform of continuous (or an almost piecewise continuous) function $f(t)$ in $[0,\infty)$ is defined as:

$$F(s) = L\{f(t);s\} = \int_0^{\infty} e^{-st} f(t) dt. \tag{12}$$

where $m-1 < \alpha \le m.$

Definition 2.5 The Laplace transform, $L\{D_*^{\alpha} f(t);s\}$ of the Caputo's fractional derivative is defined as:

$$L\{D_*^{\alpha} f(t);s\} = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), \tag{13}$$

where $m-1 < \alpha \le m,\ m \in \mathbb{N}^*.$

3 Fractional homotopy perturbation transform method

In order to elucidate the solution procedure of the fractional Laplace homotopy perturbation method [24], we consider the following nonlinear fractional differential equation:

$$D_{*t}^{\alpha} u(x,t) + Ru(x,t) + Nu(x,t) = q(x,t), \tag{14}$$
$$0 < \alpha \le 1, u(x,0) = h(x), t > 0, x \in \mathbb{R}$$

where $D^{\alpha} = \dfrac{\partial^{\alpha}}{\partial t^{\alpha}}$, $R[x]$ is the linear operator in x, $N[x]$ is the general nonlinear operator in x, and $q(x,t)$ are continuous functions. Now, the methodology consists of applying the Laplace transform first on both sides of Eq. (14). Thus, we get:

$$L[D_{*t}^{\alpha}u(x,t)] + L[Ru(x,t)] + L[Nu(x,t)] = L[q(x,t)]. \tag{15}$$

Now, using the differentiation property of the Laplace transform, we have:

$$L[u(x,t)] = s^{-1}h(x) - s^{-\alpha}L[q(x,t)] - s^{-\alpha}L[Ru(x,t) + Nu(x,t)]. \tag{16}$$

Operating the inverse Laplace transform on both sides in Eq. (16), we get:

$$u(x,t) = G(x,t) - L^{-1}(s^{-\alpha}L[Ru(x,t) + Nu(x,t)]), \tag{17}$$

where $G(x,t)$, represents the term arising from the source term and the prescribed initial conditions. Now, applying the classical perturbation technique, we can assume that the solution can be expressed as a power series in p, as given below:

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t), \tag{18}$$

where the homotopy parameter p, is considered as a small parameter ($p \in [0,1]$). The nonlinear term can be decomposed as:

$$Nu(x,t) = \sum_{n=0}^{\infty} p^n H_n(u), \tag{19}$$

where H_n are He's polynomials of $u_0, u_1, u_2, ..., u_n$, which can be calculated by the following formula:

$$H_n(u_0,...,u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N(\sum_{i=0}^{\infty} p^i u_i)\right]_{p=0}, n = 0,1,2,... \tag{20}$$

Substituting (18) and (19) in Eq. (17) and using HPM by He ([1]-[5]), we get:

$$\sum_{n=0}^{\infty} p^n u_n = G(x,t) - p(L^{-1}(s^{-\alpha}L[R\sum_{n=0}^{\infty} p^n u_n + N\sum_{n=0}^{\infty} p^n H_n(u)])), \tag{21}$$

This is a coupling of the Laplace transform and homotopy perturbation methods using He's polynomials. Now, equating the coefficient of corresponding power of p on both sides, the following approximations are obtained as:

$$p^0 : u_0(x,t) = G(x,t),$$
$$\vdots$$
$$p^n : u_n(x,t) = -L^{-1}(s^{-\alpha} L[Ru_{n-1}(x,t) + H_{n-1}(u)]), \tag{22}$$

where $n \in \mathbb{N}^*$.

Proceeding in this same manner, the rest of the components $u_n(x,t)$, can be completely obtained, and the series solution is thus entirely determined.

Finally, we approximate the analytical solution $u(x,t)$, by truncated series:

$$u(x,t) = \lim_{N\to\infty} \sum_{n=0}^{N} u_n(x,t). \tag{23}$$

The above series solutions generally converge very rapidly. A classical approach of convergence of this type of series is already presented by Abbaoui and Cherruault [6].

4 Application of the FHPTM

In this section, we apply fractional homotopy perturbation transform method for the Caputo fractional derivative to solve nonlinear time-fractional KdV, K(2,2) and Burgers equations.

Example 4.1 Consider the time-fractional partial differential KdV equation

$$D_{*t}^{\alpha} u - 3(u^2)_x + u_{xxx} = 0, 0 < \alpha \leq 1, \tag{24}$$

subject to the initial conditions

$$u(x,0) = 6x. \tag{25}$$

Applying the Laplace transform first on both sides of Eq. (24). Thus, we get:

$$L[D_{*t}^{\alpha} u] - 3L[(u^2)_x] + L[u_{xxx}] = 0. \tag{26}$$

Using the differentiation property of the Laplace transform in Eq. (26), we get:

$$L[u(x,t)] = s^{-1}(6x) + s^{-\alpha} L[3(u^2)_x - u_{xxx}]. \tag{27}$$

Applying the inverse Laplace transform on both sides in Eq. (27), we get:

$$u(x,t) = 6x + L^{-1}(s^{-\alpha} L[3(u^2)_x - u_{xxx}]). \tag{28}$$

By applying the aforesaid homotopy perturbation method, we have:

$$\sum_{n=0}^{\infty} p^n u_n = 6x + p(L^{-1}(s^{-\alpha} L[3\sum_{n=0}^{\infty} p^n (H_n(u))_x - \sum_{n=0}^{\infty} p^n u_{nxxx}])). \qquad (29)$$

Equating the coefficient of the like power of p on both sides in Eq. (29), we get:

$$\begin{aligned} p^0 &: u_0(x,t) = 6x, \\ &\vdots \\ p^n &: u_n(x,t) = L^{-1}(s^{-\alpha} L[3(H_{n-1})_x - (u_{n-1})_{xxx}]), \end{aligned} \qquad (30)$$

where $n \in \mathbb{N}^*$.

The first few components of He's polynomials [26], for example, are given by:

$$\begin{aligned} H_0 &= u_0^2, \\ H_1 &= 2u_0u_1, \\ H_2 &= 2u_0u_2 + u_1^2, \\ H_3 &= 2u_0u_3 + 2u_1u_2, \\ &\vdots \end{aligned} \qquad (31)$$

Using He's polynomials (31) and the iteration formulas (30) we obtain:

$$\begin{aligned} u_0(x,t) &= 6x, \\ u_1(x,t) &= 6x(36)\frac{1}{\Gamma(\alpha+1)} t^{\alpha}, \\ u_2(x,t) &= 6x(36)^2 \frac{2}{\Gamma(2\alpha+1)} t^{2\alpha}, \\ u_3(x,t) &= 6x(36)^3 \frac{4\Gamma^2(\alpha+1) + \Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} t^{3\alpha}, \\ u_4(x,t) &= 6x(36)^4 \left[\frac{8\Gamma^2(\alpha+1) + 2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right] \frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)} t^{4\alpha}, \\ &\vdots \end{aligned} \qquad (32)$$

The first four terms of the decomposition series solution for Eq. (24) is given by:

$$u(x,t) = 6x + 6x(36)\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + 6x(36)^2\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}$$
$$+ 6x(36)^3\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}t^{3\alpha}$$
$$+ 6x(36)^4\left[\frac{8\Gamma^2(\alpha+1)+2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha}. \tag{33}$$

The first four terms of the decomposition series solution, for the special case $\alpha = 1$, is given by:

$$u(x,t) = 6x[1 + (36)t + (36)^2t^2 + (36)^3t^3 + (36)^4t^4 + \ldots]. \tag{34}$$

That gives:

$$u(x,t) = \frac{6x}{1-36t}, |36t| < 1, \tag{35}$$

which is an exact solution to the KdV equation as presented in [25].

Example 4.2 We next consider the time-fractional partial differential K(2,2) equation

$$D_{*t}^{\alpha}u + (u^2)_x + (u^2)_{xxx} = 0, 0 < \alpha \leq 1,$$
$$u(x,0) = x. \tag{36}$$

In a similar way as above we have:

$$\sum_{n=0}^{\infty} p^n u_n = x - p(L^{-1}(s^{-\alpha}L[\sum_{n=0}^{\infty} p^n (H_n(u))_x - \sum_{n=0}^{\infty} p^n (H_n(u))_{xxx}])). \tag{37}$$

Equating the coefficient of the like power of p on both sides in Eq. (37), we get:

$$p^0 : u_0(x,t) = x,$$
$$\vdots$$
$$p^n : u_n(x,t) = -L^{-1}(s^{-\alpha}L[(H_{n-1})_x + (H_{n-1})_{xxx}]), \tag{38}$$

where $n \in \mathbb{N}^*$.

Using He's polynomials (31) and the iteration formulas (38) we obtain:

$$u_0(x,t) = x,$$
$$u_1(x,t) = -2x\frac{1}{\Gamma(\alpha+1)}t^{\alpha},$$

$$u_2(x,t) = 4x\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}, \tag{39}$$

$$u_3(x,t) = -8x\left[\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]t^{3\alpha},$$

$$u_4(x,t) = 16x\left[\frac{8\Gamma^2(\alpha+1)+2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha},$$

$$\vdots$$

The first four terms of the decomposition series solution for Eq. (36) is given as:

$$u(x,t) = x - 2x\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + 4x\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}$$
$$-8x\left[\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]t^{3\alpha}$$
$$+16x\left[\frac{8\Gamma^2(\alpha+1)+2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha} + \ldots \tag{40}$$

The first four terms of the decomposition series solution for the special $\alpha = 1$, is given by:

$$u(x,t) = x[1 - 2t + 4t^2 - 8t^3 + 16t^4 + \ldots]. \tag{41}$$

That gives:

$$u(x,t) = \frac{x}{1+2t}, |2t| < 1, \tag{42}$$

which is an exact solution to the K(2,2) equation as presented in [25].

Example 4.3 We finally consider the Burgers equation

$$D_{*t}^{\alpha}u + \frac{1}{2}(u^2)_x - u_{xx} = 0, 0 < \alpha \le 1,$$
$$u(x,0) = x. \tag{43}$$

In a similar way as above we have:

$$\sum_{n=0}^{\infty} p^n u_n = x - p(L^{-1}(s^{-\alpha}L\left[\frac{1}{2}\sum_{n=0}^{\infty} p^n (H_n(u))_x - \sum_{n=0}^{\infty} p^n u_{nxx}\right])). \tag{44}$$

Equating the coefficient of the like power of p on both sides in Eq. (44), we get:

$$p^0 : u_0(x,t) = x,$$
$$\vdots$$
$$p^n : u_n(x,t) = -L^{-1}\left(s^{-\alpha} L\left[\frac{1}{2}(H_{n-1})_x - (u_{n-1})_{xx}\right]\right), \tag{45}$$

where $n \in \mathbb{N}^*$.

Using He's polynomials (31) and the iteration formulas (45) we obtain:

$$\begin{aligned}
u_0(x,t) &= x,\\
u_1(x,t) &= -x\frac{1}{\Gamma(\alpha+1)}t^{\alpha},\\
u_2(x,t) &= x\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha},\\
u_3(x,t) &= -x\left[\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]t^{3\alpha},\\
u_4(x,t) &= x\left[\frac{8\Gamma^2(\alpha+1)+2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha},\\
&\ \vdots
\end{aligned} \tag{46}$$

The first four terms of the decomposition series solution for Eq. (43) is given as:

$$\begin{aligned}
u(x,t) &= x - x\frac{1}{\Gamma(\alpha+1)}t^{\alpha} + x\frac{2}{\Gamma(2\alpha+1)}t^{2\alpha}\\
&- x\left[\frac{4\Gamma^2(\alpha+1)+\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}\right]t^{3\alpha}\\
&+ x\left[\frac{8\Gamma^2(\alpha+1)+2\Gamma(2\alpha+1)}{\Gamma^2(\alpha+1)\Gamma(3\alpha+1)}+\frac{4}{\Gamma(\alpha+1)\Gamma(2\alpha+1)}\right]\frac{\Gamma(3\alpha+1)}{\Gamma(4\alpha+1)}t^{4\alpha} + \ldots
\end{aligned} \tag{47}$$

The first four terms of the decomposition series solution for the special case $\alpha = 1$, is given by:

$$u(x,t) = x[1 - t + t^2 - t^3 + t^4 + \ldots]. \tag{48}$$

that gives:

$$u(x,t)=\frac{x}{1+t},|t|<1, \tag{49}$$

which is an exact solution to the Burgers equation as presented in [25].

5 Conclusion

In this paper, we have studied the time-fractional KdV, K(2,2) and Burgers equations by using the fractional homotopy perturbation transform method (FHPTM). The result shown that the FHPTM is an efficient method for calculating approximate solutions for nonlinear partial differential equations of fractional order. The solution obtained using the proposed method has a very high accuracy comparing with the variational iteration method. The method produces the same solution as the variational iteration method with the proper choice of the initial condition.

6 Open Problem

In this work, the fractional homotopy perturbation transform method (FHPTM) to be effective for solving the time-fractional KdV, K(2,2) and Burgers equations. One can apply fractional homotopy perturbation Sumudu transform method (FHPSTM),...,to the same problem.

Is it possible to solve this problem for $1<\alpha\leq 2$ and for the space-fractional?

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