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# Deformation and Stresses in Long Thermoelastic Pad Supports Under Prescribed Temperature by a Boundary Integral Method

**A.Y. Al-Ali, K.H. Almutairi, E.K. Rawy, M.S. Abou-Dina and A.F. Ghaleb**

Department of Mathematics, Faculty of Science,Cairo University, Geza, Egypt.
P.O.Box 12613, Geza.
e-mail:moustafa_abodina@hotmail.com

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**Abstract**

***A previously introduced boundary integral method is used to find an approximate solution to a problem of plane, uncoupled thermoelasticity inside an ellipse with hump. Part of the boundary is under a given variable pressure, while the other part is completely fixed. The singular behavior of the solution is put in evidence at those points where the boundary conditions change. The solution is then sought for in the form of series in Cartesian harmonics, enriched with a specially chosen harmonic function with singular boundary behavior to simulate the existing singularities. The results are analyzed in detail and the functions of practical interest are represented on the boundary and also inside the domain by three-dimensional plots. This model may be useful in analyzing the stresses that arise in long elastic pad supports under real conditions.***

**Keywords:** *Plane Uncoupled Thermoelasticity, Mixed Boundary Conditions, Boundary Integral Method, Cartesian Harmonics, Singular Behavior.*
**2010 Mathematics Subject Classification:** 03F55, 46S40.

# 1 Introduction

The Theory of Thermoelasticity has received considerable attention because of its importance, not only from the point of view of Technology and Environmental Sciences, but also for its theoretical and mathematical importance as a subject that embodies the interaction of mechanical and thermal fields. The monumental work of Nowacki [45] enlightens many aspects of this theory. Complex models of thermoelastic media may be found in [33]. Different mathematical methods have been used to tackle the problem of thermoelasticity. Shanker and Dhaliwal [52] use integral representations for the basic unknowns of the problem of asymmetric thermoelasticity. Singh and Dhaliwal [53] consider mixed boundary-value problems of thermoelastostatics and electrostatics. Abou-Dina and Ghaleb [4] propose a boundary integral method for the solution of plane strain problems of thermoelasticity in stresses in real functions for homogeneous isotropic media in simply connected regions. The method is applied to a number of examples with boundary conditions of the first, or of the second type only, but the case of mixed conditions was not treated. Computational aspects of this method are considered in [2], and an application for the ellipse is treated in [14]. An approach by complex analysis may be found in [45, 26]. In this latter work, the authors present the general solutions of two-dimensional problems under uniform heat flux and under point source. Along the same pattern, Han and Hasebe [25] derive Green's function for thermal stress boundary-value problem of an infinite plane with a hole under adiabatic or isothermal conditions. Meleshko [43] addresses the problem of determining thermal stresses in a rectangle by Fourier series with examples [43]. Seremeta [51] presents integral representations for thermoelastic Green's functions for Poisson's equation with numerous examples. Thermoelasticity with electromagnetic interactions may be found in [9, 16].

Thermoelastostatics belong to a more general class of problems, the elliptic boundary-value problems. This vast subject relies heavily on results from the Theory of Potential. Elements of this theory and many of its theoretical and numerical aspects may be found in [39, 35, 31]. Properties of elliptic operators have been studied extensively. Many problems have been solved involving elliptic problems, mainly in the fields of thermostatics and elastostatics. The method of fundamental solutions was used in numerous publications ([19] and the references therein). Abou-Dina [1] treats some problems related to the harmonic and the biharmonic operators relying on Trefftz's method. Abou-Dina and Ghaleb investigate the approximate solutions to some regular and singular boundary-value problems for Laplace's operator in rectangular regions by a boundary Fourier expansion [6]. Read [47] uses analytic series to find solutions to Laplacian problems with mixed boundary conditions. Problems with mixed boundary conditions are also treated in [24, 13]. El-Dhaba' *et al.*

investigate the deformation of a rectangle by finite Fourier transform [15].

When the shape of the boundary is of complicated shape, one has to use either purely numerical techniques or to semi-analytical methods. Among these, the Boundary Integral Formulations are of prime importance. Such approaches are usually linked with the well-developed theory of Fredholm integral equations. An extensive literature exists on the subject. The use of integral equation methods in potential theory and in elastostatics is illustrated in [39, 35]. Altiero and Gavazza [7] present a unified boundary integral method for linear elastostatics. Heise [27, 28] applies boundary integral equations to treat problems of elastostatics with discontinuous boundary conditions. Koizumia *et al.* [37] present a boundary integral equation analysis for thermoelastostatics using thermoelastic potential. Constanda [11, 12] applies a boundary integral formulation to solve Dirichlet and Neumann problems of elasticity. Different applications of integral equation methods are presented in [48, 46, 49]. A special method of solution is treated in [57]. Elliotis *et al.* [17] present a boundary integral method adapted to the biharmonic equation with crack singularities. Li *et al.* [42] present a numerical solution for models of linear elastostatics involving crack singularities. A review of boundary integral methods in the theory of elasticity of hemitropic materials may be found in [44]. Cheng *et al.* [10] investigate mechanical quadrature methods and extrapolation algorithms for boundary integral equations with linear boundary conditions in elasticity.

The role of mixed boundary conditions in the correct formulation of boundary-value problems is recognized by Weaver and Sarachik [55]. The presence of such boundary conditions adds to the difficulty of solving the problem and requires special attention. Mixed boundary conditions are considered in [53]. Helsing [29] presents an integral equation method to solve Laplace's equation under mixed Dirichlet and Neumann conditions on contiguous parts of the boundary, and the problem of elastostatics under mixed conditions. The same author proposes a fast and stable algorithm for treating singular integral equations on piecewise smooth curves [30]. Boundary-value problems of mixed type with applications are also considered by Khuri [36]. Gjam *et al.* [21] give an approximate solution to the problem of the ellipse with half boundary fixed and the other half under given pressure, and use expansions involving a harmonic function with logarithmic singular behavior at the boundary.

The presence of corner boundary points, or mixed-type boundary conditions introduces singular behavior of the solution. This greatly influences the efficiency of computations. An extensive treatment of singularities exists in the literature for the Laplacian, as well as for the elastic problems. Williams [56] discusses stress singulariries in plates. An algorithm for plane potential solving problems with mixed boundary conditions involving extraction of singularities is treated in [23]. Gusenkova and Pleshchinskii [22] construct complex potentials with logarithmic singulariries for elastic bodies with defect along a smooth arc. Abou-Dina and Ghaleb [6] introduce logarithmic singularities on the boundary of rectangular domains for approximate solutions to Laplacian boundary-value problems with mixed boundary conditions. Kotousov and Lew [38] study stress singularities under various boundary conditions at angular corners of plates. El-Seadawy *et al.* [18] use boundary integrals to solve 2D problems with mixed geometry. Helsing and Ojala [32] treat corner singularities for elliptic problems by boundary integral equation methods on domains having a large number of corners and branching points. Mixed-type boundary conditions at corners are treated in [6, 42, 41, 40]. Gillman *et al.* [20] present simplified techniques for discretizing the boundary integral equations in 2D domains with corners. An interesting contribution about singular solutions of Laplace's equation may be found in [54]

In the present paper, two problems of uncoupled thermoelasticity are solved on a domain in the form of an ellipse with elliptical hump, and on a rectangular domain. The thermal boundary condition is of Dirichlet type, i.e. the temperature is prescribed on the boundary. As to the mechanical boundary conditions, they are of mixed type: One part of the boundary is subjected to a variable pressure, while the remaining part is totally fixed. Generalizing the semi-analytical scheme presented in [21], the problem is reduced to a collection of two subproblems of uncoupled thermoelasticity having common solution, one of type I (given stresses) and the other of type II (given displacements). Each of these two subproblems has given entries on part of the boundary, while the other part carries unknown values to be determined within the framework of the solution. The two subproblems are then reduced to a system of boundary integral equations following [2]. Discretization then yields a rectangular system of linear algebraic equations. The obtained results on the boundary clearly put in evidence the existence of a singular behavior of the stress components at the two separation boundary points. This is simulated by introducing a specially designed harmonic function in the cross-sectional domain, with singularities in the second derivatives at the two separation points. Based on this, a solution is proposed in the bulk for the two basic harmonic functions in the form of expansions in Cartesian harmonics, which include the singular harmonic function as part of the stress function. The coefficients of these expansions are then determined by the Boundary Collocation Method. Boundary plots, as well as three-dimensional plots are provided for the functions of practical interest. The results and the efficiency of the used scheme are discussed. All figures were produced using Mathematica 9.0 Software.

The problems under consideration model a long elastic pad support and thereby is of practical importance. The stresses applied on one part of the boundary represent the influence of the body resting on the foundation. The chosen forms of the boundary involve corner points and mixed boundary conditions. These factors are challenging from the computational point of view and the obtained results clearly indicate the efficiency of the proposed method. The boundary shapes, as well as the chosen boundary functions, are only representative. Other settings may be considered as well, provided the overall equilibrium is conserved. For example adding a shear stress on the boundary or treating a multilayer foundation [8].

# 2 Problem description.

This study is devoted to the uncoupled plane theory of thermoelasticity for long cylinders made from an isotropic, homogeneous, elastic material. The normal cross-section $D$ of the cylinder is simply connected and bounded by a contour $C$. An orthogonal system of Cartesian coordinates $(x, y)$ with origin $O$ inside the domain $D$ is used to express the necessary mathematical relations for the problem. The lateral surface of the cylinder is acted upon by forces and is subjected to thermal action which will cause deformation of the body. Body forces and bulk heating are not considered for simplicity. In the case of angular boundary points, the contour $S$ must be properly smoothened.

The parametric representation of the contour $C$ is:

$$x = x(\theta), \qquad y = y(\theta), \tag{1}$$

where $\theta$ is the angular parameter measured from the $x$-axis in the usual positive direction along the contour.
The vectors $\boldsymbol{\tau}$ and $\boldsymbol{n}$ denote the unit vector tangent to $C$ at any arbitrary point on the contour, and the unit outwards normal at this point respectively. One has:

$$\boldsymbol{\tau} = \frac{\dot{x}}{\omega}\boldsymbol{i} + \frac{\dot{y}}{\omega}\boldsymbol{j} \quad \text{and} \quad \boldsymbol{n} = \frac{\dot{y}}{\omega}\boldsymbol{i} - \frac{\dot{x}}{\omega}\boldsymbol{j}, \tag{2}$$

the 'dot' over a symbol denotes differentiation with respect to the parameter $\theta$, and

$$\omega = \sqrt{\dot{x}^2 + \dot{y}^2}. \tag{3}$$

In the particular case where the length on the curve is taken as parameter, $\omega = 1$.

# 3 Basic equations.

The basic equations governing the plane, linear uncoupled theory of Thermoelasticity are listed below without proof in accordance with [4] and [2].

## 3.1 Equation of thermostatics.

In the steady state, the temperature $T$ as measured from a reference temperature $T_0$ satisfies Laplace's equation:

$$\nabla^2 T = 0, \tag{4}$$

This harmonic function in $D$ is completely determined from the solution of the heat problem, once the thermal boundary condition has been specified.

## 3.2 Equations of equilibrium.

In the solution by stresses and in the absence of body forces, the identically non-vanishing stress components are defined through a stress function $U$ by the relations:

$$\sigma_{xx} = \frac{\partial^2 U}{\partial y^2}, \qquad \sigma_{xy} = -\frac{\partial^2 U}{\partial x\,\partial y}, \qquad \sigma_{yy} = \frac{\partial^2 U}{\partial x^2}. \tag{5}$$

and this function by the compatibility condition satisfies the biharmonic equation

$$\nabla^4 U = 0. \tag{6}$$

The generalized Hooke's law reads:

$$\begin{aligned}
\sigma_{xx} &= \frac{\nu E}{(1+\nu)\,(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{E}{(1+\nu)}\frac{\partial u}{\partial x} - \frac{\alpha E}{(1-2\nu)}T \\
\sigma_{xy} &= \frac{E}{2(1+\nu)}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \\
\sigma_{yy} &= \frac{\nu E}{(1+\nu)(1-2\nu)}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{E}{(1+\nu)}\frac{\partial v}{\partial y} - \frac{\alpha E}{(1-2\nu)}T
\end{aligned} \tag{7}$$

where $E$, $\nu$ and $\alpha$ denote Young's modulus, Poisson's ratio and the coefficient of linear thermal expansion respectively for the considered thermoelastic medium, and $u, v$ denote the displacement components.
The stress function $U$ solving the equation (6) is represented through two harmonic functions as:

$$U = x\,\phi + y\,\phi^c + \psi \tag{8}$$

where the superscript '$c$' denotes the harmonic conjugate.
The stress components are expressed in terms of $\phi$ and $\psi$ as:

$$\begin{aligned}
\sigma_{xx} &= x\,\frac{\partial^2 \phi}{\partial y^2} + 2\,\frac{\partial \phi^c}{\partial y} + y\,\frac{\partial^2 \phi^c}{\partial y^2} + \frac{\partial^2 \psi}{\partial y^2} \\
\sigma_{xy} &= -x\,\frac{\partial^2 \phi}{\partial x\,\partial y} - y\,\frac{\partial^2 \phi^c}{\partial x\,\partial y} - \frac{\partial^2 \psi}{\partial x\,\partial y} \\
\sigma_{yy} &= x\,\frac{\partial^2 \phi}{\partial x^2} + 2\,\frac{\partial \phi}{\partial x} + y\,\frac{\partial^2 \phi^c}{\partial x^2} + \frac{\partial^2 \psi}{\partial x^2}
\end{aligned} \tag{9}$$

One easily obtains the following representation of the Cartesian displacement components $u$ and $v$:

$$\frac{E}{(1+\nu)}\,u = -\frac{\partial U}{\partial x} + 4\,(1-\nu)\,\phi + \frac{E}{1+\nu}\,u_T,$$
$$\frac{E}{(1+\nu)}\,v = -\frac{\partial U}{\partial y} + 4\,(1-\nu)\,\phi^c + \frac{E}{1+\nu}\,v_T \tag{10}$$

where

$$u_T = \alpha\,(1+\nu)\int_{M_0}^{M}(T\,dx - T^c\,dy),$$
$$v_T = \alpha\,(1+\nu)\int_{M_0}^{M}(T^c\,dx + T\,dy) \tag{11}$$

are the temperature displacements. The integrals in (11) are noted in complex form in ([45], p. 323).

Rewritten in terms of $\phi$ and $\psi$, relations (10) yield:

$$2\mu\,u = (3-4\,\nu)\,\phi - x\frac{\partial\phi}{\partial x} - y\,\frac{\partial\phi^c}{\partial x} - \frac{\partial\psi}{\partial x} + 2\mu\,u_T,$$
$$2\mu\,v = (3-4\,\nu)\phi^c - x\,\frac{\partial\phi}{\partial y} - y\,\frac{\partial\phi^c}{\partial y} - \frac{\partial\psi}{\partial y} + 2\mu\,v_T \tag{12}$$

where $\mu = \frac{E}{2(1+\nu)}$ is the modulus of rigidity of the elastic material.

# 4 Necessary closure conditions

For a unique solution to the considered problem, the basic field equations are complemented by boundary conditions, the conditions for removal of rigid body motion, and other conditions which have no physical insight.

## 4.1 Boundary conditions

• **Heat problem**
In what follows, only the Robin problem will be considered.
○ **The Dirichlet Problem**
In this type of problem, the temperature function is given on the boundary $C$ of the domain $D$ by a relation of the form:

$$T(s) = g(s), \tag{13}$$

by using the boundary integral representation of harmonic function one gets:

$$T(s) - g(s) = \frac{1}{\pi} \oint_C \left( T(s') \frac{\partial \ln R}{\partial n'} - \frac{\partial T(s')}{n'} \ln R \right) ds' - g(s) = 0., \qquad (14)$$

• <u>**Mechanical problem**</u>

◦ **The first fundamental problem of elasticity**

Assuming that the density of the given distribution of the total external surface forces is:

$$\boldsymbol{f} = f_x \boldsymbol{i} + f_y \boldsymbol{j} = \sigma_{nx} \boldsymbol{i} + \sigma_{ny} \boldsymbol{j},$$

the boundary conditions take the form:

$$\begin{aligned} f_x &= (x\phi_{yy} + 2\phi^c_y + y\phi^c_{yy} + \psi_{yy}) \frac{\dot{y}}{\omega} + (x\phi_{xy} + y\phi^c_{xy} + \psi_{xy}) \frac{\dot{x}}{\omega}, \\ f_y &= -(x\phi_{xy} + y\phi^c_{xy} + \psi_{xy}) \frac{\dot{y}}{\omega} - (x\phi_{xx} + 2\phi^c_x + y\phi^c_{xx} + \psi_{xx}) \frac{\dot{x}}{\omega}. \end{aligned} \qquad (15)$$

◦ **The second fundamental problem of elasticity**

Assuming that the displacement vector is

$$\boldsymbol{d} = d_x \boldsymbol{i} + d_y \, \boldsymbol{j} = d_n \boldsymbol{n} + d_\tau \boldsymbol{\tau},$$

the boundary conditions take the form:

$$\begin{aligned} 2\mu \, d_x &= (3 - 4\,\nu)\, \phi - x\, \phi_x - y\, \phi^c_x - \psi_x + 2\mu\, u_T, \\ 2\mu \, d_y &= (3 - 4\,\nu)\, \phi^c - x\, \phi_y - y\, \phi^c_y - \psi_y + 2\mu\, v_T. \end{aligned}$$

## 4.2 Elimination of rigid body motion

This is applied for the first fundamental problem of elasticity in order to get rid of the rigid body motion in this case. It requires that the displacement and the rotation vectors vanish at an arbitrary chosen point $(a, b)$ inside $D$. Keeping in mind our choice of the coordinate system, one may write these conditions at the origin for the rigid body motion. For the second fundamental problem of elasticity, this is secured by the nature of the boundary conditions.

◦ **Eliminating the rigid body translation**

This condition is expressed at the origin by:

$$\begin{aligned} (3 - 4\nu)\, \phi(0,0) - \psi_x(0,0) + 2\mu\, u_T(0,0) &= 0 \\ (3 - 4\nu)\, \phi^c(0,0) - \psi_y(0,0) + 2\mu\, v_T(0,0) &= 0 \end{aligned} \qquad (16)$$

○ **Eliminating the rigid body rotation**
This condition is expressed at the origin by:

$$\frac{\partial u}{\partial y}(0,0)-\frac{\partial v}{\partial x}(0,0)=4\,(1-\nu)\,\phi_y(0,0)-2\,\mu\,\alpha\,(1+\nu)\,T^c(0,0)=0 \qquad (17)$$

In setting any of the closure conditions, the first and the second derivatives of any harmonic function $f$ with respect to $x$ and $y$ on the boundary may be calculated as in Appendix (B).

### 4.3 Additional simplifying conditions

The following supplementary purely mathematical conditions are adopted for simplicity at the point of the boundary where $\theta = 0$:

$$\begin{aligned}
x(0)\,\phi(0)+y(0)\,\phi^c(0)+\psi(0)&=0\\
x(0)\,\phi^c(0)-y(0)\,\phi(0)+\psi^c(0)&=0\\
x(0)\,\phi_x(0)+\phi(0)+y(0)\phi^c_x(0)+\psi_x(0)&=0\\
x(0)\,\phi_y(0)+\phi^c(0)+y(0)\,\phi^c_y(0)+\psi_y(0)&=0\\
T^c(0,0)&=0\\
T(0,0)&=0
\end{aligned} \qquad (18)$$

All the above mentioned mechanical equations and conditions can be transformed into boundary integral equations by using the boundary integral representation of the basic harmonic functions $\phi$ and $\psi$ (and their conjugates), together with the Cauchy-Riemann relations. For details, the reader is referred to [4] and [2].

## 5 Calculation of the harmonic functions at internal points

We write down expansions of the four harmonic functions involved in the calculations in terms of some adequately chosen basis. The expansion coefficients may then be determined via the well-known Boundary Collocation Method

(BCM). These expansions are taken as follows:

$$\phi(x,y) = A + a_0 x + \sum_{n=1}^{\infty} a_n \cos(nx)\cosh(ny) + \sum_{n=1}^{\infty} b_n \sin(nx)\cosh(ny),$$

$$\phi^c(x,y) = B + a_0\, y - \sum_{n=1}^{\infty} a_n \sin(nx)\sinh(ny) + \sum_{n=1}^{\infty} b_n \cos(nx)\sinh(ny),$$

$$\psi(x,y) = F + d_0 x + \sum_{n=1}^{\infty} d_n \cos(nx)\cosh(ny) + \sum_{n=1}^{\infty} e_n \sin(nx)\cosh(ny) + \psi_s(x,y),$$

$$T(x,y) = H + j_0 x + \sum_{n=1}^{\infty} j_n \cos(nx)\cosh(ny) + \sum_{n=1}^{\infty} k_n \sin(nx)\cosh(ny),$$

$$T^c(x,y) = L + j_0 y - \sum_{n=1}^{\infty} j_n \sin(nx)\sinh(ny) - \sum_{n=1}^{\infty} k_n \sin(nx)\cosh(ny).$$

where $\psi_s(x,y)$ is defined in Appendix (C).

# 6 Numerical treatment

In order to write down the discretized form of the basic equations and conditions, the complete angle $2\pi$ is uniformly divided into a sufficiently large number of sections, $p$. Consequently, the contour $C$ is approximated to a broken closed contour with unequal side lengths. Any contour integration on $D$ will be approximated by a finite sum as usual. Derivatives of functions on $C$ are also approximated in a proper way. Details of the calculations may be found in [2, 5, 21]. After discretization of all the basic equations and conditions, a linear rectangular algebraic system of equations is obtained for the boundary values of the unknown functions.

# 7 The ellipse with elliptical hump.

Consider a normal cross-section in the form of an ellipse with hump as shown on Fig.(1). This is obtained from the intersection of two identical ellipses with semi-major and semi-minor axes lengths $a_1 = b_2 = 1$ and $b_1 = a_2 = 0.5$ respectively. The origin of coordinates $O$ is taken at the common center of the two ellipses, the $x$-axis being directed along the major axis of the elliptical hump.

For an efficient application of the method, the cross-sectional contour must be properly smoothened. Figure (1) shows the original contour, the smoothed contour, and the comparison. The smoothed boundary is formed by curve fitting using sines and cosines of multiples of the angular parameter. The parametric equations for the original contour are given in Appendix C.

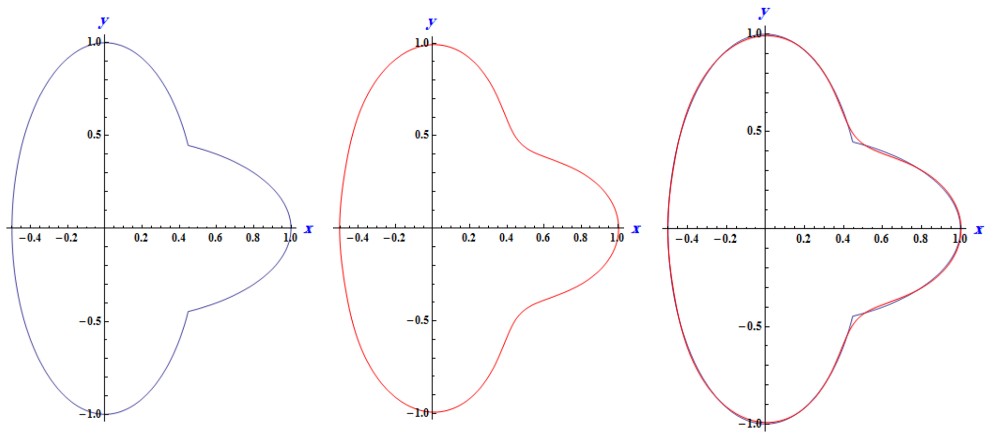


Figure 1: Ellipse with elliptical hump. Original and smoothed boundaries.

The boundary of the domain is subjected to the following conditions:

- Dirichlet thermal condition

$$T(\theta) = h_1 \cos 2\theta, \qquad\qquad h_1 = 0.05.$$

- The right half of the boundary is subjected to a pressure $p$ with intensity

$$p(\theta) = h_2 \sin(\theta_1 - \theta)\,\sin(\theta_2 - \theta), \quad 0 \le \theta < \theta_1 \wedge \theta_2 < \theta \le 2\pi,$$

  and we have taken $h_2 = 0.05$. This choice makes the pressure distribution tend to zero smoothly enough at both ends of its interval of definition.

- The left half of the boundary is completely fixed,

$$u = 0, \quad v = 0, \quad \theta_1 \le \theta \le \theta_2, \tag{19}$$

  where

$$\theta_1 = 1.5719547718343965 \approx \frac{\pi}{2}, \qquad \theta_2 = 4.71123053534519 \approx \frac{3\,\pi}{2}.$$

Once the solution for temperature has been obtained, the mechanical problem is replaced by two subproblems, one of type I and the other of type II, having a common solution (Cf. [21]). For each of these two subproblems, the boundary conditions are given on one part of the boundary and complemented with unknown values on the other part, to be determined within the solution of the problem. Following the scheme presented in [3, 5], the equations for each of these two subproblems are reduced to a system of boundary integral equations which are then discretized as explained above. The singular behavior of the stress components at the two separation boundary points is put in evidence. Accordingly, a harmonic function with singular boundary behavior is proposed and added to the basic harmonic function $\psi$ in order to obtain the solution in the bulk. It is worth noting that this function gives rise to the same type of behavior of the two stress components $\sigma_{xx}$ and $\sigma_{yy}$. If this is not the case, then suitable singular terms have to be added to the functions $\phi$ and $\phi^c$. The coefficients in the expansions given above are determined by the Boundary Collocation Method. Plots are provided for the boundary values of the unknown functions, and three-dimensional plots for the solution in the bulk. The efficiency of the used numerical scheme is discussed. All figures were produced using Mathematica 9.0 Software. The following figures were obtained with a number of nodal points $p = 155$. A further increase in the value of $p$ caused a deterioration of the results. The expansions for the unknown functions were truncated to 4 terms for the temperature, and to 12 for the other functions. The Least Squares method was used to solve all the resulting systems of equations. After a solution is obtained, it is substituted back into the equations for error analysis. For the boundary analysis, the errors did not exceed $7.8 \times 10^{-4}$. For the determination of the expansion coefficients, the error did not exceed $3.1 \times 10^{-2}$.

Fig.(2) expresses the boundary displacement due to temperature only. Such displacement does not satisfy any boundary conditions. The obtained results seem to be compatible with the heat exchange on the boundary.

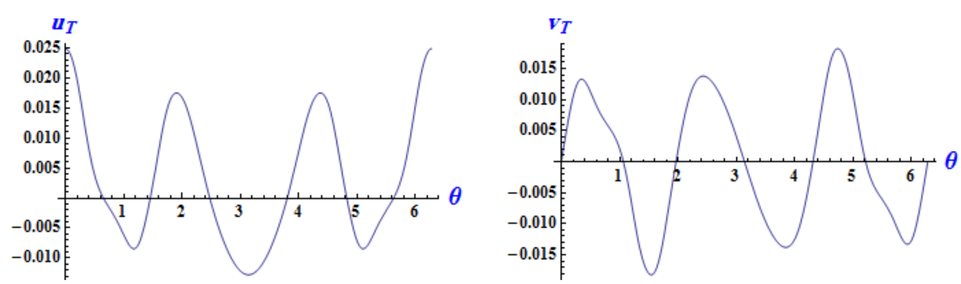


Figure 2: Temperature displacements $u_T$ and $v_T$ on the boundary.

The plots given on Fig.(3), (4), (5), (6), (7) and (8) show the values of the unknown functions as obtained from the boundary analysis (dotted curves), together with the values calculated from the expansions (line curves) for comparison. While the boundary curves of the basic harmonic functions and the stress function seem smooth enough, the fluctuations increase for the displacement components which include first derivatives, and increase even more for the stress components which involve the second derivatives of these functions. One also notices the discontinuities occurring in the stress components at the boundary separation points. Based on these observations, we have enriched the above expansion of the harmonic function $\psi$ with a harmonic function that has a singular boundary behavior at the separation points. The way to build such a function is explained in detail in Appendix (D). The stresses resulting from this function considered as stress function are shown on Fig.(28).

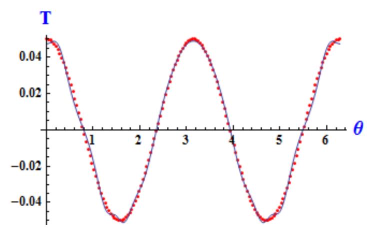


Figure 3: Temperature $T$ on the boundary.

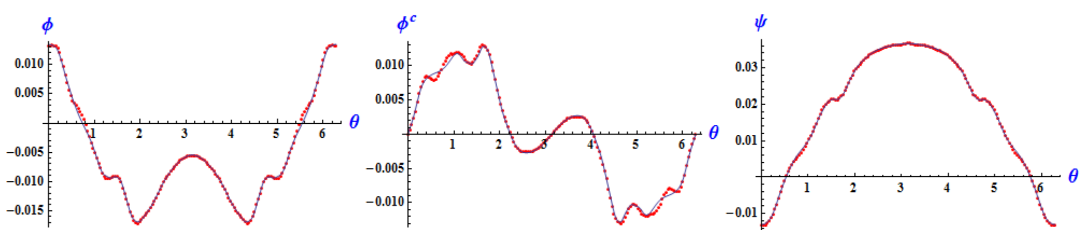


Figure 4: The harmonic functions on the boundary

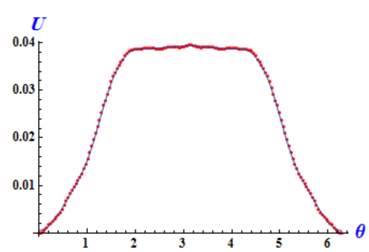


Figure 5: Stress function on the boundary

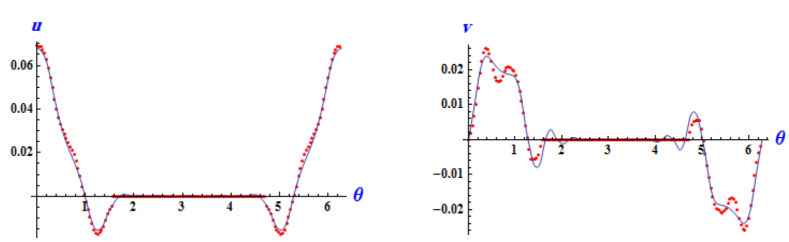


Figure 6: Displacement on the boundary

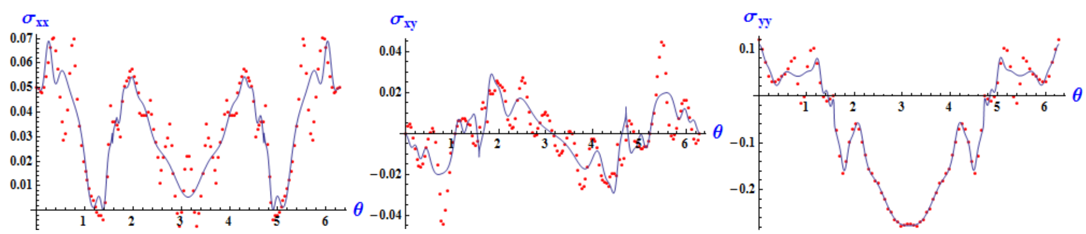

Figure 7: Components of the stress tensor on the boundary

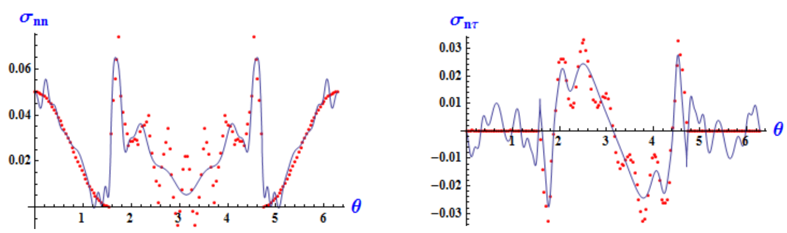

Figure 8: Tangential and normal components of the stress tensor on $C$

The deformed contour is shown on Fig.(9). It represents the combined action of external mechanical and thermal factors. The same figure contains the boundary displacement due to temperature alone.

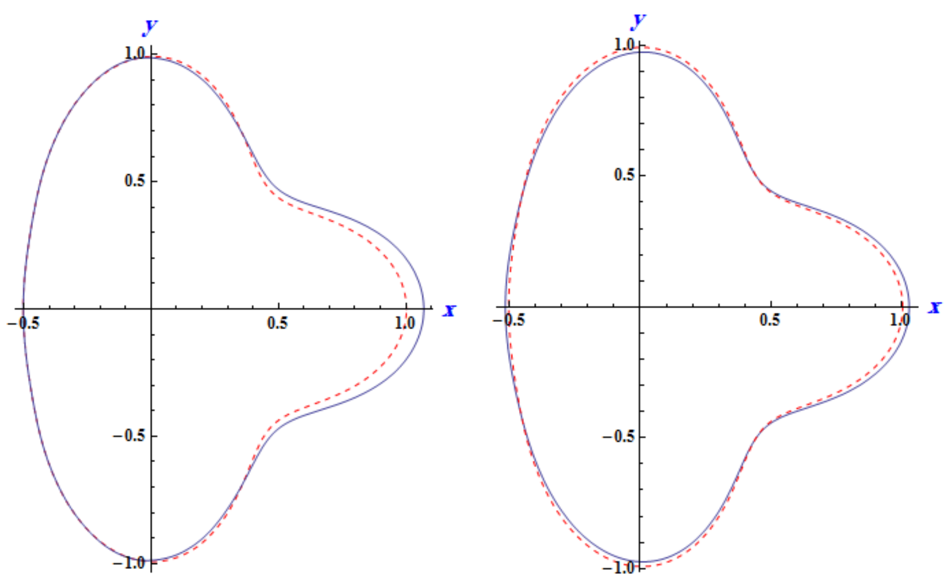

Figure 9: Total displacement (left) and temperature displacement (right), each compared with the original boundary (dashed curve).

The stress vector distribution on the boundary is shown on Fig.(10). It permits to appreciate the direction of this vector, as well as its magnitude as compared with the applied pressure. It is interesting to note that this vector is directed inwards everywhere on the boundary, except at two locations on the fixed part, close to the separation points. It is at these two points that a debonding can potentially take place. These two emplacements correspond to two humps with positive values on the curve for $\sigma_{nn}$ on Fig.(28). One also notices the larger stresses on the central part of the fixed boundary, just opposite to the given larger stresses on the hump.

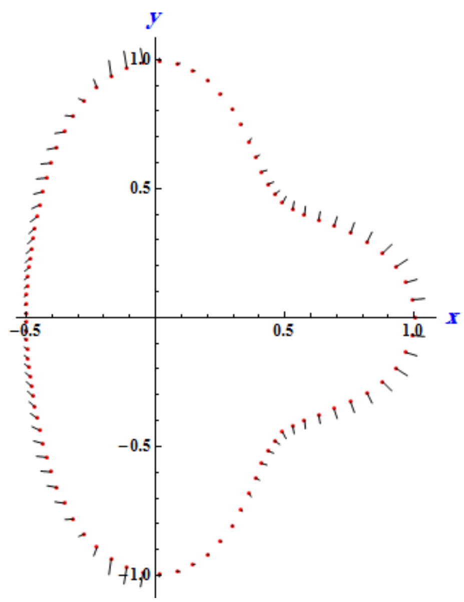


Figure 10: Stress vector distribution on the boundary.

The bulk distributions of functions of practical interest are shown on Figures. (11), (12), (13) and (14). The cross-sectional domain is also shown for convenience. The larger displacements along the $x$-axis takes place around the tip of the hump, as expected, while the larger displacements along the $y$-axis occur around the points of intersection of the two ellipses.

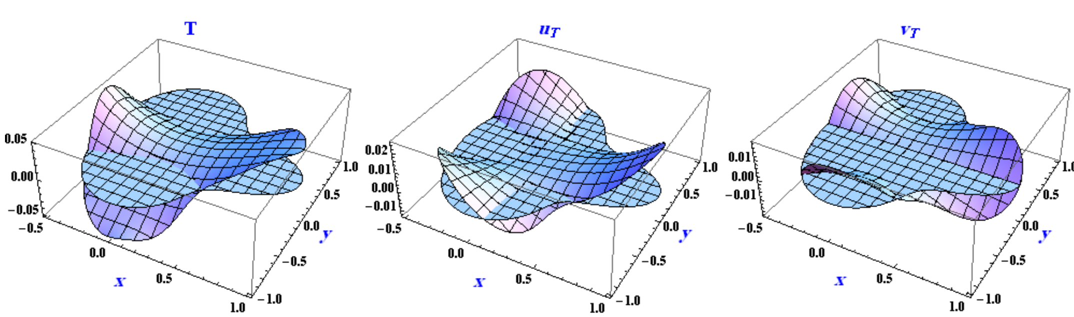


Figure 11: $T(x, y), u_T(x, y)$ and $v_T(x, y)$

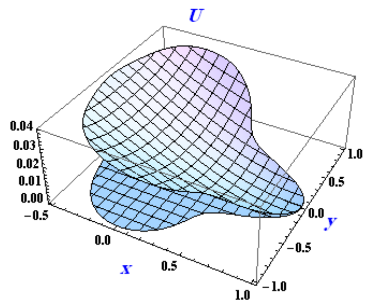


Figure 12: $U(x,y)$

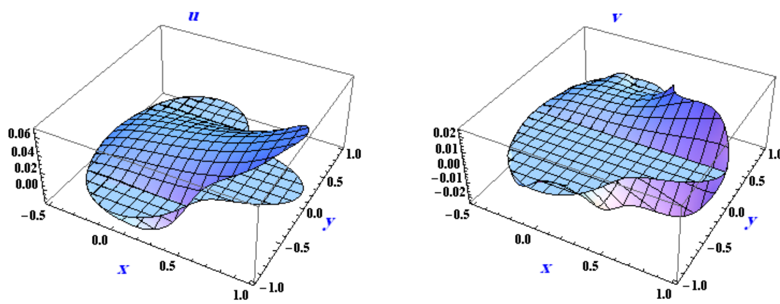


Figure 13: $u(x,y)$ and $v(x,y)$

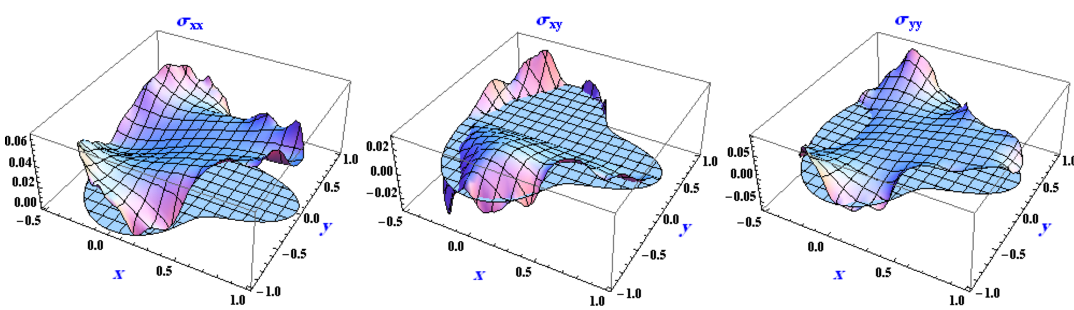


Figure 14: $\sigma_{xx}(x,y), \sigma_{xy}(x,y)$ and $\sigma_{yy}(x,y)$

# 8 The rectangle.

Consider an infinitely long cylinder of rectangular normal cross-section from an isotropic, homogeneous, elastic material. A system of orthogonal Cartesian coordinates is used, with origin 0 at the center of the rectangle, $x-$axis as shown on Fig.(15). The portion of the rectangular boundary lying in the first quadrant $(0 \leq \theta \leq \frac{\pi}{2})$ may be expressed parametrically as follows:

$$x(\theta) = \begin{cases} a & 0 \le \theta \le \Theta, \\ b\,\cot\theta & \Theta \le \theta \le \frac{\pi}{2}, \end{cases}$$

and

$$y(\theta) = \begin{cases} a\,\tan\theta & 0 \le \theta \le \Theta, \\ b & \Theta \le \theta \le \frac{\pi}{2}, \end{cases}$$

with $\Theta = \tan^{-1}(\frac{b}{a})$. Here, $2a$ and $2b$ are respectively the length and width of the rectangle, while $\theta$ denotes the polar angle of a general point on the rectangle. For dimension analysis purposes, the half-length of the major axis is taken to be the characteristic length, i.e. $a$ is taken to be equal to unity, $a = 1$. Also, we take $b = 0.7$ for definiteness.
As for the previous problem, the cross-sectional contour must be properly smoothened. Fig.(15) shows the original contour, the smoothed contour, and the comparison. The smoothed boundary is again formed by replacing sections around the corners by smooth curves, then by curve fitting using sines and cosines of multiples of the angular parameter.

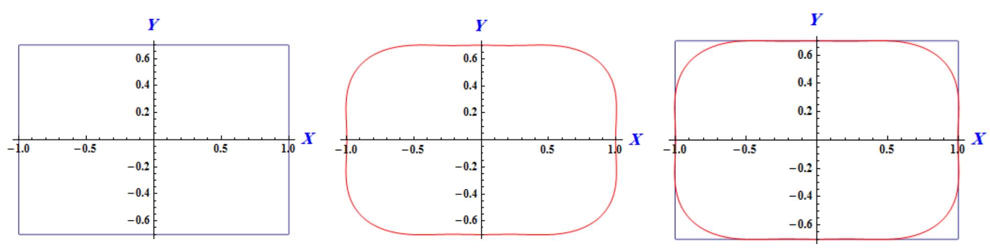

Figure 15: Rectangle. Original and smoothed boundaries.

The boundary of the domain is subjected to a prescribed heat flux and is fixed on one half, while the other half is under given variable pressure. Such conditions may take place in parts of electrical instruments:

- Dirichlet thermal condition

$$T(\theta) = h_1\,(1 + \cos 2\theta), \qquad h_1 = 0.05.$$

- The right half of the boundary is subjected to a pressure $p$ with intensity

$$p(\theta) = h_2 \cos\,\theta, \quad 0 \le \theta < \theta_1 \,\wedge\, \theta_2 < \theta \le 2\pi,$$

  and we have taken $h_2 = 0.05$. This choice makes the pressure distribution tend to zero smoothly enough at both ends of its interval of definition.

- The left half of the boundary is completely fixed,

$$u = 0, \quad v = 0, \quad \theta_1 \leq \theta \leq \theta_2, \tag{20}$$

where

$$\theta_1 = \frac{\pi}{2}, \qquad \theta_2 = \frac{3\,\pi}{2}.$$

The solution proceeds exactly as for the first problem. A singular solution is added to the basic harmonic function $\psi$ in order to obtain the solution in the bulk as series expansions in Cartesian harmonics. The coefficients in these expansions are determined by Boundary Collocation Method. Plots are provided for the boundary values of the unknown functions, and three-dimensional plots for the solution in the bulk. The efficiency of the used numerical scheme is discussed, as well as the obtained results.

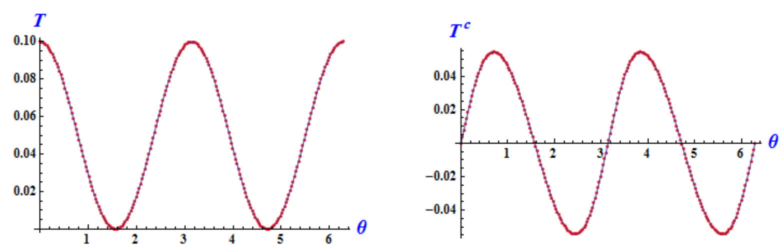


Figure 16: $T(x, y)$ and $T^c(x, y)$ in $C$

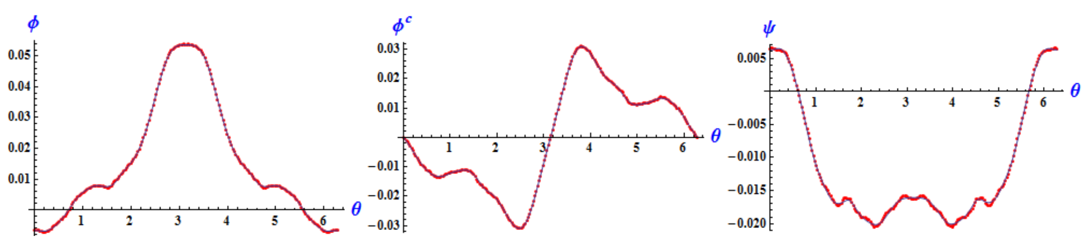


Figure 17: The harmonic function in $C$

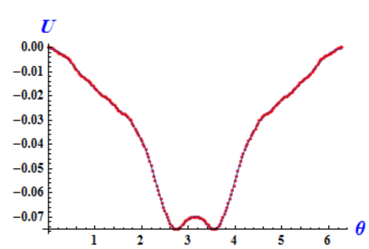


Figure 18: Stress function in $C$

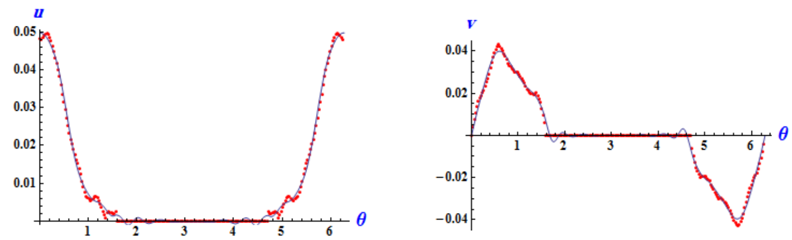


Figure 19: Displacement in $C$

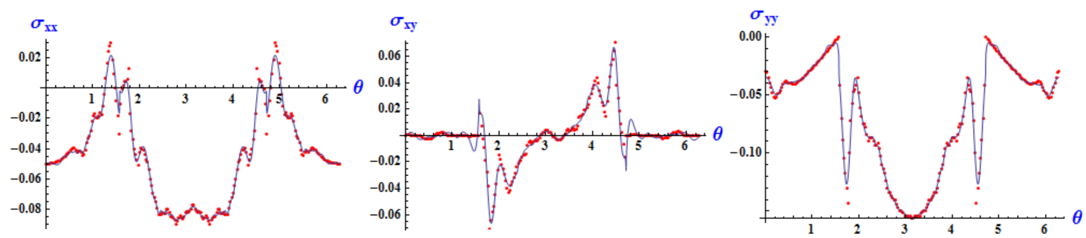


Figure 20: Components of the stress tensor in $C$

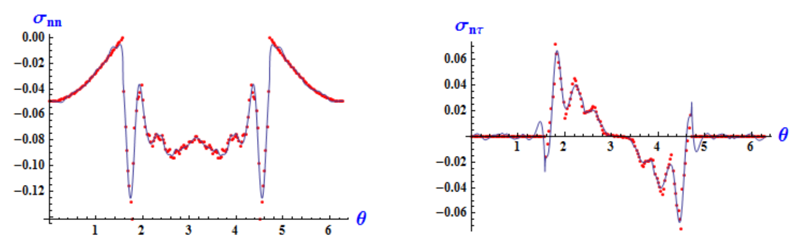


Figure 21: Tangential and normal components of the stress tensor on $C$

The plots on the Figs.(16), (17), (18), (19), (20) and (21) show the values of the unknown functions as obtained from the boundary analysis (dotted curves), together with the values calculated from the expansions (line curves). The boundary distribution of temperature is seen to be in accordance with the given heat flux. While the boundary curves of the basic harmonic functions seem smooth enough, the fluctuations increase for the displacement components which include first derivatives, and increase even more for the stress components which involve the second derivatives. The calculation of the second derivatives on the boundary is a major source of error in the proposed method. Different methods of calculations were used. The present results correspond to calculations involving the nearest 15 points from both sides of the considered node. One notices the discontinuities occurring in the stress components at the boundary transition points. Based on these observations, we have enriched the expansion of the basic harmonic function $\psi$ with a harmonic function that has a singular boundary behavior at the separation points. Details are presented in Appendix (D). The stresses resulting from this singular function taken as stress function are shown on Fig.(28).

The deformed contour is shown on Fig.(22). It represents the combined action of external mechanical and thermal factors.

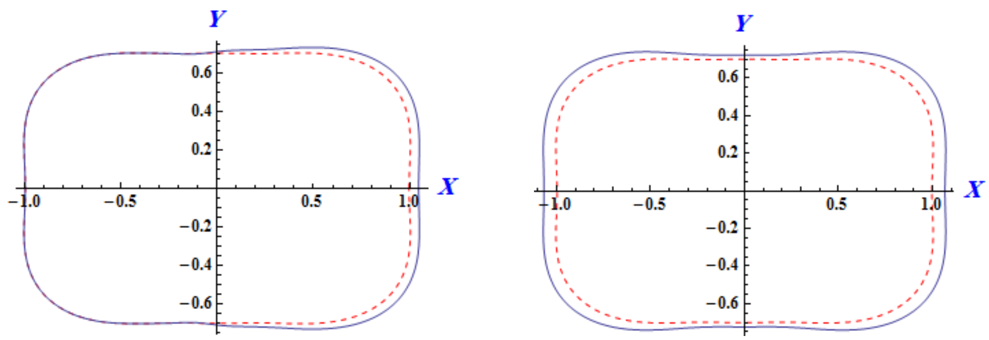


Figure 22: Total displacement (left) and temperature displacement (right), each compared with the original boundary (dashed curve).

The stress vector distribution on the boundary is shown on Fig.(23). It permits to appreciate the direction of this vector, as well as its magnitude as compared with the applied pressure. It is clearly seen that this vector is directed inwards everywhere on the boundary, except at two locations on the fixed part, close to the transition points. It is at these two points that debonding can potentially take place. These two emplacements correspond to two humps with positive values on the curve for $\sigma_{nn}$ on Fig.(28). The stresses are relatively lower around the transition points, and tend to increase on that part of the boundary facing the applied pressure.

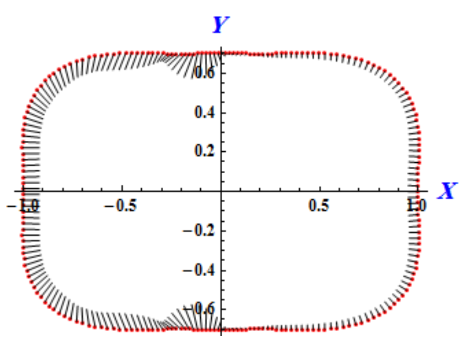


Figure 23: Stress vector distribution on the boundary.

The bulk distributions of functions of practical interest are shown on Figures (11), (25) (26) and (27). The cross-sectional domain is also shown for convenience.

The obtained results may be of interest in evaluating the displacements and stresses occurring in long thermoelastic pad supports, in those cases when heat effects cannot be neglected.

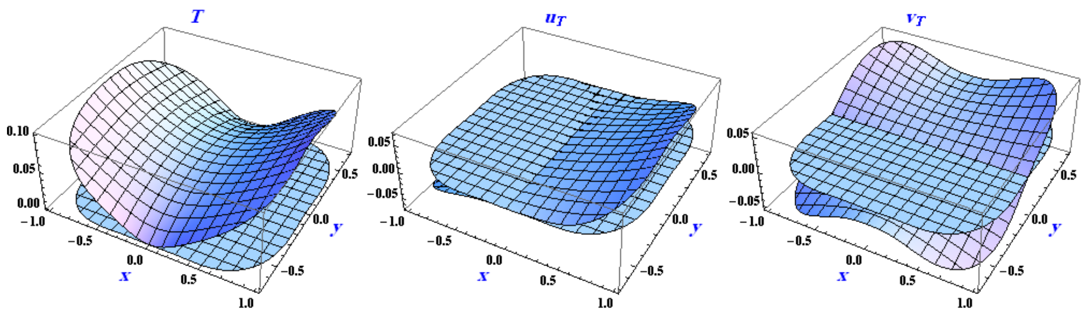


Figure 24: $T(x,y), U_T(x,y)$ and $V_T(x,y)$

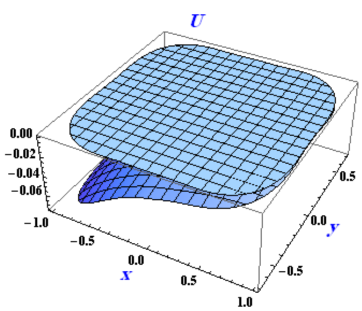


Figure 25: $U(x,y)$

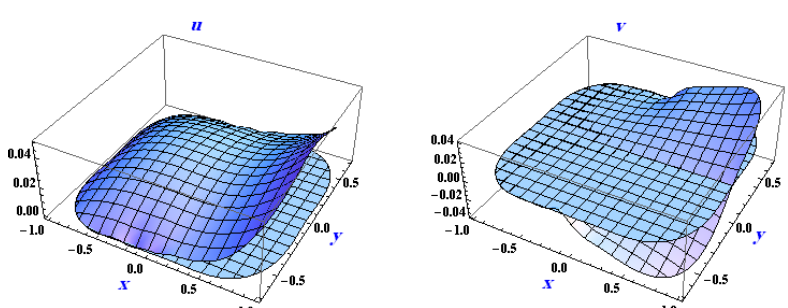


Figure 26: $u(x,y)$ and $v(x,y)$

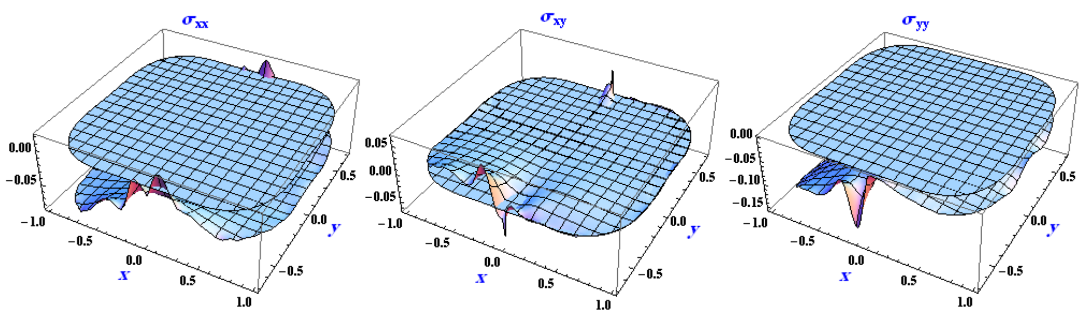

Figure 27: $\sigma_{xx}(x,y), \sigma_{xy}(x,y)$ and $\sigma_{yy}(x,y)$

# 9 Conclusions.

Two plane problems of linear, uncoupled thermoelasticity for an isotropic and homogeneous medium filling a long cylinder has been solved by a boundary integral method. The boundary of the normal cross-section is in the form of an ellipse with hump, or a rectangle, and is subjected to a prescribed temperature and to mixed mechanical boundary conditions. The displacements and stresses were obtained on the boundary and in the bulk. The computational difficulties encountered at the boundary transition points have been largely overcome by introducing a specially built harmonic function in the cross-sectional area, with singular behavior at the transition points. An efficient application of the method requires smoothening of the boundary. This has been achieved by replacement of sections around the corners by smooth curves and by curve fitting. A careful calculation of the derivatives along the boundary of unknown functions is necessary for an efficient application of the method. After discretization, for a given boundary point, this has been carried out using 15 neighbouring nodes on each side of it. This procedure leads to a smoothening of the finite jumps occurring in the second derivatives of some boundary functions due to the mixed nature of the boundary conditions. The final outcome of the boundary analysis produces errors in satisfying the solved linear algebraic systems of equations which do not exceed $10^{-3}$. For the unknown functions in the bulk, expansions in harmonic functions have been used. The coefficients in these expansions have been determined by Boundary Collocation Method. The errors here did not exceed 0.02. The form of the deformed boundary is shown for each problem. The obtained results express the fact that debonding of the fixed part of the boundary may occur near the transition points. Other types of thermal or mechanical boundary conditions may be treated by the same method. The singular function, however, may vary from case to case. The results may be of interest in evaluating the stresses arising in long pad supports in those cases, when thermal effects are important.

# 10 Future work.

The present research will be extended to the case when the cylinder is an electrical conductor carrying a steady, axial current. This is thermo-magnetoelasticity which is expected to have several applications in the electrical industry. In this case, the basic equations are enriched with the equations of Magnetostatics, inside and outside the conductor. The unique component of the magnetic vector potential is along the cylinder's generators and yields a harmonic part. Thus, the number of harmonic unknown functions rises from 3 in thermoelasticity to 5. Another application will involve the case of no electric current, when the material is magnetizable.

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# Appendices

## A Boundary representation of harmonic functions.

For an arbitrary point $(x, y) \in D$, the boundary integral representation of harmonic functions reads:

$$f(x,y) = \frac{1}{2\pi} \oint_C \left( f \, \frac{\partial \ln R}{\partial n'} - \ln R \, \frac{\partial f}{\partial n'} \right) ds'.$$

Here $R$ is the distance between the point $(x, y)$ and the current integration point. When the point $(x, y)$ tends to a boundary point, this relation transforms into an integral equation for the boundary values of the function $f$. Using integration by parts, this may be rewritten as:

$$f(x,y) = \frac{1}{\pi} \oint_C \left( f(x',y') \frac{\partial \ln R}{\partial n'} + (f^c(x',y') - f^c(x,y)) \frac{\partial \ln R}{\partial \tau'} \right) ds',$$

where $(x', y')$ is the current integration point.

This last equation contains removable singularities which can be treated as explained in [3].

## B The first and second derivatives of harmonic functions with respect to $x$ and $y$ on the boundary.

For a general harmonic function $f$ in the cross-section, the following formulae

$$\begin{aligned}
\left.\frac{\partial f}{\partial x}\right|_i &= \frac{1}{\omega_i^2} (\dot{f}_i^c \, \dot{y}_i + \dot{f}_i \, \dot{x}_i) \\
\left.\frac{\partial f}{\partial y}\right|_i &= \frac{1}{\omega_i^2} (\dot{f}_i \, \dot{y}_i - \dot{f}_i^c \, \dot{x}_i) \\
\left.\frac{\partial^2 f}{\partial y^2}\right|_i &= \frac{\alpha_i}{\omega_i^4} \, \ddot{f}_i - \frac{\beta_i}{\omega_i^4} \, \ddot{f}_i^c + \frac{\varrho_i}{\omega_i^6} \, \dot{f}_i + \frac{\rho_i}{\omega_i^6} \, \dot{f}_i^c \\
\left.\frac{\partial^2 f}{\partial x \, \partial y}\right|_i &= \frac{\beta_i}{\omega_i^4} \, \ddot{f}_i + \frac{\alpha_i}{\omega_i^4} \, \ddot{f}_i^c - \frac{\rho_i}{\omega_i^6} \, \dot{f}_i - \frac{\varrho_i}{\omega_i^6} \, \dot{f}_i^c \\
\left.\frac{\partial^2 f}{\partial x^2}\right|_i &= -\frac{\alpha_i}{\omega_i^4} \, \ddot{f}_i + \frac{\beta_i}{\omega_i^4} \, \ddot{f}_i^c + \frac{\varrho_i}{\omega_i^6} \, \dot{f}_i - \frac{\rho_i}{\omega_i^6} \, \dot{f}_i^c
\end{aligned}$$

where $f$ stands for any one of the used harmonic functions, and

$$\rho_i = \alpha_i \, \gamma_i + \beta_i \, \delta_i,$$
$$\varrho_i = \alpha_i \, \delta_i - \beta_i \, \gamma_i,$$

with

$$\alpha_i = \dot{y}_i^2 - \dot{x}_i^2,$$
$$\beta_i = 2 \, \dot{x}_i \, \dot{y}_i,$$
$$\gamma_i = \dot{x}_i \, \ddot{y}_i - \dot{y}_i \, \ddot{x}_i,$$
$$\delta_i = \dot{x}_i \, \ddot{x}_i + \dot{y}_i \, \ddot{y}_i.$$

# C Treating the singularities.

In order to simulate the singular behavior of stresses at the two truncation point, we consider the following harmonic function defined on the upper half-space, with singular behavior at the origin:

$$f(x, y) = \frac{1}{2\pi} \left[ y + 2Re \left( \frac{i \, c_1^2}{2} \, e^{c_1} \, E_1(c_1) \right) \right].$$

where

$$E_1(z) = -\gamma - \ln(z) - \sum_{n=1}^{\infty} \frac{(-1)^n \, z^n}{n \, n!} \tag{C.1}$$

is the integral exponential function defined in ([34], P.62) and $\gamma = 0.5772156649$ is the well-known Euler constant.
The obtained function will now be centered at each of the two boundary truncation points in order to simulate the behavior of stresses there. The sum of the resulting two functions is now added to the function $\psi$ in the above formulation, and will be denoted $\psi S$. The following three-dimensional plots on Fig.(28) show the harmonic function with singular boundary behavior, and the singular stresses as calculated from it as stress function.

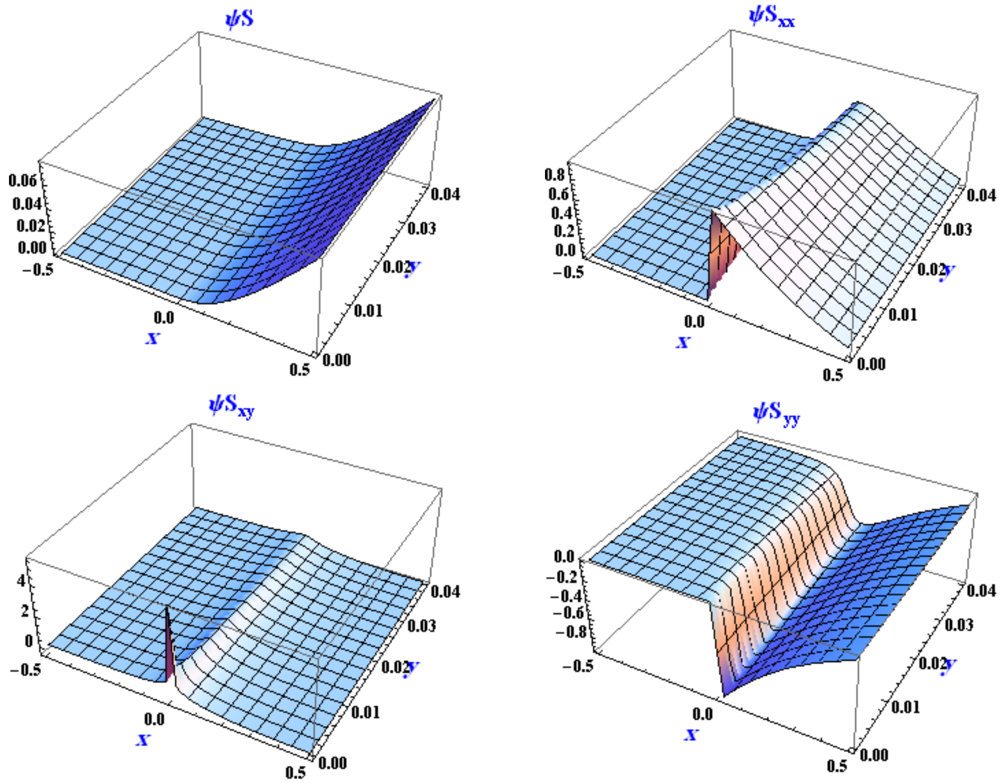


Figure 28: Singular stresses