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Generalized quartic fractional spline interpolation with applications

Faraidun K. Hamasalh

Faculty of Science and Science Education,
School of Science Education, Sulaimani Univ., Sulaimani, Iraq
e-mail: faraidunsalh@gmail.com

Pshtiwan O. Muhammad

Faculty of Science and Science Education,
School of Science Education, Sulaimani Univ., Sulaimani, Iraq
e-mail: pshtiwansangawi@gmail.com

Abstract

In this paper, a new fractional spline function of polynomial form with the idea of the lacunary interpolation is considered to find approximate solution for fractional differential equations (FDEs). The proposed method is applicable for $\alpha \in (0, 1]$, where α denotes the order of the fractional derivative in the Caputo sense. Convergence analysis of the method is considered. Some illustrative examples are presented and the obtained results reveal that the proposed technique is very effective, convenient and quite accurate to such considered problems. The study is conducted through illustrative examples and error analysis.

keywords: *Fractional integral and derivative, Caputo Derivative, Taylor's expansion, Error bound, Spline functions.*

1 Introduction

Fractional calculus (referring to the integration and differentiation of non-integer order) has been developed more than three centuries ago [1,2] and has been applied to an increasing number of fields such as physics [3,9,19] , chemistry and/or biochemistry [4], control [5-8], medicine [9], etc. In the last few decades, several numerical and analytical methods have been proposed in the literature to solve fractional differential equations. The most commonly used ones are fractional difference method [10], Adomian decomposition method [11], variational iteration method [12,13], and and Adams-Bashforth-Moulton method [14-16]. Moreover, the analytic results on the existence and uniqueness of solutions to the fractional differential equations have been investigated by many authors [17,18].

In the present work we introduce a new fractional spline of a polynomial form relying on the idea of lacunary interpolation problem which is applicable for each $\alpha \in (0,1]$. For details on lacunary interpolation refer to [22-24]. The fractional differential equations are solved by using our new fractional spline. Some numerical examples are given to to illustrate the accuracy of the method.

2 Preliminaries

In this section, some definitions and Taylor's Theorem, used in our work, will be presented. There are many definitions for fractional derivatives, the most commonly used ones are the Riemann-Liouville and the Caputo derivatives, especially the Caputo derivative are involved in our work. Suppose that $\alpha > 0, x > a$, $\alpha, a, x \in \mathbb{R}$, then

Definition 1. [21] The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined by

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}$$

where Γ is the gamma function.

Definition 2. [21] The Riemann-Liouville fractional derivative of order $\alpha > 0$ is defined by

$$D^\alpha f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_a^x (x-\xi)^{n-\alpha-1} \; f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}$$

Definition 3. [21] The Caputo fractional derivative of order $\alpha > 0$ is defined by

$$D_*^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-\xi)^{n-\alpha-1} \frac{d^n}{d\xi^n} f(\xi) d\xi, \quad n-1 < \alpha < n \in \mathbb{N}$$

Definition 4. [20] Let $\alpha \in \mathbb{R}^+$, $\Omega \subset \mathbb{R}$ an interval such that $a \in \Omega$, $a \leq x,\ \forall x \in \Omega$. Then the following set of functions are defined:

$${}_a\mathcal{I}_{\alpha} = \{ f \in C(\Omega) : I^{\alpha} f(x) \,\text{exist and is finite in}\, \Omega \}$$

$${}_a\mathcal{D}_{\alpha} = \{ f \in C(\Omega) : D_*^{\alpha} f(x) \,\text{exist and is finite in}\, \Omega \}$$

In view of these definitions we can conclude the following therm :

Theorem 1. [20] Let $\alpha \in (0,1]$, $p \in \mathbb{N}$ and $f(x)$ a continuous function in $[a,b]$ satisfying the following conditions:

(1) $D_*^{m\alpha} f \in C([a,b])$ and $D_*^{m\alpha} f \in {}_a\mathcal{I}_{\alpha}([a,b])$, $\forall m = 1, 2, ..., p$.

(2) $D_*^{(p+1)\alpha} f(x)$ is continuous on $[a,b]$.

Then for each $x \in [a,b]$,

$$f(x) = \sum_{m=0}^{p} D_*^{m\alpha} f(a) \frac{(x-a)^{m\alpha}}{\Gamma(m\alpha+1)} + R_p(x,a),$$

with

$$R_p(x,a) = D_*^{(p+1)\alpha} f(\xi) \frac{(x-a)^{(p+1)\alpha}}{\Gamma((p+1)\alpha+1)}, \qquad a \leq \xi \leq x$$

Remark 1. For simplicity we will use the operator D instead of D_* from now on.

3 Description of the Method

Given the mesh points, $\triangle : 0 = x_0 < x_1 < \cdots < x_n = 1$ with $x_{k+1} - x_k = h$, $k = 0, 1, \cdots, n-1$, and real numbers $\{f_k, D^{\alpha} f_k, D^{4\alpha} f_k\}_{k=0}^{n}$ associated with the knots. We are going to construct spline interpolant $S_{\triangle}$ for which $D^{m\alpha} S_{\triangle}(x_i) = D^{m\alpha} f_i$, $i = 0, 1, ..., n$, and $m = 0, 1, 4$. This construction is given in the following two cases:

Case 1

In this case we suppose that the conditions of theorem 1 are satisfied with $p = 4$, and then we can define the spline interpolant as follows:

$$\begin{aligned} S_{\triangle} = S_k(x) =& y_k + \frac{(x - x_k)^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k + a_k \frac{(x - x_k)^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ b_k \frac{(x - x_k)^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{(x - x_k)^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k \end{aligned} \tag{1}$$

where $x_k \le x \le x_{k+1}$ and $k = 0, 1, ..., n-1$.

4 Existence and Uniqueness

If we require that $S_{\triangle}(x)$ and $D^{\alpha} S_{\triangle}(x)$ is continuous on $[0, 1]$, then it is easy to prove that formula (1) *exists and is unique*. That is, clear from the continuity conditions of $S_{\triangle}(x)$ and $D^{\alpha} S_{\triangle}(x)$ from which we get :

$$y_{k+1} = y_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k + a_k \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} + b_k \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k \tag{2}$$

and

$$D^{\alpha} y_{k+1} = D^{\alpha} y_k + \frac{h^{\alpha}}{\Gamma(\alpha+1)} a_k + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} b_k + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y_k \tag{3}$$

The coefficients a_k and b_k are determined in terms of the given data using the continuity conditions of $S_{\triangle}(x)$ and $D^{\alpha} S_{\triangle}(x)$. Thus we have

$$a_k = \frac{\frac{1}{\Gamma(2\alpha+1)} A_k - \frac{h^{\alpha}}{\Gamma(3\alpha+1)} B_k}{k_1 h^{2\alpha}} \tag{4}$$

$$b_k = \frac{\frac{1}{\Gamma(2\alpha+1)} B_k - \frac{h^{-\alpha}}{\Gamma(\alpha+1)} A_k}{k_1 h^{2\alpha}} \tag{5}$$

where

$$k_1 = \frac{1}{\Gamma(2\alpha+1)\Gamma(2\alpha+1)} - \frac{1}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}$$

$$A_k = y_{k+1} - y_k - \frac{h^\alpha}{\Gamma(\alpha+1)} D^\alpha y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k, \tag{6}$$

$$\text{and} \quad B_k = D^\alpha y_{k+1} - D^\alpha y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y_k. \tag{7}$$

5 Error Bounds

Suppose that the conditions of theorem 1 are satisfied with $p = 4$ and $D^{m\alpha} S_\triangle(x_i) = D^{m\alpha} y_i$, $\alpha \in (0, 1]$, $m = 0, 1, 4$; i=0(1)n-1. We shall prove the following :

Theorem 2. Let $S_k(x)$ be the fractional spline interpolant of the polynomial form (1) solving the lacunary case $(0, \alpha, 4\alpha)$. Then for all $x \in [0, 1]$ the inequality

$$|D^{m\alpha} S_\triangle(x) - D^{m\alpha} y(x)| \le c_{m\alpha} h^{(4-m)\alpha} \omega_{4\alpha}(h)$$

holds for all $m = 0, 1, ..., 4$, and $\alpha \in (0, 1]$. Where $\omega_{4\alpha}(h)$ is the modulus of continuity of $D^{4\alpha} y(x)$, and

$$c_0 = \frac{k_2}{k_1\Gamma(2\alpha+1)} + \frac{k_3}{k_1\Gamma(3\alpha+1)} + \frac{1}{\Gamma(4\alpha+1)}, \quad c_\alpha = \frac{k_2}{k_1\Gamma(\alpha+1)} + \frac{k_3}{k_1\Gamma(2\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)},$$

$$c_{2\alpha} = \frac{k_2}{k_1} + \frac{k_3}{k_1\Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)}, \quad c_{3\alpha} = \frac{k_3}{k_1} + \frac{1}{\Gamma(\alpha+1)}, \quad c_{4\alpha} = 1.$$

To proof this theorem we shall need the following lemma,

Lemma 1. The following estimates are valid for all $k = 0(1)n-1$.

$$\left|a_k - D^{2\alpha} y_k\right| \le \frac{k_2}{k_1} h^{2\alpha}\, \omega_{4\alpha}(h), \tag{8}$$

$$\left|b_k - D^{3\alpha} y_k\right| \le \frac{k_3}{k_1} h^{\alpha}\, \omega_{4\alpha}(h), \tag{9}$$

for $k = 0, 1, ..., n-1$, where

$$k_2 = \left(\frac{1}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)\Gamma(3\alpha+1)}\right), \quad \text{and}$$

$$k_3 = \left(\frac{1}{\Gamma(2\alpha+1)\Gamma(3\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(4\alpha+1)}\right).$$

Proof. From (6) we can find

$$\begin{aligned}
\left|a_k - D^{2\alpha}y_k\right| &= \left|\frac{1}{k_1 h^{2\alpha}}\left[\frac{1}{\Gamma(2\alpha+1)}\left(y_{k+1} - y_k - \frac{h^\alpha}{\Gamma(\alpha+1)}D^\alpha y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha}y_k\right)\right.\right. \\
&\quad \left.\left. - \frac{h^\alpha}{\Gamma(3\alpha+1)}\left(D^\alpha y_{k+1} - D^\alpha y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}D^{4\alpha}y_k\right)\right] - D^{2\alpha}y_k\right| \\
&= \left|\frac{1}{k_1 h^{2\alpha}}\left[\frac{1}{\Gamma(2\alpha+1)}\left(y_{k+1} - y_k - \frac{h^\alpha}{\Gamma(\alpha+1)}D^\alpha y_k - \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}D^{2\alpha}y_k\right.\right.\right. \\
&\quad \left. - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha}y_k\right) - \frac{h^\alpha}{\Gamma(3\alpha+1)}\left(D^\alpha y_{k+1} - D^\alpha y_k\right. \\
&\quad \left.\left.\left. - \frac{h^\alpha}{\Gamma(\alpha+1)}D^{2\alpha}y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}D^{4\alpha}y_k\right)\right]\right| \qquad (10)
\end{aligned}$$

taking:

$$y_{k+1} = y_k + \frac{h^\alpha}{\Gamma(\alpha+1)}D^\alpha y_k + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}D^{2\alpha}y_k + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}D^{3\alpha}y_k + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha}y(\xi_k),$$

and

$$D^\alpha y_{k+1} = D^\alpha y_k + \frac{h^\alpha}{\Gamma(\alpha+1)}D^{2\alpha}y_k + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)}D^{3\alpha}y_k + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)}D^{4\alpha}y(\eta_k),$$

where $x_k < \xi_k, \eta_k < x_{k+1}$. Then (10) becomes

$$\begin{aligned}
\left|a_k - D^{2\alpha}y_k\right| &\le \frac{1}{k_1 h^{2\alpha}}\left[\frac{h^{4\alpha}}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)}\left|D^{4\alpha}y(\xi_k) - D^{4\alpha}y_k\right|\right. \\
&\quad \left. + \frac{h^{4\alpha}}{\Gamma(3\alpha+1)\Gamma(3\alpha+1)}\left|D^{4\alpha}y(\eta_k) - D^{4\alpha}y_k\right|\right] \\
&\le \frac{h^{2\alpha}}{k_1}\left[\frac{1}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)\Gamma(3\alpha+1)}\right]\omega_{4\alpha}(h) \\
&= \frac{k_2}{k_1}\,h^{2\alpha}\,\omega_{4\alpha}(h)
\end{aligned}$$

Similarly, after using (7), we can easily the second part of the lemma. Thus we have proved the lemma. □

Proof of theorem 2. In view of the above lemma we can see that, for $x_k \le x \le x_{k+1}$ and $k = 0, 1, ..., n-1$,

$$\begin{aligned}
|S_k(x) - y(x)| &= \Bigg| y_k + \frac{(x-x_k)^\alpha}{\Gamma(\alpha+1)} D^\alpha y_k + a_k \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} + b_k \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} \\
&\quad + \frac{(x-x_k)^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k - y_k - \frac{(x-x_k)^\alpha}{\Gamma(\alpha+1)} D^\alpha y_k - D^{2\alpha} y_k \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} \\
&\quad - D^{3\alpha} y_k \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} - \frac{(x-x_k)^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y(\xi_k) \Bigg| \\
&\le \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \left| a_k - D^{2\alpha} y_k \right| + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \left| b_k - D^{3\alpha} y_k \right| + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} \omega_{4\alpha}(h) \\
\text{using (8),(9),} \quad &\le \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \cdot \frac{k_2}{k_1} h^{2\alpha}\, \omega_{4\alpha}(h) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \cdot \frac{k_3}{k_1} h^{\alpha}\, \omega_{4\alpha}(h) + \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} \omega_{4\alpha}(h) \\
&= \left(\frac{k_2}{k_1 \Gamma(2\alpha+1)} + \frac{k_3}{k_1 \Gamma(3\alpha+1)} + \frac{1}{\Gamma(4\alpha+1)} \right) h^{4\alpha}\, \omega_{4\alpha}(h), \quad (11)
\end{aligned}$$

$$\begin{aligned}
|D^\alpha S_k(x) - D^\alpha y(x)| &= \Bigg| D^\alpha y_k + \frac{(x-x_k)^\alpha}{\Gamma(\alpha+1)} a_k + \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} b_k + \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y_k \\
&\quad - D^\alpha y_k - \frac{(x-x_k)^\alpha}{\Gamma(\alpha+1)} D^{2\alpha} y_k - \frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} D^{3\alpha} y_k - \frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y(\xi_k) \Bigg| \\
&\le \frac{h^{\alpha}}{\Gamma(\alpha+1)} \left| a_k - D^{2\alpha} y_k \right| + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \left| b_k - D^{3\alpha} y_k \right| + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} \omega_{4\alpha}(h) \\
&\le \left(\frac{k_2}{k_1 \Gamma(\alpha+1)} + \frac{k_3}{k_1 \Gamma(2\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)} \right) h^{3\alpha}\, \omega_{4\alpha}(h), \quad (12)
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left| D^{2\alpha} S_k(x) - D^{2\alpha} y(x) \right| &\le \left| a_k - D^{2\alpha} y_k \right| + \frac{h^{\alpha}}{\Gamma(\alpha+1)} \left| b_k - D^{3\alpha} y_k \right| + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} \omega_{4\alpha}(h) \\
&\le \left(\frac{k_2}{k_1} + \frac{k_3}{k_1 \Gamma(\alpha+1)} + \frac{1}{\Gamma(2\alpha+1)} \right) h^{2\alpha}\, \omega_{4\alpha}(h), \quad (13)
\end{aligned}$$

$$\begin{aligned}\left|D^{3\alpha}S_k(x) - D^{3\alpha}y(x)\right| &\le \left|b_k - D^{3\alpha}y_k\right| + \frac{h^{\alpha}}{\Gamma(\alpha+1)}\omega_{4\alpha}(h) \\ &\le \left(\frac{k_3}{k_1} + \frac{1}{\Gamma(\alpha+1)}\right) h^{\alpha}\,\omega_{4\alpha}(h), \qquad (14)\end{aligned}$$

and finally,

$$\left|D^{4\alpha}S_k(x) - D^{4\alpha}y(x)\right| \le \omega_{4\alpha}(h). \qquad (15)$$

This completes the proof. □

Case 2

In this case we suppose that the conditions of theorem 1 are fulfilled with $p = 5$, and then we can define the spline interpolant as follows :

$$\begin{aligned}S_{\triangle} = S_k(x) =& y_k + \frac{(x-x_k)^{\alpha}}{\Gamma(\alpha+1)}D^{\alpha}y_k + a_k\frac{(x-x_k)^{2\alpha}}{\Gamma(2\alpha+1)} \\ &+ b_k\frac{(x-x_k)^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{(x-x_k)^{4\alpha}}{\Gamma(4\alpha+1)}D^{4\alpha}y_k + c_k\frac{(x-x_k)^{5\alpha}}{\Gamma(5\alpha+1)}, \qquad (16)\end{aligned}$$

where $x_k \le x \le x_{k+1}$ and $k = 0, 1, ..., n-1$. Let

$$c_k = \Gamma(\alpha+1)h^{-\alpha}\left[D^{4\alpha}y_{k+1} - D^{4\alpha}y_k\right].$$

It can be easily shown that

$$\left|c_k - D^{5\alpha}y_k\right| \le \omega_{5\alpha}(h), \qquad (17)$$

where $\omega_{5\alpha}(h)$ is the modulus of continuity of $D^{5\alpha}y(x)$.
Now, if $S_{\triangle}(x) \in C[0,1]$ and $S_{\triangle}^{\alpha}(x) \in C[0,1]$ then the *existence* and *uniqueness* of $S_{\triangle}(x)$ is easy to be proved, since here a_k and b_k are uniquely determined by

$$a_k = \frac{\frac{1}{\Gamma(2\alpha+1)}A_k - \frac{h^{\alpha}}{\Gamma(3\alpha+1)}B_k}{k_1 h^{2\alpha}} \qquad (18)$$

$$b_k = \frac{\frac{1}{\Gamma(2\alpha+1)}B_k - \frac{h^{-\alpha}}{\Gamma(\alpha+1)}A_k}{k_1 h^{2\alpha}} \qquad (19)$$

where

$$k_1 = \frac{1}{\Gamma(2\alpha+1)\Gamma(2\alpha+1)} - \frac{1}{\Gamma(\alpha+1)\Gamma(3\alpha+1)}$$

$$A_k = y_{k+1} - y_k - \frac{h^{\alpha}}{\Gamma(\alpha+1)} D^{\alpha} y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} D^{4\alpha} y_k - \frac{h^{5\alpha}}{\Gamma(5\alpha+1)} c_k, \quad (20)$$

$$\text{and} \quad B_k = D^{\alpha} y_{k+1} - D^{\alpha} y_k - \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D^{4\alpha} y_k - \frac{h^{4\alpha}}{\Gamma(4\alpha+1)} c_k. \quad (21)$$

Then we have the following lemma :

Lemma 2. The following estimates can be obtained for $k = 0, 1, ..., n-1$.

$$\left| a_k - D^{2\alpha} y_k \right| \le \frac{k_4}{k_1} h^{2\alpha} \, \omega_{4\alpha}(h), \quad (22)$$

$$\left| b_k - D^{3\alpha} y_k \right| \le \frac{k_5}{k_1} h^{\alpha} \, \omega_{4\alpha}(h), \quad (23)$$

where

$$k_4 = \left(\frac{1}{\Gamma(2\alpha+1)\Gamma(5\alpha+1)} + \frac{1}{\Gamma(3\alpha+1)\Gamma(4\alpha+1)} \right), \quad \text{and}$$

$$k_5 = \left(\frac{1}{\Gamma(2\alpha+1)\Gamma(4\alpha+1)} + \frac{1}{\Gamma(\alpha+1)\Gamma(5\alpha+1)} \right).$$

Proof. Use (17), (18), (20), and (21) to obtain the first inequality and use (17), (19), (20), and (21) to obtain the second inequality and follow the steps of the lemma 1. Thus we can prove the lemma. □

And then we can conclude the following theorem :

Theorem 3. Let $S_k(x)$ be the fractional spline interpolant of the polynomial form (1) solving the lacunary case $(0, \alpha, 4\alpha)$ for which the conditions of theorem 1 are satisfied with $p = 5$. Then for all $x \in [0, 1]$ the inequality

$$|D^{m\alpha} S_{\triangle}(x) - D^{m\alpha} y(x)| \le c_{m\alpha} h^{(5-m)\alpha} \omega_{5\alpha}(h)$$

holds for all $m = 0, 1, ..., 5$, and $\alpha \in (0, 1]$. Where $\omega_{5\alpha}(h)$ is the modulus of continuity of $D^{5\alpha}y(x)$, and

$$c_0=\frac{k_4}{k_1\Gamma(2\alpha+1)}+\frac{k_5}{k_1\Gamma(3\alpha+1)}+\frac{1}{\Gamma(5\alpha+1)},\quad c_\alpha=\frac{k_4}{k_1\Gamma(\alpha+1)}+\frac{k_5}{k_1\Gamma(2\alpha+1)}+\frac{1}{\Gamma(4\alpha+1)},$$
$$c_{2\alpha}=\frac{k_4}{k_1}+\frac{k_5}{k_1\Gamma(\alpha+1)}+\frac{1}{\Gamma(3\alpha+1)},\quad c_{3\alpha}=\frac{k_5}{k_1}+\frac{1}{\Gamma(2\alpha+1)},\quad c_{4\alpha}=\frac{1}{\Gamma(\alpha+1)},\quad c_{5\alpha}=1.$$

Proof. Proceed as in theorem 2. □

6 Numerical Illustrations

We now consider some numerical examples illustrating the solution using our fractional spline method. All calculations are implemented with MATLAB 12.

Example 1. Consider the linear fractional differential equation

$$D^2y(x)+2\,D^\alpha y(x)+y(x)=2x+\frac{4}{\Gamma(4-\alpha)}x^{3-\alpha}+\frac{1}{3}x^3,\quad 0<\alpha\le 1,\tag{24}$$

subject to

$$y(0)=y'(0)=0.$$

It is easily verified that the exact solution of this problem is

$$y(x)=\frac{1}{3}x^3.$$

The maximal absolute errors obtained, for $\alpha = 0.5$, and for $0 \le x \le 1$, are shown in Table 1 and Table 2, to illustrate the accuracy of the spline method of polynomial form. We have shown the maximal error's values in each case. Note that $|e^{m\alpha}(x)| = |D^{m\alpha}S_k(x) - D^{m\alpha}y(x)|$, for $\alpha = 0.5$, $m = 0, 1, ..., 4$ for case 1, and $m = 0, 1, ..., 5$ for case 2.

Table 1: Maximal absolute errors in case 1 for Example 1.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	5.4910×10^{-2}	1.1015×10^{-1}	2.7105×10^{-1}	6.2211×10^{-1}	2×10^{-1}
0.01	5.4910×10^{-5}	3.4832×10^{-4}	2.7105×10^{-3}	1.9673×10^{-2}	2×10^{-2}
0.001	5.4910×10^{-8}	1.1015×10^{-6}	2.7105×10^{-5}	6.2211×10^{-4}	2×10^{-3}

Table 2: Maximal absolute errors in case 2 for Example 1.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$
0.1	4.2117×10^{-2}	1.1394×10^{-1}	3.8320×10^{-1}
0.01	4.2117×10^{-5}	3.6031×10^{-4}	3.8320×10^{-3}
0.001	1.3318×10^{-7}	1.1394×10^{-6}	3.8320×10^{-5}
h	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$	$\lvert e^{5\alpha}(x)\rvert$
0.1	1.5609×10^{-1}	2.5464×10^{-1}	7.1364×10^{-1}
0.01	4.9362×10^{-3}	2.5464×10^{-2}	2.2567×10^{-1}
0.001	1.5609×10^{-4}	2.5464×10^{-3}	7.1364×10^{-2}

Example 2. Consider the fractional differential equation

$$D^{\alpha}y(x)=x^{4}-\frac{1}{2}x^{3}+\frac{24}{\Gamma(4-\alpha)}x^{3-\alpha}+\frac{3}{\Gamma(5-\alpha)}x^{4-\alpha}-y(x),\quad 0<\alpha<1,\quad (25)$$

With the initial condition $y(0)=0$.
The exact solution is

$$y(x)=x^{4}-\frac{1}{2}x^{3}.$$

Thus the maximal absolute errors obtained, for case 1, case 2 and for $\alpha=0.5$, are shown in Table 3 and Table 4, respectively, with $|e^{m\alpha}(x)|=|D^{m\alpha}S_k(x)-D^{m\alpha}y(x)|$, for $\alpha=0.5$, $m=0,1,...,4$ for case 1, and $m=0,1,...,5$ for case 2.

Table 3: Maximal absolute error in case 1 for Example 2.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$
0.1	74.1286×10^{-2}	0.1487×10^{-1}	0.3659×10^{-1}	83.9857×10^{-1}	27×10^{-1}
0.01	74.1286×10^{-5}	47.0239×10^{-4}	36.5918×10^{-3}	26.5586×10^{-2}	27×10^{-2}
0.001	74.1286×10^{-8}	14.8702×10^{-6}	36.5918×10^{-5}	83.9857×10^{-4}	27×10^{-3}

Table 4: Maximal absolute error in case 2 for Example 2.

h	$\lvert e(x)\rvert$	$\lvert e^{\alpha}(x)\rvert$	$\lvert e^{2\alpha}(x)\rvert$
0.1	44.8919×10^{-2}	12.1447×10^{-1}	40.8441×10^{-1}
0.01	44.8919×10^{-5}	38.4049×10^{-4}	40.8441×10^{-3}
0.001	14.1960×10^{-7}	12.1447×10^{-6}	40.8441×10^{-5}
h	$\lvert e^{3\alpha}(x)\rvert$	$\lvert e^{4\alpha}(x)\rvert$	$\lvert e^{5\alpha}(x)\rvert$
0.1	16.6379×10^{-1}	27.1421×10^{-1}	76.0656×10^{-1}
0.01	52.6138×10^{-3}	27.1421×10^{-2}	24.0540×10^{-1}
0.001	16.6379×10^{-4}	27.1421×10^{-3}	76.0656×10^{-2}

7 Conclusion

In this paper, we introduced a new kind of the fractional spline of polynomial form to be applicable for the case $0 < \alpha \leq 1$. The method is tested by considering two test problems for two fractional ordinary differential equations. The two examples are of fractional order α, $0 < \alpha \leq 1$, and maximal absolute errors are obtained, to illustrate the accuracy of the method.

8 Open Problems

Actually the present paper deals with the generalized of the quartic spline of fractional order, with a new idea for fractional interpolation model and applied for fractional differential equations with real order, and therefore using this growth indicator one may evaluate error estimations the above model rates of composite entire fractional spline functions under some boundary conditions. In this construction, the following natural questions may arise for the doer of this branch.

1. These theories can be modified by the treatment of the notions, as change the fractional spline model for quintic with a new boundary value fractional order.

2. Further, same model of fractional spline can be applied for differential equations, such as PDE's and ODE's.

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