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Parameter variation in a third order singular boundary value problem

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Abstract

A third order non-linear boundary value problem that arises from the problem of boundary-layer flows of an incompressible non-Newtonian fluid modelled by a power-law rheology is considered. The shear stress parameter (curvature at the origin) is computed for different values of the power-law index n ***and different values of*** ϵ ***(the initial rates of change at the origin). The interrelationships between these parameters are examined and regions of linear/non-linear interaction/dependence are revealed.***

Keywords: *Singular non-linear boundary value problem, Shooting method, Parameter sensitivity, Power-law fluid, Boundary-layer flow.*

1 Introduction

The problem of boundary layer flow of non-Newtonian fluids has established its presence and significance in many applications in mechanics, engineering and industry. It has attracted much attention from mathematicians working in differential equations as well as applied and computational mathematics. The problem is modelled by a boundary value problem on a semi-infinite domain which involves a third order non-linear ordinary differential equation. A transformation to a finite domain results in a non-linear singular boundary value problem of the third order.

Earlier studies of the problem can be found in [2, 3] where both Newtonian and non-Newtonian fluids were considered. In fact, this singular problem has several variations and forms when it comes to the exact constants/parameters within the equation itself as well as the boundary conditions used. However, even though much research has been done on the existence and uniqueness of the problem [8, 9, 11, 14], obtaining analytic and approximate solutions [4, 13] as well as numerical solutions [7, 10] for both Newtonian and non-Newtonian fluids, very little research has been done on the parameter dependence and interrelationships between the different parameters within the problem.

The most commonly used model in non-Newtonian fluid mechanics is the Ostwald-de Waele model with a power-law rheology and which is characterized by a power-law index n. The value $n = 1$ corresponds to a Newtonian fluid, while $n > 1$ describes a dilatant or shear-thickening fluid and $0 < n < 1$ describes pseudo-plastic or shear-thinning fluid (cf. [1, 5, 6, 12]).

2 The problem and the solution approach

We have the following power-law problem:

$$(|f''|^{n-1}f'')' + \frac{1}{n+1}ff'' = 0; f(0) = 0, f'(0) = \epsilon, f'(\eta) \to 1 \text{ as } \eta \to \infty. \quad (1)$$

Here the derivatives (primes) are taken with respect to η. Observe that if $f'' > 0$ then we can drop the absolute value and the above equation will have the form

$$n(f'')^{n-1}f''' + \frac{1}{n+1}ff'' = 0. \quad (2)$$

We make the transformation $x = \dfrac{1}{\eta+1}$. This transforms the domain of the problem from $[1, \infty)$ to $(0, 1]$ and yields the following equations
$f'(\eta) = -x^2\dfrac{df}{dx}; f''(\eta) = x^4\dfrac{d^2f}{dx^2} + 2x^3\dfrac{df}{dx}; f'''(\eta) = -x^6\dfrac{d^3f}{dx^3} - 6x^5\dfrac{d^2f}{dx^2} - 6x^4\dfrac{df}{dx}$.

Now apply a transformation on f with $g(x) = xf(x)$; $(f(x) = g(x)/x)$. With all derivatives (primes) now taken with respect to x we have
$f'(x) = \dfrac{g'(x)}{x} - \dfrac{g(x)}{x^2}; f''(x) = \dfrac{g''(x)}{x} - \dfrac{2g'(x)}{x^2} + \dfrac{2g(x)}{x^3};$
$f'''(x) = \dfrac{g'''(x)}{x} - \dfrac{3g''(x)}{x^2} + \dfrac{6g'(x)}{x^3} - \dfrac{6g(x)}{x^4}.$
Combining these two transformations we have (primes on f now with respect to η but those on g with respect to x):

$$f'(\eta) = g(x) - xg'(x); f''(\eta) = x^3g''(x); f'''(\eta) = -x^5g'''(x) - 3x^4g''(x).$$

Hence the transformed equation takes the form

$$\frac{d(|x^3 g''|^{n-1}x^3 g'')}{d\eta} + \frac{1}{n+1}x^2 g g'' = 0. \tag{3}$$

If $f'' > 0$ then substituting the transformation identities directly into (2) one obtains: $n(x^3 g''(x))^{n-1}(-x^5 g'''(x) - 3x^4 g''(x)) + \dfrac{1}{n+1}\dfrac{g(x)}{x}x^3 g''(x) = 0$, which leads to:

$$g'''(x) = \frac{g(x)(g''(x))^{2-n}}{n(n+1)x^{3n}} - \frac{3g''(x)}{x}.$$

On the other hand if $f'' < 0$ then equation (3) takes the form

$$g'''(x) = -\frac{g(x)(-g''(x))^{2-n}}{n(n+1)x^{3n}} - \frac{3g''(x)}{x}.$$

The boundary conditions are

$$g(0) = 1, g'(1) = -\epsilon, g(1) = 0.$$

The transformed power-law problem is a non-linear *singular* boundary value problem. Integration of this problem takes place from $x = 1$ to $x = 0$ where one replaces the condition at 0 (namely $g(0) = 1$) with a condition at $x = 1$ (namely $g''(1) = \alpha$) and examine the values of α that lead to the condition $g(0) = 1$, in which case the solution of the problem is determined. This is known as a shooting method.

To simplify this problem we make one last transformation by transforming x into $1 - x$. This reverses the direction of the integration process so that it would be from $x = 0$ to $x = 1$. The problem then takes the form:

$$y''' = -\frac{y(y'')^{2-n}}{n(n+1)(1-x)^{3n}} + \frac{3y''}{1-x}$$

for solutions with positive curvature and

$$y''' = \frac{y(-y'')^{2-n}}{n(n+1)(1-x)^{3n}} - \frac{3y''}{1-x}$$

for solutions with negative curvature. The boundary conditions are then given by

$$y(0) = 0, \quad y'(0) = \epsilon, \quad y(1) = 1.$$

Now we integrate from $x = 0$ to $x = 1$ with the conditions $y(0) = 0, y'(0) = \epsilon, y''(0) = \alpha$ where we find the value of α that will satisfy the boundary condition $y(1) = 1$.

3 Parameter dependence and interrelationships

We seek to study the dependence of the shear stress parameter $|\alpha| = |f''(0)|$ (observe that $\alpha = y''(0) = f''(0)$ in light of the transformation identities discussed earlier) on the power-law index n. In particular we look for regions where the relationship is linear and where it can be reasonably approximated by polynomials. We also look for cases where the relationship is almost constant and may result in very small sensitivity of solutions and shear stress to changes (or errors in the computation in a physical problem or industrial applications) of n.

Another feature that is of interest is the dependence of the smallest possible shear stress (and the value of n where it occurs) on ϵ. We investigate that relationship and find regions/intervals where the relationship is linear or can be predicted easily in order to have simple equations that can approximate this relationship with small error.

First consider the dependence of the shear stress parameter $|\alpha| = |f''(0)|$ on the value of the power-law index n for different values of ϵ: ones that correspond to solutions with positive curvature ($\alpha > 0, \epsilon < 1$) as well as ones that correspond to solutions with negative curvature ($\alpha < 0, \epsilon > 1$).

For $\epsilon = 0.5$ the relationship between α and n is shown in figure 1. The fact that $\alpha \to \infty$ as $n \to 0$ and as $n \to \infty$ is natural (see the analysis in [14] concerning the interaction between the two terms on the right hand side of the governing differential equation). Observe that there is a minimum for α and it occurs at $n = 0.5$. One crucial observation here is that for $n > 1$ the relationship can be approximated by a linear relationship between the two parameters $|\alpha| = |f''(0)|$ and n over small intervals of n. Those intervals can be chosen to be larger for larger values of n where the curve has a smaller curvature and in fact smaller inclination.

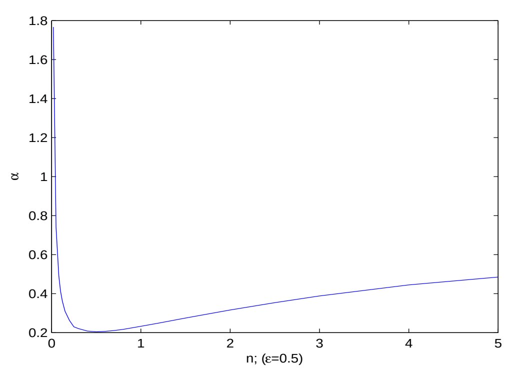

Figure 1: $\epsilon = 0.5$ A case of positive curvature solutions

The smaller inclination on the other hand (for larger values of n) shows that the sensitivity of shear stress on the value of n becomes much smaller (an error in the calculation of n does *not* result in a significant change in shear stress which is a crucial result in industrial applications). On the other hand for $n < 0.5$ the shear stress is very sensitive to changes/miscalculations in n and the relationship cannot be approximated linearly except for extremely small ranges of n within that region.

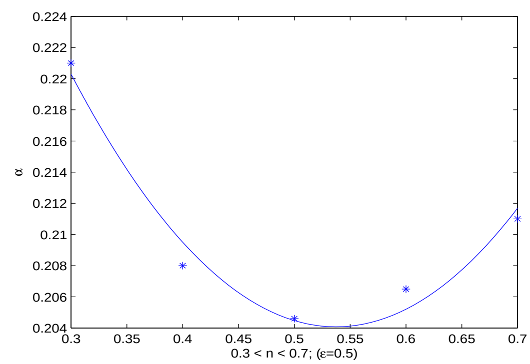

Figure 2: $\epsilon = 0.5$ Central Region

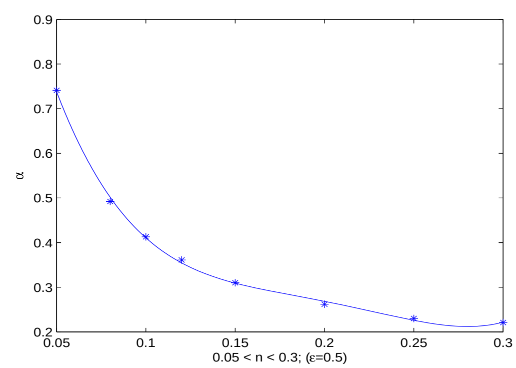

Figure 3: $\epsilon = 0.5$ Left Region

To further approximate the relationship between α and n within the range given in the figure itself it is best to subdivide the figure into three regions: a central region around $n = 0.5$, another one to the left, and the last one to the right. The central region can be reasonably approximated with a second degree polynomial

$$p_2(x) = 0.2879x^2 - 0.3094x + 0.2872$$

for $0.3 < n < 0.7$ with error not exceeding 0.002 as shown in figure 2. The region on the left can be approximated by a third degree polynomial

$$p_3(x) = 688x^4 - 564.9x^3 + 171.7x^2 - 23.71x + 1.5605$$

for an error less than 0.01 (see figure 3). Lastly the third degree polynomial

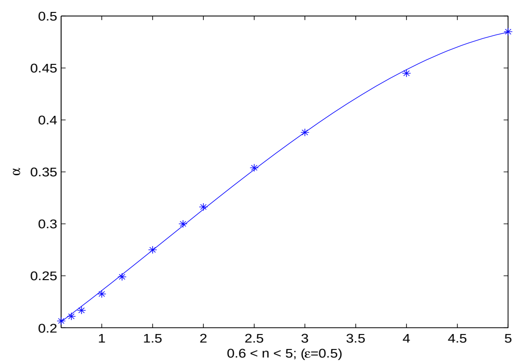

Figure 4: $\epsilon = 0.5$ Right Region

$$p_3(x) = -0.0017x^3 + 0.0083x^2 + 0.0651x + 0.164$$

approximates the region on the right for an error less than 0.005 as illustrated in figure 4. (Note that p_3 here approximates a larger range of n with smaller error than that in figure 3. This is just a suggested approach and an illustration on how this relationship may be approximated. One can always use higher degree polynomials, however we have shown that good approximations are possible with smaller degree polynomials as long as we choose the ranges of n carefully.) For $n > 5$ on the other hand one can use *linear* approximations as long as one does not choose an extremely large range of n so as to maintain very small errors.

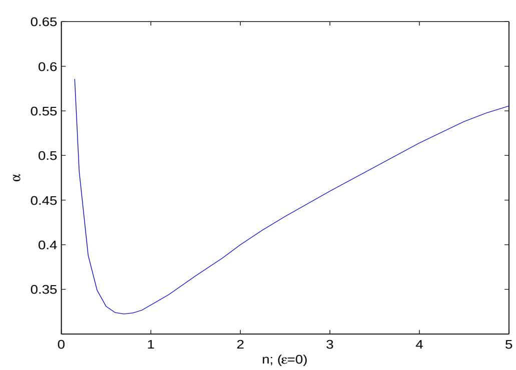

Figure 5: $\epsilon = 0$ (A case of positive curvature solutions)

Figures five and six illustrate that for positive curvature solutions (with $\epsilon = 0$ and $\epsilon = 0.75$ we have a similar situation to that in the case of $\epsilon = 0.5$ with just some differences in the details. It remains an open question to find

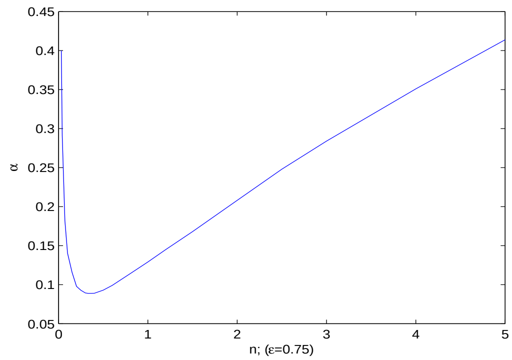

Figure 6: $\epsilon = 0.75$ (A case of positive curvature solutions)

one relationship governing the values of α depending on both parameters n and ϵ for as large range as possible for these two parameters.

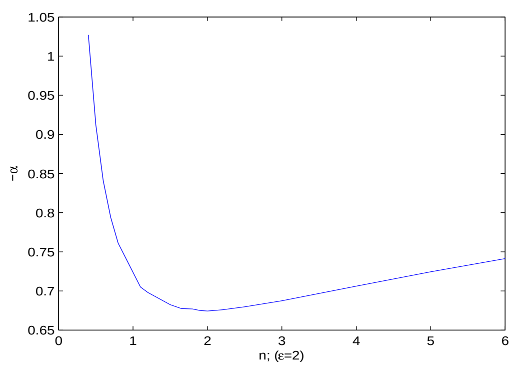

Figure 7: $\epsilon = 2$ (A case of negative curvature solutions)

Turning to the case of negative curvature solutions of problem (1), observe that while figure 7 ($\epsilon = 2$) shows a similar trend to that already observed for the positive curvature cases, we have a strongly different trend for $\epsilon = 4$ as shown in figure 8 with a large delay in the minimum value of α, where it occurs here in the vicinity of $n = 40$. This is a very important observation since it shows that for larger values of ϵ shear stress becomes almost constant with respect to n for larger values of n. So a linear relationship may not even be needed here and shear stress sensitivity to changes/variations in n (or errors in a precise calculation of n) is virtually zero since shear stress becomes a virtual constant over a large range of n.

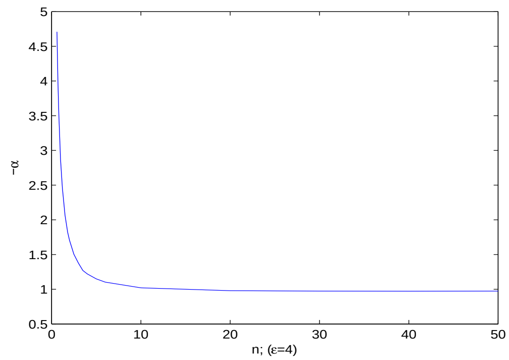

Figure 8: $\epsilon = 4$ (A case of negative curvature solutions)

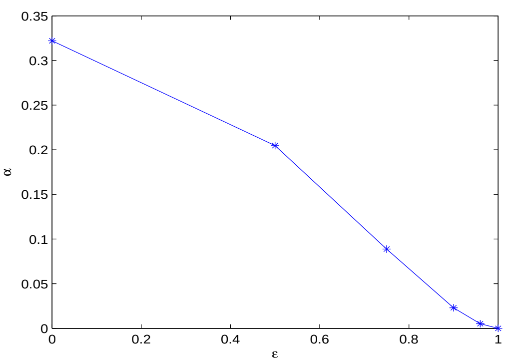

Figure 9: Minimum value of α as it depends on $0 < \epsilon < 1$

3.1 Shear stress minimum values

Figure 9 shows the dependence of the minimum value of α as it depends on ϵ (for each ϵ there is a value of n where a minimum value of the shear stress occurs as was shown in figures 1, 5, 6, 7 and 8) for values of $0 \leq \epsilon < 1$. Observe in particular the strongly linear relationship for the range $0.5 < \epsilon < 0.9$. The relationship is relaxed as it gets close to $\epsilon = 1$, however it remains predictable (can be approximated by polynomials of smaller degrees but requires closer investigation first).

In figure 10 we observe a similar trend to that in figure 9 but now for $1 \leq \epsilon < 4$. Again notice the strongly linear relationship for $1.1 < \epsilon < 2$. The relationship is relaxed in the vicinity of $\epsilon = 1$.

Figure 11 illustrates the relationship between the values of n where the minimum value of α occurs for $0 \leq \epsilon < 1$. Once again there is a strongly linear relationship for $0.5 < \epsilon < 0.9$. The figure suggests that this relationship takes a sharp turn as it gets close to $\epsilon = 1$. However once again the relationship can be very predictable around that point and needs further investigation. On

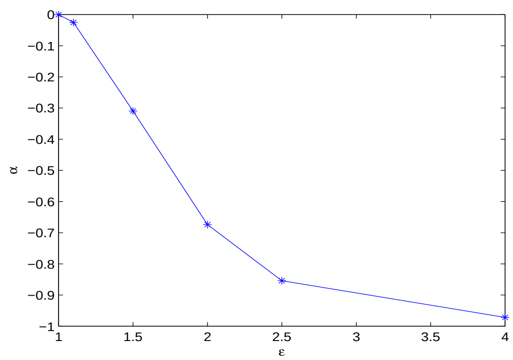

Figure 10: Minimum value of $|\alpha|$ as it depends on $1 < \epsilon < 4$

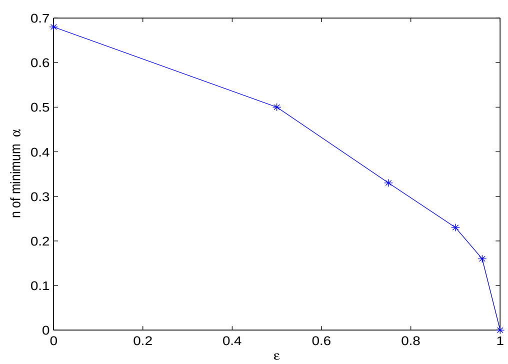

Figure 11: n where the minimum value of α occurs vs. $0 < \epsilon < 1$

the other hand the relationship deviates from the linear nature around $\epsilon = 0$ where it has a negative curvature, however the curvature remains small in that region and therefore the relationship is still close to a linear one.

On the other hand figure 12 suggests a relationship close to linear between n where the minimum value of $|\alpha|$ occurs and ϵ for $1.1 < \epsilon < 2$. For $1.1 < \epsilon < 2.5$ a second order polynomial gives an excellent approximation. The curve takes a stronger turn for $\epsilon > 2.5$, however it remains predictable for $1 < \epsilon < 1.1$ where in fact the curvature of the relationship changes sign, but once again this requires further investigation.

4 Conclusions

We have examined the nature of the relationships between $|\alpha|, n$ and ϵ (as well as the minimum values of $|\alpha|$) using numerical results obtained from integrating the transformed power-law problem (which consists of a third order non-linear singular boundary value problem) with the aid of a shooting method. The used

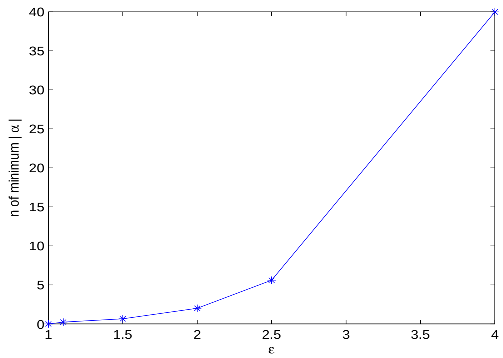

Figure 12: n where the minimum value of $|\alpha|$ occurs vs. $1 < \epsilon < 4$

transformation facilitated the integration process which was used to determine the values of α. We were able to derive reasonable approximations for the interrelationships between those parameters and to reveal regions of linear relationship or almost constant values of shear stress which in turn indicates less sensitivity between the parameters (less sensitivity with respect to n is especially important in applications).

5 Open problem

It remains an open problem and an unanswered question to obtain the exact analytical interrelationships between the different parameters that we have examined for this third order singular non-linear boundary value problem arising from industrial applications of non-Newtonian fluids. Integration of the governing differential equation leads to an expression for $y''(0)$ for values of $n > 1/3$ since $y''(1) \to 0$ for $n > 1/3$ as was discussed in [14] (while $y''(1)$ becomes unbounded for $0 < n < 1/3$). Such an approach involves an integration of the right-hand side of the governing equation (which depends in a non-trivial way on n) and may in particular be very useful in finding an analytical relationship for the exact values of n for which the minimum value of $|\alpha|$ occurs, and that's for a wide range of ϵ where we have found numerically that the minimum value of $|\alpha|$ occurs at values of $n > 1/3$.

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