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New $A-$Generalized Sequence Spaces Defined by Ideal Convergence and a Sequence of Modulus Functions on Multiple Normed Spaces

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Abstract

The object of this paper is to introduce a new class of ideal convergent (briefly I-convergent) sequence spaces using an infinite matrix, sequence of modulus functions and difference operator defined on n-normed spaces. We study these spaces for some linear topological structures and algebraic properties.

Keywords: *I-convergence, n-norm, modulus function , difference sequence space*

1 Introduction

The concept of 2-normed spaces was initially introduced by Gahler [19], in the mid of 1960's, while that of n-normed spaces can be found in Misiak [1]. Since then, many others have studied this concept and obtained various results, (see Gunawan [7], Gunawan and Mashadi [8, 9]). The notion of ideal-convergence in 2-normed spaces was introduced and studied in [14, 2] and [4]. Later on it was extended to n-normed spaces by Gurdal and Sahiner [15] , Hazarika [3] and Savas [5].

Let n be a non-negative integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\|.. ..., .\|$ on X^n satisfying the following conditions:

(1) $\|(x_1, x_2, ..., x_n)\| = 0$ if and only if $x_1, x_2, ..., x_n$ are linearly dependent.

(2) $\|(x_1, x_2, ..., x_n)\|$ is invariant under permutation,

(3) $\|\alpha x_1 x_2, ..., x_n\| = |\alpha| \|x_1, x_2, ..., x_n\|$, for any $\alpha \in R$,

(4) $\|(x_1 + \bar{x}, x_2, ..., x_n)\| \leq \|(x_1, x_2, ..., x_n)\| + \|(\bar{x}, x_2, ..., x_n)\|$

is called an n-norm on X and the pair $(X, \|., ..., .\|)$ is called an n-normed space.

A trivial example of an n-normed space is $X = R^n$, equipped with the Euclidean n-norm $\|(x_1, x_2, ..., x_n)\|_E =$ volume of the n-dimensional parallelepiped spanned by the vectors $x_1, x_2, ..., x_n$ which may be given explicitly by the formula

$\|(x_1, x_2, ..., x_n)\|_E = |\det(x_{ij})| = abs(\det(\langle x_i, x_j \rangle))$

Where $x_i = (x_{i1}, x_{i2}, ..., x_{in}) \in R^n$ for each $i = 1, 2, 3 ..., n$.

The standard n-norm on X a real inner product space of dimension $d \geq n$ is as follows:

$\|(x_1, x_2, ..., x_n)\|_S = [\det(\langle x_i, x_j \rangle)]^{\frac{1}{2}}$,

where $\langle \; , \; \rangle$ denotes the inner product on X.

The notion of modulus function was introduced by Nakano [11]. We recall that a modulus $f : [0, \infty) \to [0, \infty)$, which is such that

(i) $f(x) = 0$, if and only if $x = 0$;

(ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$;

(iii) f is increasing ;

(iv) f is continuous from right at zero.

It follows from (ii) and (iv) that f must be continuous everywhere on $[0, \infty)$.

For a sequence of modulus function $F = (f_k)$, we give the following conditions:

(v) $\sup_k f_k(x) < \infty$ for all $x > 0$,

(vi) $\lim_{x \to 0} f_k(x) = 0$ uniformly in $k \geq 1$.

We remark that in case $f = (f_k)$ for all k, where f is a modulus, the conditions (v) and (vi) are automatically fulfilled. The modulus function may be bounded or unbounded. For example, if we take $f(x) = \frac{1}{x+1}$, then $f(x)$ is bounded. If $f(x) = x^p, 0 < p < 1$, then the modulus $f(x)$ is unbounded.

The concept of I-convergence was introduced by Kastyrko et al. [18] as a generalization of statistical convergence. A family $I \subset 2^Y$ of subsets of nonempty set Y is said to be an ideal in Y if

(i) $\Phi \notin I$

(ii) for each $A, B \in I$, we have $A \bigcup B \in I$

(iii) $A \in I$ and each $B \subset A$, we have $B \in I$. While an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$.

Given $I \subset 2^N$ be a non-trivial ideal in N. A sequence $(x_n)_{n \in N}$ in X is said to be I- convergent to $x \in X$ if for each $\varepsilon > 0$ the set $A(\varepsilon) = \{n \in N : \| x_n - x \| \geq \varepsilon\}$ belongs to I (see [17]) . Savas [6] and Jalal [20] used an Orlicz function to construct I-convergent sequence spaces.

The notion of difference sequence spaces was introduced by Kizmaz [10]. It was further generalized by Et and Colak [12] . Later Et. and Esi [13] defined the sequence spaces

$$X(\Delta_m^s) = \{ x = (x_k) \in w : (\Delta_m^s x_k) \in X \}$$

where $m = (m_k)$ is any fixed sequence of nonzero complex numbers and $s \in N$, then $\Delta_m^0 x_k = m_k x_k$, $\Delta_m x = m_k x_k - m_{k+1} x_{k+1}$ are non-negative ,

$\Delta_m^s x = (\Delta_m^s x_k) = (\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+1})$ and so that

$$\Delta_m^s x_k = \sum_{i=0}^{s} (-1)^i \binom{s}{i} m_{k+i} x_{k+i}$$

Taking $m = s = 1$, we get the spaces of $\ell_\infty(\Delta), c(\Delta), c_0(\Delta)$, introduced and studied by Kizmaz [10].

2 Definitions and Inclusion Theorems

In this section, we define new Ideal convergent sequence spaces on n-normed spaces by using a sequence of modulus functions. We also give inclusion relations between these spaces.

Let I be an admissible ideal of N, and let $p=(p_k)$ be a bounded sequence of positive real numbers for all $k \in N$. Let $A=(a_{nk})$ be an infinite matrix, $F=(f_k)$ be a sequence of modulus functions and $(X, \|., ...,.\|)$ be an n-normed space. Further $w(n-X)$ denotes the spaces of X-valued sequence spaces. For every $z_1, z_2, ..., z_{n-1} \in X$, and for every $\varepsilon > 0$ we define the following sequence spaces:

$$w^I[A, \Delta_m^s, F, p, \|.,...,.\|] =$$
$$\left\{ x=(x_k) \in w(n-X): for\ every\ \varepsilon>0, \left\{ n \in N : \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\| \Delta_m^s x_k - L, z_1, z_2, ..., z_{n-1} \| \right) \right]^{p_k} \geq \varepsilon \right\} \in I, L \in X \right\}$$

$$w_0^I[A, \Delta_m^s, F, p, \|.,...,.\|] =$$
$$\left\{ x=(x_k) \in w(n-X) : \text{for every } \varepsilon>0, \left\{ n \in N : \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\| \Delta_m^s x_k, z_1, z_2, ..., z_{n-1} \| \right) \right]^{p_k} \geq \varepsilon \right\} \in I \right\},$$

$$w_\infty^I[A, \Delta_m^s, F, p, \|.,..,...,.\|] =$$
$$\left\{ x=(x_k) \in w(n-X) : \exists K>0\ s.t. \left\{ n \in N : \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\| \Delta_m^s x_k, z_1, z_2, ..., z_{n-1} \| \right) \right]^{p_k} \geq K \right\} \in I \right\}.$$

The following well known inequality will be used throughout the article. Let $p=(p_k)$ be any sequence of positive real numbers with $0 \leq p_k \leq \sup_k p_k = H$, $D = \max\{1, 2^{H-1}\}$, then

$$|a_k + b_k|^{p_k} \leq D\left(|a_k|^{p_k} + |b_k|^{p_k} \right)$$

for all $k \in N$ and $a_k, b_k \in C$. Also $|a|^{p_k} \leq \max\{1, |a|^H\}$ for all $a \in C$. (see [16])

Theorem 2.1. The sequence spaces $w^I[A, \Delta_m^s, F, p, \|.,....,.\|]$, $w_0^I[A, \Delta_m^s, F, p, \|.,...,.\|]$ and $w_\infty^I[A, \Delta_m^s, F, p, \|.,...,.\|]$ are linear.

Proof. We shall prove the theorem for the space $w_0^I[A, \Delta_m^s, F, p, \|.,...,.\|]$ only and the others can be proved in a similar manner. Let $x=(x_k)$ and $y=(y_k)$ be two elements in $w_0^I[A, \Delta_m^s, F, p, \|.,...,.\|]$ and α, β be scalars in $\Re$, so that

$$\left\{ n \in N : \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s x_k, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

and

$$\left\{ n \in N : \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s y_k, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\} \in I.$$

Since $\|., ..., .\|$ is an n-norm, f_k is a sequence of modulus functions for all k and Δ_m^s is linear the following inequality holds:

$$\sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s (\alpha x_k + \beta y_k), z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k}$$

$$\leq D\, T_\alpha^{\sup p_k} \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s x_k, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k}$$

$$+ D\, T_\beta^{\sup p_k} \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s x_k, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k}$$

where T_α and T_β are positive integers such that that $|\alpha| \leq T_\alpha$ and $|\beta| \leq T_\beta$.
From the above inequality we get

$$\left\{ n \in N : \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s (\alpha x_k + \beta y_k), z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\subseteq \left\{ n \in N : D\, T_\alpha^{\sup p_k} \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s x_k, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\}$$

$$\bigcup \left\{ n \in N : D\, T_\beta^{\sup p_k} \sum_{k=1}^{\infty} a_{nk} \left[f_k \left(\left\| \Delta_m^s y_k, z_1, z_2, ..., z_{n-1} \right\| \right) \right]^{p_k} \geq \varepsilon \right\}. \tag{1}$$

Since both the sets on the right hand of the relation (1) belong to I so the set on the left hand side of the inclusion relation belongs to I. This completes the proof of the theorem. □

Theorem 2.2. Let $F = (f_k)$ and $G = (g_k)$ be sequences of moduli and $h = \inf p_k > 0$. Then

(i) $w^I\left[A, \Delta_m^s, F, p, \|.,...,.\|\right] \cap w^I\left[A, \Delta_m^s, F, p, \|.,...,.\|\right] \subseteq$
$w^I\left[A, \Delta_m^s, F + G, p, \|.,...,.\|\right]$

(ii) $w_0^I\left[A, \Delta_m^s, F, p, \|.,...,.\|\right] \cap w_0^I\left[A, \Delta_m^s, F, p, \|.,...,.\|\right] \subseteq$
$w_0^I\left[A, \Delta_m^s, F + G, p, \|.,...,.\|\right]$.

Proof.(i) Let $x=(x_k)\in w^I\left[A,\Delta_m^s,F,\ p,\|\ .,\ldots,.\|\right]\cap w^I\left[A,\Delta_m^s,G,\ p,\|\ .,\ldots,.\|\right]$.
Then by the following inequality the result follows

$$\sum_{k=1}^{\infty}a_{nk}\left[(f_k+g_k)\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|\right)\right]^{p_k}$$
$$\le D\sum_{k=1}^{\infty}a_{nk}\left[f_k\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|\right)\right]^{p_k}$$
$$+D\sum_{k=1}^{\infty}a_{nk}\left[g_k\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|\right)\right]^{p_k}.$$

Hence

$$\left\{n\in N:\sum_{k=1}^{\infty}a_{nk}\left[(f_k+g_k)\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|\right)\right]^{p_k}\ge\varepsilon\right\}$$
$$\subseteq\left\{n\in N:D\sum_{k=1}^{\infty}a_{nk}\left[f_k\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|\right)\right]^{p_k}\ge\frac{\varepsilon}{2}\right\}$$
$$\bigcup\left\{n\in N:D\sum_{k=1}^{\infty}a_{nk}\left[g_k\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|\right)\right]^{p_k}\ge\frac{\varepsilon}{2}\right\}.$$

Since both the sets on the right hand belong to I so the set on the left hand side of the inclusion relation belongs to I. This completes the proof of the Theorem (i). (ii) can be proved similarly. □

Theorem 2.3. The inclusions $X\left[A,\Delta_m^{s-1},F,\ p,\|\ .,\ldots,.\|\right]\subseteq X\left[A,\Delta_m^s,F,\ p,\|\ .,\ldots,.\|\right]$ are strict for $s\ge1$. In general $X\left[A,\Delta_m^j,F,\ p,\|\ .,\ldots,.\|\right]\subseteq X\left[A,\Delta_m^s,F,\ p,\|\ .,\ldots,.\|\right]$ for $j=0,1,2,\ldots,s-1$ and the inclusions are strict, where $X=w_0^I,w^I$ *and* w_∞^I.

Proof. We give the proof for $w_0^I\left[A,\Delta_m^{s-1},F,\ p,\|\ .,\ldots,.\|\right]$ only, others can be proved by a similar argument. Let $x=(x_k)$ be any element in the space $w_0^I\left[A,\Delta_m^{s-1},F,\ p,\|\ .,\ldots,.\|\right]$. Let $\varepsilon>0$ be given, then

$$\left\{n\in N:\sum_{k=1}^{\infty}a_{nk}\left[f_k\left(\|\Delta_m^s x_k,z_1,z_2,\ldots,z_{n-1}\|\right)\right]^{p_k}\ge\varepsilon\right\}\in I.$$

Since f_k is a modulus function for all k, it follows that

$$\sum_{k=1}^{\infty} a_{nk}\left[f_k\left(\left\|\Delta_m^s x_k, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k}$$
$$= \sum_{k=1}^{\infty} a_{nk}\left[f_k\left(\left\|\Delta_m^{s-1} x_k - \Delta_m^{s-1} x_{k+1}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k}$$
$$\leq D\sum_{k=1}^{\infty} a_{nk}\left[f_k\left(\left\|\Delta_m^{s-1} x_k, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} +$$
$$D\sum_{k=1}^{\infty} a_{nk}\left[f_k\left(\left\|\Delta_m^{s-1} x_{k+1}, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k}.$$ Hence the proof. □

The inclusion is strict follows from the following example.

Example 2.4. Let $F(x) = x$, for all $x \in [0, \infty)$, $p_k = 1$, for all $k \in N$ and sequence $A = (C,1)$ that is the Cesaro matrix. Consider a sequence $x = (x_k) = (k^s)$. Then $x = (x_k)$ belongs to $w_0^I\left[\Delta_m^s, F, p, \|.,\ldots,.\|\right]$ but does not belong to $w_0^I\left[\Delta_m^{s-1}, F, p, \|.,\ldots,.\|\right]$, because $\Delta_m^s x_k = 0$ and $\Delta_m^{s-1} x_k = (-1)^{s-1}(s-1)!$.

Definition 2.5. Let X be a sequence space. Then X is called solid if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$ for all (α_k) for all sequences of scalars $|\alpha_k| \leq 1$ for all $k \in N$.

Theorem 2.6. The sequence spaces $w_0^I\left[A, \Delta_m^s, F, p, \|.,\ldots,.\|\right]$ and $w_\infty^I\left[A, \Delta_m^s, F, p, \|.,\ldots,.\|\right]$ are solid.

Proof. We give the proof for $w_0^I\left[A, \Delta_m^s, F, p, \|.,\ldots,.\|\right]$ only. Let $x = (x_k) \in w_0^I\left[A, \Delta_m^s, F, p, \|.,\ldots,.\|\right]$ and $\alpha = (\alpha_k)$ be a sequence of scalars such that $|\alpha_k| \leq 1$ for all $k \in N$. Then for given $\varepsilon > 0$ we have

$$\left\{ n \in N : \sum_{k=1}^{\infty} a_{nk}\left[f_k\left(\left\|\Delta_m^s(\alpha_k x_k), z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq \varepsilon \right\}$$
$$\subseteq \left\{ n \in N : E\sum_{k=1}^{\infty} a_{nk}\left[f_k\left(\left\|\Delta_m^s x_k, z_1, z_2, \ldots, z_{n-1}\right\|\right)\right]^{p_k} \geq \varepsilon \right\} \in I,$$

where $E = \max\left\{1, |\alpha_k|^H\right\}$. Hence $(\alpha_k x_k) \in w_0^I\left[A, \Delta_m^s, F, p, \|.,\ldots,.\|\right]$. Thus the space

$w_0^I\left[A, \Delta_m^s, F, p, \|.,\ldots,.\|\right]$ is solid . This completes the proof. □

Theorem 2 .7. Let $(X,\|.,\ldots,.\|_E)$and $(X,\|.,\ldots,.\|_S)$ be standard and Euclid n-normed space spaces respectively, then

$$w^I\left\lfloor A,\Delta_m^s,F,\ p,\|\ .,\ldots,.\|_E\right\rfloor\cap w^I\left\lfloor A,\Delta_m^s,F,\ p,\|\ .,\ldots,.\|_S\right\rfloor$$
$$\subseteq w^I\left\lfloor A,\Delta_m^s,F,\ p,\left(\|\ .,\ldots,.\|_E+\|.,\ldots,.\|_S\right)\right\rfloor.$$

Proof. The following inequality that gives us the desired inclusion

$$\sum_{k=1}^{\infty}a_{nk}\left[f_k\left(\left(\|.,\ldots,.\|_E+\|.,\ldots,.\|_S\right)+\left(\Delta_m^s x_k-L,z_1,z_2,\ldots z_{n-1}\right)\right)\right]^{p_k}$$
$$\leq D\sum_{k=1}^{\infty}a_{nk}\left[f_k\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|_E\right)\right]^{p_k}$$
$$+D\sum_{k=1}^{\infty}a_{nk}\left[f_k\left(\|\Delta_m^s x_k-L,z_1,z_2,\ldots,z_{n-1}\|_S\right)\right]^{p_k}.$$

□

3 Conclusion

We introduce a new class of I-convergent sequence spaces using an infinite matrix, sequence of modulus functions and difference operator defined on n-normed spaces and proved their several algebraic and topological properties.

4 Open Problems

In this paper we have introduced a new class of ideal convergent (briefly I-convergent) sequence spaces using an infinite matrix, sequence of modulus functions and difference operator defined on n-normed spaces. Some linear topological structures and algebraic properties for these spaces has also been discussed. In the sequel the next step will be to:

-introduce some new classes of ideal convergent sequence spaces using an infinite matrix, Orlicz functions and difference operator defined on n-normed spaces.

- further some linear topological structures and algebraic properties for these spaces can also been discussed also.

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