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Multiple positive solutions for a nonlinear three-point integral boundary-value problem

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Abstract

We investigate the existence of positive solutions to the nonlinear second-order three-point integral boundary value problem.

$$u''(t) + f(t, u(t)) = 0,\ 0 < t < T,$$
$$u(0) = \beta u(\eta),\ u(T) = \alpha \int_0^\eta u(s)ds,$$

where $0 < \eta < T$, $0 < \alpha < \frac{2T}{\eta^2}$, $0 < \beta < \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$ ***are given constants. We establish the existence of at least three positive solutions by using the Leggett-Williams fixed-point theorem.***

Keywords: *Positive solutions, Three-point boundary value problems, multiple solutions, Fixed points, Cone.*

1 Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by II'in and Moiseev [16]. Then Gupta [6] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. We refer the reader to [5, 7, 8, 9, 10, 11, 12, 13, 18, 19, 20, 21, 22, 23, 25, 26, 27, 28, 30, 31, 32, 33, 34, 35, 36, 37, 39, 40, 41] and the references therein.

This paper is a continuation of our study in [15] and is concerned with the existence and multiplicity of positive solutions of the problem

$$u''(t) + f(t, u(t)) = 0,\ \ t \in (0, T), \tag{1}$$

with the three-point integral boundary condition

$$u(0) = \beta u(\eta),\ \ u(T) = \alpha \int_0^\eta u(s)ds, \tag{2}$$

Throughout this paper, we assume the following hypotheses:

(H1) $f \in C([0, T] \times [0, \infty), [0, \infty))$ and $f(t, .)$ does not vanish identically on any subset of $[0, T]$ with positive measure.

(H2) $\eta \in (0, T)$, $0 < \alpha < \frac{2T}{\eta^2}$ and $0 < \beta < \frac{2T - \alpha\eta^2}{\alpha\eta^2 - 2\eta + 2T}$.

In this paper, by using the Leggett-Williams fixed-point theorem [17], we will show the existence of at least three positive solutions for a three-point integral boundary value problem. Some papers in this area include [24, 29, 38, 1, 2, 3, 14, 4].

2 Background and definitions

Definition 2.1 *Let E be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:*
(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$ implies $x = 0$.
Every cone $P \subset E$ induces an ordering in E given by $x \leq y$ if and only if $y - x \in E$.

Definition 2.2 *An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.*

Definition 2.3 *A map ψ is said to be a nonnegative continuous concave functional on a cone P of a real Banach space E if $\psi : P \to [0, \infty)$ is continuous and*

$$\psi(tx + (1 - t)y) \geq t\psi(x) + (1 - t)\psi(y)$$

for all $x, y \in P$ and $t \in [0,1]$. Similarly we say the map φ is a nonnegative continuous convex functional on a cone P of a real Banach space E if $\varphi : P \to [0,\infty)$ is continuous and

$$\varphi(tx + (1-t)y) \le t\varphi(x) + (1-t)\varphi(y)$$

for all $x, y \in P$ and $t \in [0,1]$.

Definition 2.4 *Let ψ be a nonnegative continuous concave functional on the cone P. Define the convex sets P_c and $P(\psi, a, b)$ by*

$$P_c = \{x \in P : \|x\| < c\}, \quad for \quad c > 0$$

$$P(\psi, a, b) = \{x \in P : a \le \psi(x), \|x\| \le b\}, \quad for \quad 0 < a < b.$$

Next we state the Leggett-Williams fixed-point theorem.

Theorem 2.5 ([17]) *Let $A : \overline{P}_c \to \overline{P}_c$ be a completely continuous operator and let ψ be a nonnegative continuous concave functional on P such that $\|\psi(x)\| \le \|x\|$ for all $x \in \overline{P}_c$. Suppose that there exist $0 < a < b < d \le c$ such that the following conditions hold,*

(C1) $\{x \in P(\psi, b, d) : \psi(x) > b\} \ne \emptyset$ and $\psi(Ax) > b$ for all $x \in P(\psi, b, d)$;

(C2) $\|Ax\| < a$ for $\|x\| \le a$;

(C3) $\psi(Ax) > b$ for $x \in P(\psi, b, c)$ with $\|Ax\| > d$.

Then A has at least three fixed points x_1, x_2 and x_3 in $\overline{P}_c$ satisfying
$\|x_1\| < a$, $\psi(x_2) > b$, $a < \|x_3\|$ with $\psi(x_3) < b$.

3 Some preliminary results

In order to prove our main result, we need some preliminary results. Let us consider the following boundary value problem

$$u''(t) + y(t) = 0, \ t \in (0, T), \tag{3}$$

$$u(0) = \beta u(\eta), \ u(T) = \alpha \int_0^{\eta} u(s)ds \tag{4}$$

For problem (3), (4), we have the following conclusions which are derived from [15].

Lemma 3.1 (See [15]) *Let $\beta \neq \frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. Then for $y \in C([0,T],\mathbb{R})$, the problem* (3)-(4) *has the unique solution*

$$\begin{aligned} u(t) &= \frac{\beta(2T-\alpha\eta^2)-2\beta(1-\alpha\eta)t}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^\eta(\eta-s)y(s)ds \\ &\quad+\frac{\alpha\beta\eta-\alpha(\beta-1)t}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^\eta(\eta-s)^2y(s)ds \\ &\quad+\frac{2(\beta-1)t-2\beta\eta}{(\alpha\eta^2-2T)-\beta(2\eta-\alpha\eta^2-2T)}\int_0^T(T-s)y(s)ds \\ &\quad-\int_0^t(t-s)y(s)ds. \end{aligned}$$

Lemma 3.2 (See [15]) *Let $0<\alpha<\frac{2T}{\eta^2}$, $0\leq\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. If $y\in C([0,T],[0,\infty))$, then the unique solution u of problem* (3), (4) *satisfies $u(t)\geq 0$ for $t\in[0,T]$.*

Lemma 3.3 (See [15]) *Let $0<\alpha<\frac{2T}{\eta^2}$, $0\leq\beta<\frac{2T-\alpha\eta^2}{\alpha\eta^2-2\eta+2T}$. If $y\in C([0,T],[0,\infty))$, then the unique solution u of the problem* (3), (4) *satisfies*

$$\min_{t\in[\eta,T]}u(t)\geq\gamma\|u\|,\ \ \|u\|=\max_{t\in[0,T]}|u(t)|, \tag{5}$$

where

$$\gamma:=\min\left\{\frac{\eta}{T},\frac{\alpha(\beta+1)\eta^2}{2T},\frac{\alpha(\beta+1)\eta(T-\eta)}{2T-\alpha(\beta+1)\eta^2}\right\}\in(0,1)\,. \tag{6}$$

4 Existence of triple solutions

In this section, we discuss the multiplicity of positive solutions for the general boundary-value problem (1), (2)

In the following, we denote

$$\Lambda:=(2T-\alpha\eta^2)-\beta(\alpha\eta^2-2\eta+2T), \tag{7}$$

$$m:=\Big(\frac{T^2(2T(\beta+1)+\beta\eta(\alpha\eta+2)+\alpha\beta T^2)}{2\Lambda}\Big)^{-1}, \tag{8}$$

$$\delta:=\min\Big\{\frac{\eta(T-\eta)^2}{\Lambda},\frac{\alpha\eta^2(1+\beta)(T-\eta)^2}{2\Lambda}\Big\}. \tag{9}$$

Using Theorem 2.5, we established the following existence theorem for the boundary-value problem (1), (2).

Theorem 4.1 *Assume* (H1) *and* (H2) *hold. Suppose there exists constants* $0 < a < b < b/\gamma \le c$ *such that*

(D1) $f(t,u) < ma$ *for* $t \in [0,T]$, $u \in [0,a]$;

(D2) $f(t,u) \ge \frac{b}{\delta}$ *for* $t \in [\eta,T]$, $u \in [b,\frac{b}{\gamma}]$;

(D3) $f(t,u) \le mc$ *for* $t \in [0,T]$, $u \in [0,c]$,

where γ, m, δ *are as defined in* (6), (8) *and* (9), *respectively. Then the boundary-value problem* (1)-(2) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\| < a, \quad \min_{t\in[0,T]} u_2(t) > b, \quad a < \|u_3\| \quad \textit{with} \ \min_{t\in[0,T]} u_3(t) < b.$$

Proof.

Let $E = C([0,T],\mathbb{R})$ be endowed with the maximum norm, $\|u\| = \max_{t\in[0,T]} u(t)$, define the cone $P \subset C([0,T],\mathbb{R})$ by

$$P = \{u \in C([0,T],\mathbb{R}) : u \text{ concave down and } u(t) \ge 0 \text{ on } [0,T]\}. \tag{10}$$

Let $\psi : P \to [0,\infty)$ be defined by

$$\psi(u) = \min_{t\in[\eta,T]} u(t), \quad u \in P. \tag{11}$$

then ψ is a nonnegative continuous concave functional and $\psi(u) \le \|u\|, u \in P$.

Define the operator $A : P \to C([0,T],\mathbb{R})$ by

$$\begin{aligned} Au(t) \;=\; & -\frac{\beta(2T-\alpha\eta^2)-2\beta(1-\alpha\eta)t}{\Lambda}\int_0^\eta (\eta-s)f(s,u(s))ds \\ & -\frac{\alpha\beta\eta-\alpha(\beta-1)t}{\Lambda}\int_0^\eta (\eta-s)^2 f(s,u(s))ds \\ & -\frac{2(\beta-1)t-2\beta\eta}{\Lambda}\int_0^T (T-s)f(s,u(s))ds \\ & -\int_0^t (t-s)f(s,u(s))ds. \end{aligned}$$

Then the fixed points of A just are the solutions of the boundary-value problem (1)-(2) from Lemma 3.1. Since $(Au)''(t) = -f(t,u(t))$ for $t \in (0,T)$, together with (H1) and Lemma 3.2, we see that $Au(t) \ge 0,\ t \in [0,T]$ and $(Au)''(t) \le 0,\ t \in (0,T)$. Thus $A : P \to P$. Moreover, A is completely continuous.

We now show that all the conditions of Theorem 2.5 are satisfied. From (11), we know that $\psi(u) \leq \|u\|$, for all $u \in P$.

Now if $u \in \overline{P_c}$, then $0 \leq u \leq c$, together with (D3), we find $\forall\ t \in [0,T]$,

$$\begin{aligned}
Au(t) &\leq \frac{2\beta(1-\alpha\eta)t-\beta(2T-\alpha\eta^2)}{\Lambda}\int_0^\eta(\eta-s)f(s,u(s))ds \\
&\quad +\frac{\alpha(\beta-1)t-\alpha\beta\eta}{\Lambda}\int_0^\eta(\eta-s)^2f(s,u(s))ds \\
&\quad +\frac{2\beta\eta-2(\beta-1)t}{\Lambda}\int_0^T(T-s)f(s,u(s))ds \\
&\leq \frac{2\beta T+\alpha\beta\eta^2}{\Lambda}\int_0^\eta(\eta-s)f(s,u(s))ds+\frac{\alpha\beta T}{\Lambda}\int_0^\eta(\eta-s)^2f(s,u(s))ds \\
&\quad +\frac{2\beta\eta+2T}{\Lambda}\int_0^T(T-s)f(s,u(s))ds \\
&\leq \frac{2T(\beta+1)+\beta\eta(\alpha\eta+2)}{\Lambda}\int_0^T(T-s)f(s,u(s))ds \\
&\quad +\frac{\alpha\beta T}{\Lambda}\int_0^\eta(\eta-s)^2f(s,u(s))ds \\
&\leq \frac{2T(\beta+1)+\beta\eta(\alpha\eta+2)}{\Lambda}\int_0^T(T-s)f(s,u(s))ds \\
&\quad +\frac{\alpha\beta T}{\Lambda}\int_0^T T(T-s)f(s,u(s))ds \\
&= \frac{2(\beta+1)+T^{-1}\beta\eta(\alpha\eta+2)+\alpha\beta T}{\Lambda}\int_0^T T(T-s)f(s,u(s))ds \\
&\leq mc\frac{2(\beta+1)+T^{-1}\beta\eta(\alpha\eta+2)+\alpha\beta T}{\Lambda}\int_0^T T(T-s)ds \\
&= mc\frac{T^2(2T(\beta+1)+\beta\eta(\alpha\eta+2)+\alpha\beta T^2)}{2\Lambda} \\
&= c
\end{aligned}$$

Thus, $A:\overline{P_c}\to\overline{P_c}$.

By (D1) and the argument above, we can get that $A:\overline{P_a}\to P_a$. So, $\|Au\|<a$ for $\|u\|\leq a$, the condition (C2) of Theorem 2.5 holds.

Consider the condition (C1) of Theorem 2.5 now. Since $\psi(b/\gamma)=b/\gamma>b$, let $d=b/\gamma$, then $\{u\in P(\psi,b,d):\psi(u)>b\}\neq\emptyset$. For $u\in P(\psi,b,d)$, we have $b\leq u(t)\leq b/\gamma,\ t\in[\eta,T]$. Combining with (D2), we get

$$f(t,u)\geq\frac{b}{\delta},\quad t\in[\eta,T].$$

Since $u\in P(\psi,b,d)$, then there are two cases, (i) $\psi(Au)(t)=Au(T)$ and (ii) $\psi(Au)(t)=Au(\eta)$. In case (i), we have

$$\begin{aligned}
\psi(Au)(t) &= Au(T)\\
&= -\frac{\beta(2T-\alpha\eta^2)-2\beta(1-\alpha\eta)T}{\Lambda}\int_0^\eta(\eta-s)f(s,u(s))ds\\
&\quad -\frac{\alpha\beta\eta-\alpha(\beta-1)T}{\Lambda}\int_0^\eta(\eta-s)^2f(s,u(s))ds\\
&\quad -\frac{2(\beta-1)T-2\beta\eta}{\Lambda}\int_0^T(T-s)f(s,u(s))ds\\
&\quad -\int_0^T(T-s)f(s,u(s))ds\\
&= \frac{\alpha\beta\eta(\eta-2T)}{\Lambda}\int_0^\eta(\eta-s)f(s,u(s))ds\\
&\quad +\frac{\alpha(\beta-1)T-\alpha\beta\eta}{\Lambda}\int_0^\eta(\eta-s)^2f(s,u(s))ds\\
&\quad +\frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_0^T(T-s)f(s,u(s))ds\\
&= \frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_0^T(T-s)f(s,u(s))ds-\frac{\alpha\eta^2T(\beta+1)}{\Lambda}\int_0^\eta f(s,u(s))ds\\
&\quad +\frac{\alpha\eta(\beta\eta+2T)}{\Lambda}\int_0^\eta sf(s,u(s))ds\\
&\quad +\frac{\alpha(\beta-1)T-\alpha\beta\eta}{\Lambda}\int_0^\eta s^2f(s,u(s))ds\\
&= \frac{\alpha\eta^2T(\beta+1)}{\Lambda}\int_\eta^T f(s,u(s))ds-\frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_0^\eta sf(s,u(s))ds\\
&\quad -\frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_\eta^T sf(s,u(s))ds+\frac{\alpha\eta(\beta\eta+2T)}{\Lambda}\int_0^\eta sf(s,u(s))ds\\
&\quad +\frac{\alpha(\beta-1)T-\alpha\beta\eta}{\Lambda}\int_0^\eta s^2f(s,u(s))ds\\
&= \frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_\eta^T(T-s)f(s,u(s))ds+\frac{\alpha\eta(2T-\eta)}{\Lambda}\int_0^\eta sf(s,u(s))ds\\
&\quad +\frac{\alpha\beta(T-\eta)-\alpha T}{\Lambda}\int_0^\eta s^2f(s,u(s))ds\\
&> \frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_\eta^T(T-s)f(s,u(s))ds+\frac{\alpha\eta T}{\Lambda}\int_0^\eta sf(s,u(s))ds\\
&\quad -\frac{\alpha T}{\Lambda}\int_0^\eta s^2f(s,u(s))ds\\
&= \frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_\eta^T(T-s)f(s,u(s))ds+\frac{\alpha T}{\Lambda}\int_0^\eta s(\eta-s)f(s,u(s))ds
\end{aligned}$$

$$
\begin{aligned}
&> \frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_\eta^T (T-s)f(s,u(s))ds\\
&\geq \frac{b}{\delta}\frac{\alpha\eta^2(\beta+1)}{\Lambda}\int_\eta^T (T-s)ds\\
&= \frac{b}{\delta}\frac{\alpha\eta^2(\beta+1)(T-\eta)^2}{2\Lambda}\\
&\geq b.
\end{aligned}
$$

In case (ii), we have

$$
\begin{aligned}
\psi(Au)(t) &= Au(\eta)\\
&= -\frac{\beta(\alpha\eta^2-2\eta+2T)}{\Lambda}\int_0^\eta(\eta-s)f(s,u(s))ds-\int_0^\eta(\eta-s)f(s,u(s))ds\\
&\quad+\frac{2\eta}{\Lambda}\int_0^T(T-s)f(s,u(s))ds-\frac{\alpha\eta}{\Lambda}\int_0^\eta(\eta-s)^2f(s,u(s))ds\\
&= \frac{2\eta}{\Lambda}\int_0^T(T-s)f(s,u(s))ds-\frac{2T-\alpha\eta^2}{\Lambda}\int_0^\eta(\eta-s)f(s,u(s))ds\\
&\quad-\frac{\alpha\eta}{\Lambda}\int_0^\eta(\eta-s)^2f(s,u(s))ds\\
&= \frac{2\eta}{\Lambda}\int_0^T(T-s)f(s,u(s))ds-\frac{\eta(2T-\alpha\eta^2)}{\Lambda}\int_0^\eta f(s,u(s))ds\\
&\quad+\frac{2T-\alpha\eta^2}{\Lambda}\int_0^\eta sf(s,u(s))ds-\frac{\alpha\eta^3}{\Lambda}\int_0^\eta f(s,u(s))ds\\
&\quad+\frac{2\alpha\eta^2}{\Lambda}\int_0^\eta sf(s,u(s))ds-\frac{\alpha\eta}{\Lambda}\int_0^\eta s^2f(s,u(s))ds\\
&= \frac{2\eta}{\Lambda}\int_0^T(T-s)f(s,u(s))ds-\frac{2\eta T}{\Lambda}\int_0^\eta f(s,u(s))ds\\
&\quad+\frac{2T+\alpha\eta^2}{\Lambda}\int_0^\eta sf(s,u(s))ds-\frac{\alpha\eta}{\Lambda}\int_0^\eta s^2f(s,u(s))ds\\
&= \frac{2\eta T}{\Lambda}\int_\eta^T f(s,u(s))ds-\frac{2\eta}{\Lambda}\int_0^\eta sf(s,u(s))ds-\frac{2\eta}{\Lambda}\int_\eta^T sf(s,u(s))ds\\
&\quad+\frac{2T+\alpha\eta^2}{\Lambda}\int_0^\eta sf(s,u(s))ds-\frac{\alpha\eta}{\Lambda}\int_0^\eta s^2f(s,u(s))ds\\
&= \frac{2\eta}{\Lambda}\int_\eta^T(T-s)f(s,u(s))ds+\frac{2(T-\eta)+\alpha\eta^2}{\Lambda}\int_0^\eta sf(s,u(s))ds\\
&\quad-\frac{\alpha\eta}{\Lambda}\int_0^\eta s^2f(s,u(s))ds
\end{aligned}
$$

$$\begin{aligned}
&> \frac{2\eta}{\Lambda}\int_\eta^T (T-s)f(s,u(s))ds + \frac{\alpha\eta^2}{\Lambda}\int_0^\eta sf(s,u(s))ds \\
&\quad -\frac{\alpha\eta}{\Lambda}\int_0^\eta s^2 f(s,u(s))ds \\
&= \frac{2\eta}{\Lambda}\int_\eta^T (T-s)f(s,u(s))ds + \frac{\alpha\eta}{\Lambda}\int_0^\eta s(\eta-s)f(s,u(s))ds \\
&> \frac{2\eta}{\Lambda}\int_\eta^T (T-s)f(s,u(s))ds \\
&\geq \frac{b}{\delta}\frac{2\eta}{\Lambda}\int_\eta^T (T-s)ds \\
&= \frac{b}{\delta}\frac{\eta(T-\eta)^2}{\Lambda} \\
&\geq b.
\end{aligned}$$

So, $\psi(Au) > b$; $\forall\ u \in P(\psi, b, b/\gamma)$.

For the condition (C3) of the Theorem 2.5, we can verify it easily under our assumptions using Lemma 3.3. Here

$$\psi(Au) = \min_{t\in[\eta,T]} Au(t) \geq \gamma\|Au\| > \gamma\frac{b}{\gamma} = b$$

as long as $u \in P(\psi, b, c)$ with $\|Au\| > b/\gamma$.

Since all conditions of Theorem 2.5 are satisfied. Then problem (1)-(2) has at least three positive solutions u_1, u_2, u_3 with

$$\|u_1\| < a, \quad \psi(u_2) > b, \quad a < \|u_3\| \quad \text{with } \psi(u_3) < b.$$

5 Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 5.1 *Consider the boundary value problem*

$$u''(t) + \frac{40u^2}{u^2+1} = 0, \quad 0 < t < 1, \tag{12}$$

$$u(0) = \frac{1}{2}u(\frac{1}{3}), \quad u(1) = 3\int_0^{\frac{1}{3}} u(s)ds. \tag{13}$$

Set $\beta = 1/2$, $\alpha = 3$, $\eta = 1/3$, $T = 1$, *and*

$$f(t,u) = f(u) = \frac{40u^2}{u^2+1}, \quad u \geq 0.$$

It is clear that $f(.)$ *is continuous and increasing on* $[0,\infty)$. *We can also show that*

$$0 < \alpha = 3 < 18 = \frac{2T}{\eta^2}, \quad 0 < \beta = \frac{1}{2} < 1 = \frac{2T - \alpha\eta^2}{\alpha\eta^2 - 2\eta + 2T}.$$

Now we check that (D1), (D2) *and* (D3) *of Theorem 4.1 are satisfied. By* (6), (8), (9), *we get* $\gamma = 1/4$, $m = 1/3$, $\delta = 2/15$. *Let* $c = 124$, *we have*

$$f(u) \leq 40 < mc = \frac{124}{3} \approx 41,33, \ u \in [0,c],$$

from $\lim_{u\to\infty} f(u) = 40$, *so that* (D3) *is met. Note that* $f(2) = 32$, *when we set* $b = 2$,

$$f(u) \geq \frac{b}{\delta} = 15, \ u \in [b, 4b],$$

holds. It means that (D2)*is satisfied. To verify* (D1), *as* $f(\frac{1}{120}) = \frac{40}{14401}$, *we take* $a = \frac{1}{120}$, *then*

$$f(u) < ma = \frac{1}{360}, \ u \in [0,a],$$

and (D1) *holds. Summing up, there exists constants* $a = 1/120$, $b = 2$, $c = 124$ *satisfying*

$$0 < a < b < \frac{b}{\gamma} \leq c,$$

such that (D1), (D2) *and* (D3) *of Theorem 4.1 hold. So the boundary-value problem* (12)-(13) *has at least three positive solutions* u_1, u_2 *and* u_3 *satisfying*

$$\|u_1\| < \frac{1}{120}, \quad \min_{t\in[0,T]} u_2(t) > 2, \quad \frac{1}{120} < \|u_3\| \ \ \textit{with} \ \min_{t\in[0,T]} u_3(t) < 2.$$

Example 5.2 *Consider the boundary value problem*

$$u''(t) + f(t,u) = 0, \quad 0 < t < 1, \tag{14}$$

$$u(0) = u(\frac{1}{2}), \quad u(1) = \int_0^{\frac{1}{2}} u(s)ds. \tag{15}$$

Set $\beta = 1$, $\alpha = 1$, $\eta = 1/2$, $T = 1$, $f(t,u) = e^{-t}h(u)$ *where*

$$h(u) = \begin{cases} \frac{2}{25}u & 0 \le u \le 1 \\ \frac{2173}{75}u - \frac{2167}{75} & 1 \le u \le 4 \\ 87 & 4 \le u \le 544 \\ \frac{87}{544}u & 544 \le u \le 546 \\ \frac{39(3u+189)}{u+270} & u \ge 546. \end{cases} \tag{16}$$

By (6), (8), (9) *and after a simple calculation, we get* $\gamma = 1/4$, $m = 4/25$, $\delta = 1/8$.

We choose $a = 1/4$, $b = 4$, *and* $c = 544$*; consequently,*

$$\begin{aligned} f(t,u) &= e^{-t}\frac{2}{25}u \le \frac{2}{25}u < \frac{4}{25} \times \frac{1}{4} = ma, \quad 0 \le t \le 1, \quad 0 \le u \le 1/4, \\ f(t,u) &= e^{-t}87 \ge \frac{87}{e} > 32 = \frac{b}{\delta}, \quad 1/2 \le t \le 1, \quad 4 \le u \le 16, \\ f(t,u) &= e^{-t}h(u) \le 87 < \frac{4}{25} \times 544 = mc, \quad 0 \le t \le 1, \quad 0 \le u \le 544. \end{aligned}$$

That is to say, all the conditions of Theorem 4.1 are satisfied. Then problem (14), (15) *has at least three positive solutions* u_1 u_2*, and* u_3 *satisfying*

$$\|u_1\| < \frac{1}{4}, \quad \psi(u_2) > 4, \quad \|u_3\| > \frac{1}{4} \ \text{ with } \ \psi(u_3) < 4.$$

6 Open Problem

Is it possible to generalize the above results for multipoint integral boundary value problems?

In this work, we have assumed that f is continuous function, it will be interesting to consider the same problem but with singularities.

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