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A fixed point method for a class of nonlinear evolution systems modeling a mechanical phenomenon

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Abstract

This paper deals with the study of a class of nonlinear evolution systems with parameter t which may interpreted as the time or the absolute temperature. Such type of problems arise in the study of quasistatic problem in viscoplasticity. The existence and uniqueness of the solution is obtained using standard arguments for elliptic equations followed by a fixed point technique. The continuous dependence of the solution with respect to the data is also given. Finally, a mechanical example is presented in order to illustrate this result.

Keywords: *fixed point technique, nonlinear evolution system, variational equality, viscoplastic material.*

1 Introduction

Let H be a real Hilbert space and let X, Y be two orthogonal subspaces of H such that $H = X \oplus Y$.Let $T > 0, A : [0, T] \to H$ and $B : [0, T] \times X \times Y \to H$ be a nonlinear operator.We are interested in the following evolution problem:

$$\dot{y}(t) = A(t)\,\dot{x}(t) + B(t, x(t), y(t)) \text{ for all } t \in [0, T] \tag{1}$$

$$x(0) = x_0, y(0) = y_0 \tag{2}$$

in which unknowns are the functions $x : [0, T] \rightarrow X$ and $y : [0, T] \rightarrow Y$.In (1) and everywhere in this paper the dot above represents the derivative with respect to the time variable.

In the case when A is a linear operator (does not depend on t),some results of existence and uniqueness for problem (1)–(2) were obtained by [7],[16] using different functional methods.

The purpose of this paper is to extend the technique presented by [16] in the case $A = A(t)$ and we prove the existence and uniqueness of the solution for the problem (1)–(2) in the case when the operator A depends on t ,(where t is interpreted as time or the absolute temperature) , using standard arguments for elliptic equations followed by a fixed point technique .

The aim of this paper is to give a new demonstration for the existence and uniqueness result for the problem (1)–(2). This demonstration is based only standard arguments for elliptic equations followed by a fixed point technique

The paper is organized as follows: in the second section we prove an existence and uniqueness result using standard arguments for elliptic equations followed by a fixed point technique (*Theorem* 2.1) ; in section 3 the continuous dependence of the solution with respect to the data is given (*Theorem* 3.1) and a finite time stability result is obtained (*Remark* 3.1) ; in section 4 we use the previous results in order to study a mechanical problem.Finally,in section 5 we propose an open problem for generalisation of (*Theorem* 2.1).

2 An existence and uniqueness result

In this paper ,we utilise the following notations

$|.|_H$: the norm on H , $\langle ., . \rangle_H$: the inner product of H.
$C^0(0, T, H)$ the space of continuous functions on $[0, T]$ with values in H.
$C^1(0, T, H)$ the space of derivable functions with continuous derivative on $[0, T]$ with values in H .

Let us recall that $C^j(0, T, H), (j = 0, 1)$ are real Banach spaces endowed with the norm

$$|x|_{0,T,H} = \max_{t \in [0,T]} |x(t)|_H$$

$$|x|_{1,T,H} = |x|_{0,T,H} + |\dot{x}|_{0,T,H}$$

In the study of the problem (1)–(2) , we consider the following assumptions:

A is supposed a positive defined symmetric operator i.e:

$$\exists \alpha > 0 \text{ such that } \langle A(t)\, x, x\rangle_H \geq \alpha\, |x|_H^2 \,, \forall x \in H \tag{3}$$

$$\langle A(t)x, y\rangle_H = \langle y, Ax\rangle_H \,, \forall x, y \in H \tag{4}$$

$$\exists K > 0; |A(t_1) - A(t_2)| \leq K\, |t_1 - t_2| \tag{5}$$

$$A(t) \text{ is continuous function} \tag{6}$$

$$\exists \lambda > 0 \; ; \; |A(t)| \leq \lambda \tag{7}$$

We also suppose that B satisfies:$\exists L > 0$,such that :

$$\begin{aligned} |B(t, x_1, y_1) - B(t, x_2, y_2)| \leq L\, (|x_1 - x_2|_H \\ + |y_1 - y_2|_H)\, ; \forall t \in [0, T]\, ; x_1, x_2 \in X; y_1, y_2 \in Y \end{aligned} \tag{8}$$

$$t \to B(t, x, y) : [0, T] \to H \text{ is a continuous function for all } x \in X \text{ and } y \in Y \tag{9}$$

and for the initial data we consider the following assumptions:

$$x_0 \in X \; , \; y_0 \in Y \tag{10}$$

The main result of this section is the following :

Theorem 2.1 *Let* (3)*–*(10) *hold. Then the problem* (1)*–*(2) *has a unique solution* $x \in C^1(0, T, X)\,, y \in C^1(0, T, Y)\,.$

Remark 2.2 *In the case when A dose not depend on t , Theorem* 2.1 *is proved by [16],[7] using different functional methods.Here we extend the technique presented by [16], in the case* $A = A(t)$ *.*

In order to prove Theorem 2.1 , we need some preliminaries results .
We start with the following Lemma whose proof can be easily obtained .

Lemma 2.3 *Let (3)–(7) hold and let* $a : [0,T] \times X \times X \to \mathbb{R}$ *be given by :*

$$a\left(t,u,v\right) = \left\langle A\left(t\right)u,v\right\rangle_H \ ;\forall u,v \in H$$

Then $a\left(t,u,v\right)$ *is a bilinear,continuous ,symmetric and coercive form on* H .

Proof (of Lemma 2.3**).**

we can easily proved that a is a bilinear and symmetric form on H .

Using (7) , we have :

$$\left|a\left(t,u,v\right)\right| = \left|\left\langle A\left(t\right)u,v\right\rangle_H\right| \leq \left|A\left(t\right)\right|\left|u\right|\left|v\right| \leq \lambda\left|u\right|\left|v\right|;\forall u,v \in H$$

which implies a is a continuous form on H

Using (3) , we have :

$$\left|a\left(t,u,u\right)\right| = \left|\left\langle A\left(t\right)u,u\right\rangle_H\right| \geq \alpha\left|u\right|^2;\forall u \in H$$

which implies a is a coercive form on H

Hence $a\left(t,u,v\right)$ is a bilinear,continuous ,symmetric and coercive form on H . ■

Proof (of Theorem 2.1**).** Let $\eta \in C^0\left(0,T,H\right)$ and $z_\eta \in C^1\left(0,T,H\right)$.Be the function defined by :

$$z_\eta(t) = \int_0^t \eta(s)ds + z_0 \ ;\forall t \in [0,T] \tag{11}$$

$$z_0 = y_0 - A\left(0\right)x_0 \tag{12}$$

Using standard arguments for elliptic equations we obtain the existence and uniqueness of two functions $x_\eta \in C^1\left(0,T,X\right)$,$y_\eta \in C^1\left(0,T,Y\right)$ such that :

$$y_\eta\left(t\right) = A\left(t\right)x_\eta + z_\eta\left(t\right) \ ;\forall \ t \in [0,T] \tag{13}$$

Moreover,the function $x_\eta$is characterized by the equality

$$a\left(x_{\eta}\left(t\right), x'\right) + \left(z_{\eta}\left(t\right), x'\right) = 0, ; \forall t \in [0, T] ; \forall x' \in X \tag{14}$$

where a is a bilinear,continuous ,symmetric and coercive form on H defined by:

$$a\left(t, u, v\right) = \left\langle A\left(t\right) u, v\right\rangle_{H} \;; \forall u, v \in H \tag{15}$$

Let us remark that by (11)–(13) and the orthogonality of the spaces X and Y ,we get :

$$x_{\eta}\left(0\right) = x_{0}, y_{\eta}\left(0\right) = y_{0} \tag{16}$$

Moreover by (13):

$$y_{\eta_1}\left(t\right) = A\left(t\right) x_{\eta_1} + z_{\eta_1}\left(t\right)$$
$$y_{\eta_2}\left(t\right) = A\left(t\right) x_{\eta_2} + z_{\eta_2}\left(t\right)$$

By the orthogonality of the spaces X and Y ,we get :

$$\left\langle y_{\eta_1}\left(t\right) - y_{\eta_2}\left(t\right), x_{\eta_1}\left(t\right) - x_{\eta_2}\left(t\right)\right\rangle = 0$$

$$\left\langle A\left(t\right) x_{\eta_1}\left(t\right) - A\left(t\right) x_{\eta_2}\left(t\right), x_{\eta_1}\left(t\right) - x_{\eta_2}\left(t\right)\right\rangle = -\left\langle z_{\eta_1} - z_{\eta_2}, x_{\eta_1}\left(t\right) - x_{\eta_2}\left(t\right)\right\rangle$$

and by (3),(4) it results

$$\left|x_{\eta_1}\left(t\right) - x_{\eta_2}\left(t\right)\right|_{C(0,T,X)} + \left|y_{\eta_1}\left(t\right) - y_{\eta_2}\left(t\right)\right|_{C(0,T,Y)} \leq C\left|z_{\eta_1}\left(t\right) - z_{\eta_2}\left(t\right)\right|_{C(0,T,H)} \tag{17}$$

Where $C > 0$ depends only on the operator A .

Using (8),(9) , we obtain that $B\left(t, x_{\eta}\left(t\right), y_{\eta}\left(t\right)\right)$ is a continuous function on $[0, T]$ with values in H hence we can define the operator $\Lambda : C^{0}\left(0, T, H\right) \to C^{0}\left(0, T, H\right)$ in the following way :

$$\Lambda_{\eta}\left(t\right) = B\left(t, x_{\eta}\left(t\right), y_{\eta}\left(t\right)\right) - \dot{A}\left(t\right) x_{\eta}\left(t\right) \;; \forall \eta \in C^{0}\left(0, T, H\right), t \in [0, T] \tag{18}$$

Whe shall prove that Λ has a unique fixed point , Indeed , let $\eta_1, \eta_2 \in C^{0}\left(0, T, H\right)$, using (8),(17) and (11) , we have :

$$\begin{aligned}\left|\Lambda_{\eta_1}\left(t\right) - \Lambda_{\eta_2}\left(t\right)\right| \leq & L\Big(\left|x_{\eta_1}\left(t\right) - x_{\eta_2}\left(t\right)\right| + \left|y_{\eta_1}\left(t\right) - y_{\eta_2}\left(t\right)\right| \\ & + \left|\dot{A}\left(t\right)\right|\left|x_{\eta_1}\left(t\right) - x_{\eta_2}\left(t\right)\right|\Big)\end{aligned}$$

Using now (5) , it results

$$|\Lambda_{\eta_1}(t) - \Lambda_{\eta_2}(t)| \leq L\left(|x_{\eta_1}(t) - x_{\eta_2}(t)| + |y_{\eta_1}(t) - y_{\eta_2}(t)|\right) + K\,|x_{\eta_1}(t) - x_{\eta_2}(t)|$$

Using now (17),(11) , we get :

$$|\Lambda_{\eta_1}(t) - \Lambda_{\eta_2}(t)| \leq k\int_0^t |\eta_1(s) - \eta_2(s)|\, ds \tag{19}$$

Where $k > 0$ depends only on the operator A .

By recurrence , denoting by Λ^p the powers of the operator Λ , (19) implies :

$$\left|\Lambda^p_{\eta_1}(t) - \Lambda^p_{\eta_2}(t)\right| \leq k^p \underbrace{\int_0^t\int_0^s \cdots \int_0^q}_{p\ integrats} |\eta_1(r) - \eta_2(r)|\, dr \cdots ds\ ;\ \forall t \in [0,T], p \in \mathbb{N}$$

It results

$$\left|\Lambda^p_{\eta_1}(t) - \Lambda^p_{\eta_2}(t)\right|_{C(0,T,H)} \leq \frac{(kT)^p}{p!}\,|\eta_1 - \eta_2|_{C(0,T,H)}$$

and since $\lim_{p\to\infty}\frac{(kT)^p}{p!} = 0$,(19) implies that for p large enough the operator Λ^p is a contraction in $C^0(0,T,H)$.Then there exists a unique $\eta^* \in C^0(0,T,H)$ such that $\Lambda^p\eta^* = \eta^*$.Moreover η^* is the unique fixed point of Λ .

Using now (11),(13),(16) and (18) we get that $x_{\eta*} \in C^1(0,T,X)$, $y_{\eta*} \in C^1(0,T,Y)$ is solution of (1)–(2).

The uniqueness part of $Theorem$ 2.1 may be obtained from the uniqueness of the fixed point of Λ . $\blacksquare$

3 The continuous dependence with respect to the data

In this section two solutions of the problem (1)–(2) for two different data are considered .

An estimation of the difference of these solution is obtained that give the continuous dependence of the solution upon the input data . In this way the finite time stability of the solution is obtained .

We have the following result :

Theorem 3.1 *Let* (3)*–*(10) *hold and let* $x_i \in C^1(0,T,X), y_i \in C^1(0,T,Y)$ *be the solution of* (1)*–*(2) *for the data* x_{0i}, y_{0i} *satisfing* (10)*;*$(i = 1,2)$*. Then there exists* $\tilde{C} > 0$ *such that*

$$|x_1 - x_2|_{C^1(0,T,X)} + |y_1 - y_2|_{C^1(0,T,Y)} \leq \tilde{C}\left(|x_{01} - x_{02}|_X + |y_{01} - y_{02}|_Y\right) \quad (20)$$

Proof. Let $t \in [0,T]$ .As it results from the proof of Theorem 2.1 we have :

if $x_i \in C^1(0,T,X), y_i \in C^1(0,T,Y)$ be the solution of the problem (1)–(2) for the data η_i ; $(i = 1,2)$, then :

$$y_i(t) = A(t)\, x_i(t) + z_i(t)\,; i = 1,2 \quad (21)$$

Where z_i are defined by :

$$z_i(t) = \int_0^t \eta_i(s)ds + z_{0i} \;;\forall t \in [0,T]\,; i = 1,2 \quad (22)$$

$$z_{0i} = y_{0i} - A(0)\, x_{0i}; i = 1,2 \quad (23)$$

Using now (17) , we have :

$$|x_1(t) - x_2(t)|_{C(0,T,X)} + |y_1(t) - y_2(t)|_{C(0,T,Y)} \leq C\, |z_1(t) - z_2(t)|_{C(0,T,H)}$$

We deduce :

$$|x_1(t) - x_2(t)|_{C^1(0,T,X)} + |y_1(t) - y_2(t)|_{C^1(0,T,Y)} \leq kC\, |z_1(0) - z_2(0)|$$

Using now (22),(23) , it results

$$|x_1(t) - x_2(t)|_{C^1(0,T,X)} + |y_1(t) - y_2(t)|_{C^1(0,T,Y)} \leq \tilde{C}\left(|x_{01} - x_{02}|_X + |y_{01} - y_{02}|_Y\right) \quad (24)$$

■

Remark 3.1 *From* (24)*, we deduce the finite time stability of every solution of* (1)*–*(2)

4 Example arising from rate-type viscoplasticity

The aim of this section is to investigate a nonlinear quasistatic problem for viscoplastic materials , using the abstract result given in section 2 . In this case the unknowns x and y of the evolution problem (1)–(2) are the small deformation tensor and the stress tensor and (1) involves the constitutive law of the material .

In the case when t is interpreted as the absolute temperature the following problem represents uncoupled thermo-viscoplastic process.

Let us consider a viscoplastic body whose material particles fulfil a bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 1, 2, 3)$ with a smooth boundary $\partial\Omega = \Gamma$ and Γ_1 is an open subset of Γ such that $meas\ \Gamma_1 > 0$. We denote by $\Gamma_2 = \Gamma - \overline{\Gamma_1}$, ν the outward unit normal vector on Γ and by S_N the set of second order symmetric tensor on $\mathbb{R}^N$. Let T be a real positive constant

We assume that the body forces f act in $\Omega \times [0, T]$, that the displacement g act on $\Gamma_1 \times [0, T]$ and that surface traction h act on $\Gamma_2 \times [0, T]$.

With these assumptions we have :

$$\dot{\sigma} = \xi(t)\,\varepsilon(\dot{u}) + G(\sigma, \varepsilon(u)) \qquad \text{in} \quad \Omega \times [0, T] \tag{25}$$

$$Div\sigma + f = 0 \qquad \text{in} \quad \Omega \times [0, T] \tag{26}$$

$$u = g \qquad \text{on} \quad \Gamma_1 \times [0, T] \tag{27}$$

$$\sigma\nu = h \qquad \text{on} \quad \Gamma_2 \times [0, T] \tag{28}$$

$$u(0) = u_0, \sigma(0) = \sigma_0 \qquad \text{in} \quad \Omega \tag{29}$$

in which the unknowns are the displacement function $u : \Omega \times [0, T] \to \mathbb{R}^N$, the stress function $\sigma : \Omega \times [0, T] \to S_N$

Here $\varepsilon(u)$ denotes the small strain tensor . In (25) , ξ and G are given constitutive functions and u_0, σ_0 are the initial data .

The problem (25)–(29) models a quasistatic problem for rate type viscoplasticity

In order to study the problem (25)–(29) , we firstly present some preliminaries results.

We utilise the following notations :

$$H = \left[L^2(\Omega)\right]^N \ , \ \mathcal{H} = \left[L^2(\Omega)\right]_S^{N\times N}$$

$$H_1 = \{u \in H \ / \ \varepsilon(u) \in \mathcal{H}\}$$

$$\mathcal{H}_1 = \{\sigma \in \mathcal{H} \ / \ Div\sigma \in H\}$$

The spaces $H, \mathcal{H}, H_1$and $\mathcal{H}_1$are Hilbert spaces endowed with the following inner products given by :

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx; \ \ \forall u, v \in H$$

$$\langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \ \ ; \ \forall \sigma, \tau \in \mathcal{H}$$

$$\langle u, v\rangle_{H_1} = \langle u, v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \forall u, v \in H_1$$

$$\langle \sigma, \tau\rangle_{\mathcal{H}_1} = \langle \sigma, \tau\rangle_{\mathcal{H}} + \langle Div\sigma, Div\tau\rangle_H, \ \ \forall \sigma, \tau \in \mathcal{H}_1$$

Let $H_\Gamma = \left[H^1(\Gamma)\right]^N$ and $\gamma : H_1 \to H_\Gamma$ be there trace map . we donote by :

$$V = \{u \in H_1/\gamma\nu = 0 \ on \ \Gamma_1\}$$

$$\vartheta = \{\tau \in \mathcal{H}_1/Div(\sigma) = 0, \sigma\nu = 0 \ on \ \Gamma_2\}$$

The operator $\varepsilon : H_1 \to \mathcal{H}$ defined by :

$$\varepsilon(u) = \frac{1}{2}\left(\nabla u + \nabla^T u\right) = (\varepsilon_{ij}(u)) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

is linear and continuous operator .

$\varepsilon(V)$ is the orthogonal complement of ϑ in $\mathcal{H}$, hence

$$\langle \sigma, \varepsilon(v)\rangle_{\mathcal{H}} = 0, \forall v \in V, \forall \sigma \in \vartheta$$

In the study of problem (25)–(29) , we consider the following assumptions :

ξ satisfies :

$$\exists \alpha > 0 \text{ such that } \langle \xi(t)x, x\rangle_H \geq \alpha |x|_H^2, \forall x \in H \tag{30}$$

$$\langle \xi(t)x, y\rangle_H = \langle y, \xi x\rangle_H, \forall x, y \in H \tag{31}$$

$$\exists K > 0; |\xi(t_1) - \xi(t_2)| \leq K|t_1 - t_2| \tag{32}$$

$$\xi(t) \text{ is continuous function} \tag{33}$$

$$\exists \lambda > 0 ; |\xi(t)| \leq \lambda \tag{34}$$

We also suppose that G satisfies :

$$\exists\ L > 0 \text{ such that} : |G(x, \sigma_1, \varepsilon_1) - G(x, \sigma_2, \varepsilon_2)| \leq L(|\sigma_1 - \sigma_2| + |\varepsilon_1 - \varepsilon_2|) \tag{35}$$

$$x \to G(x, \sigma, \varepsilon) \text{ is continuous function for all } \sigma, \varepsilon \in S_N \tag{36}$$

$$f \in C^1(0, T, H), g \in C^1(0, T, H_\Gamma), h \in C^1\left(0, T, H_\Gamma'\right) \tag{37}$$

$$u_0 \in H_1, \sigma_0 \in \mathcal{H}_1 \tag{38}$$

The main result of this section is the following :

Theorem 4.1 *Let (*30*)–(*38*) hold, Then there exists a unique solution* $u \in C^1(0, T, H_1)$, $\sigma \in C^1(0, T, \mathcal{H}_1)$ *of the problem (*25*)–(*29*)*

Proof. In order to prove *Theorem* 4.1 we need some preliminaries .

Let $\tilde{u} \in C^1(0, T, H_1), \tilde{\sigma} \in C^1(0, T, \mathcal{H}_1)$ be two functions such that :

$$\tilde{\sigma} = \xi(t)\varepsilon(\tilde{u}) \tag{39}$$

$$\tilde{u} = g \tag{40}$$

$$\tilde{\sigma}\nu = h \tag{41}$$

(The existence of this couple follows from the properties of the trace maps)

Considering the functions defined by :

$$\bar{u} = u - \tilde{u}, \bar{\sigma} = \sigma - \tilde{\sigma} \tag{42}$$

$$\dot{\bar{u}} = u_0 - \tilde{u}(0), \bar{\sigma}_0 = \sigma_0 - \tilde{\sigma}(0) \tag{43}$$

It is easy to see that the pair $(u, \sigma) \in C^1(0, T, H_1 \times \mathcal{H}_1)$ is a solution of (25)–(29) if and only if $(\bar{u}, \bar{\sigma}) \in C^1(0, T, V \times \vartheta)$ is a solution of the problem :

$$\dot{\bar{\sigma}}(t) = \xi(t)\,\varepsilon(\dot{\bar{u}}(t)) + G(\bar{\sigma}(t) + \tilde{\sigma}(t), \varepsilon(\bar{u}(t)) + \varepsilon(\tilde{u}(t))) - \dot{\tilde{\sigma}}(t) \tag{44}$$

$$\bar{u}(0) = \bar{u}_0, \bar{\sigma}(0) = \bar{\sigma}_0 \tag{45}$$

So , with the previous notations , it results that the problem (25)–(29) may be written under the form (1)–(2) with the following notations :

$$x = \varepsilon(\bar{u}), y = \bar{\sigma}, x_0 = \varepsilon(\bar{u}_0), y_0 = \bar{\sigma}_0$$

$$A = \xi, G = B - \dot{A}$$

So, the result of *Theorem* 2.1 is applicable here . Hence , under assumptions on the functions ξ, G, f, g, h, u_0 and σ_0 we obtain the existence and uniqueness of the solution for the problem (44)–(45) having the regularity $\bar{u} \in C^1(0, T, V), \bar{\sigma} \in C^1(0, T, \vartheta)$. Moreover from *Theorem* 2.1 it results that (u, σ) is also the unique solution of the problem (25)–(29)having the regularity $u \in C^1(0, T, H_1), \sigma \in C^1(0, T, \mathcal{H}_1)$ ■

5 Open Problem

`Question 1` : Can we generalise this result of existence and uniqueness in the case when :

$$\dot{y}(t) = A(\lambda(t))\,\dot{x}(t) + B(\lambda(t), x(t), y(t)) \text{ for all } t \in [0, T]$$

$$x(0) = x_0, y(0) = y_0$$

in which the unknowns are the functions $x : [0, T] \to X$, $y : [0, T] \to Y$ and $\lambda : [0, T] \to E$?

`Question 2` : Under which hypothesis on λ the generalisation is true ?

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