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Integral Transforms and Dual Integral Equations to Solve Heat Equation with Mixed Conditions

Naser Hoshan

Department of Mathematics, Tafila Technical University,

P.O. Box 179, Fax 2250033, Tafila –Jordan

e-mail: dr_naserh@yahoo.com

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Abstract

The paper is devoted to determine the solution of the non-stationary heat equation in a non axial symmetric cylindrical coordinates subject to mixed discontinuous boundary conditions of the second kind and third kinds, with the aid of a finite Fourier transform and dual integral equations method. The solution of the given mixed problem is introduced to a Fredholm integral equation of the second kind.

Keywords: *Integral transforms, dual integral equations, mixed boundary Conditions, heat equation.*

1 Introduction

Integral transform method is widely used to solve several problems in heat transfer theory with different coordinate systems for unmixed boundary conditions [1,8]. In monographs [3-6] Hankel and Laplace transforms were effectively used to investigate exact solutions for Helmholtz and heat equation subject to mixed boundary conditions of the first,the second and of the third kinds for cylinder. In this paper we propose the solution of three-dimensional non-stationary heat equation in a non axially symmetrical cylindrical coordinates with discontinuous mixed boundary conditions of the second and third kind on the level surface of a semi-infinite solid cylinder. Exact solution of the given mixed boundary value problem is obtained with the use of finite Fourier , Hankel integral transforms separation of variables and based on the application of dual integral equations method.

In this paper we apply a finite Fourier integral transform and then Hankel integral transforms with respect to coordinate variables ϕ and r , moreover, an initial mixed boundary value problem is transformed to a Helmholtz boundary problem in cylindrical coordinates , next ,application of mixed boundary conditions yields new form of a dual integral equation with Bessel function of the first kind of order n as a kernel, weight and unknown functions depend on parameters. The solution of the obtained dual integral equations is introduced to a Fredholm integral equation of the second kind with kernel and free term given in form of improper integrals.

2 Formulation of the Problem

The main goal in this paper is to solve the non-stationary heat equation for semi-space in cylindrical coordinates with a non-axially symmetry

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}+\frac{1}{r^2}\frac{\partial^2 T}{\partial \phi^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{a}\frac{\partial T}{\partial \tau} \tag{2.1}$$

where $T=T(r,z,\phi,\tau)$ is the temperature distribution function, $0<r<\infty$, $0<z<\infty$, $0\le\phi\le 2\pi$, $\tau>0$ are the corresponding cylindrical coordinates $a\ne 0$ is the temperature diffusivity coefficient (constant).

The boundary conditions

$$T(0,z,\phi,\tau)=T(\infty,z,\phi,\tau)=T(r,\infty,\phi,\tau)=0, \tag{2.2}$$

$$T(0)=T(2\pi).$$

Subject to mixed discontinuous boundary conditions of the second kind and of the third kind

(2.3) $\mu\partial T(r,0,\phi,\tau)/\partial z=-f_1(r,\phi,\tau)\ , r\in S$

(2,4) $\alpha\,\partial T(r,0,\phi,\tau)/\partial z-\beta T(r,0,\phi,\tau)=-f_2(r,\phi,\tau),\quad r\in\overline{S}$

where $S=(0,R)$, $\overline{S}=(R,\infty)$, on a surface $z=0$ μ,α,β constants.

The initial condition is

(2,5) $T(r,z,\phi,0)=\theta(r,z,\phi,0)-U_0$

Where U_0 is the initial temperature (constant), the known functions $f_i, i = 1, 2$ in (2.3),(2.4) continuous and have the limited variation with respect of each variables r and τ ,with respect to r , and periodic with period 2π .

The physical significance of the given mixed boundary value problem formulated such that, find the temperature distribution function T for a semi-infinite cylinder if the inside disk S a heat flux is given according to Fourier low, whereas on the outside disk $\overline{S}$ a heat exchange obey Newton's low of cooling , on the line of discontinuity $r = R$ no boundary conditions were given. As $\sqrt{r^2 + z^2} \to \infty$ the temperature is vanished; $T \to 0$.

3 Solution of the Problem

Use a finite Fourier transform for a function T with respect to ϕ, to the initial boundary value problem (2.1)-(2.5), we have

$$\overline{T} = \int_0^{2\pi} T F_j \, d\phi, \quad K(0) = K(2\pi),$$

Where

$$F_j = \begin{cases} \cos m\phi, j = 2m \\ \sin m\phi, j = 2m - 1 \end{cases}$$

The inverse Fourier integral transform is given by the formula

$$T = \frac{1}{\pi}\left[\frac{\overline{T_0}}{2} + \sum_{m=1}^{\infty}(\overline{T}_{2m} \cos m\phi + \overline{T}_{2m-1} \sin m\phi)\right]$$

Where

$$\overline{T_0} = \overline{T}(0) = \int_0^{2\pi} T d\phi, \quad \overline{T}_{2m} = \overline{T}(2m) = \int_0^{2\pi} T_{2m} F_j \, d\phi$$

$$\overline{T}_{2m-1} = \overline{T}(2m-1) = \int_0^{2\pi} T_{2m-1} F_j \, d\phi .$$

Equation (2.1) in a Fourier transform range is

(3.1) $$\frac{\partial^2 \overline{T}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{T}}{\partial r} - \frac{m^2}{r^2}\overline{T} + \frac{\partial^2 \overline{T}}{\partial z^2} = \frac{1}{a}\frac{\partial \overline{T}}{\partial \tau}$$

To simplify the problem mentioned above ,use well known transformation [1]

(3.2) $$\overline{T}(r, z, m, t) = \exp(-\lambda\tau)\, u(r, z, m)$$

for (2.1)-(2.5)after application Fourier transform, λ is constant, we obtain a Helmholtz equation in cylindrical coordinates

(3.3) $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{m^2}{r^2}u + \frac{\partial^2 u}{\partial z^2} = -\frac{\lambda}{a}u$$

Separating variables in (3.3), the general solution of the Helmholtz boundary value problem is obtained in form of improper integral

(3.4) $$u(r,z,m)=\int_0^\infty A(p,m)\exp(-h(p,m))\,J_m(pr)dp$$

where $h(p,m)=\sqrt{p^2+m+\lambda/a}$, $J_m(pr)$ is the Bessel function of the first kind of order m , p is the parameter of separation of variables, $A(p,m)$ unknown function.

Applying a mixed boundary conditions (2.3) and (2.4) to (3.4), we obtain the dual integral equations to determine the unknown function $A(p,m)$

(3.5) $$\int_0^\infty A(p,m)J_m(pr)\;h(p,m)dp=\overline{f_1}(r,m),\;\;r\in S$$

(3.6) $$\int_0^\infty A(p,m)J_m(pr)(\alpha\,h(p,m)+\beta)dp=\overline{f_2}(r,m),\;\;r\in\overline{S}$$

$$\overline{f_i}(r,m)=e^{-\lambda\tau}\int_0^{2\pi}\mathrm{f}_i(r,\phi)F_j d\phi$$

To solve the dual equations (3.5),(3.6), rewrite the equations in the standard form

(3.7) $$\int_0^\infty B(p,m)J_m(pr)\,g(p,m)dp=\overline{f_1}(r,m),\quad r\in S$$

(3.8) $$\int_0^\infty B(p,m)J_m(pr)\;dp=\overline{f_2}(r,m),\quad r\in\overline{S}$$

where $w(p,m)=\dfrac{h(p,m)}{\alpha h(p,m)+\beta}$, $\quad \lim\limits_{p\to\infty}w(p,m)=1/\alpha,\;\;\alpha>0$

$B(p,m)=(\alpha h(p,m)+\lambda)A(p,m)$, $g(p,m)=w(p,m)-1/\alpha=\dfrac{-\beta}{\alpha h(p,m)+\beta}$.

Rewrite (3.8) in form

(3.9) $$\int_0^\infty B(p,m)J_m(pr)dp=\begin{cases}\overline{\psi}(r,m), & r\in S\\ \overline{f_2}(r,m), & r\in\overline{S}.\end{cases}$$

where $\overline{\psi}(r,m)$ is unknown continuous function defined outside the disk in $\overline{S}$.

Applying to (3.9) the inverse Hankel integral transform [7] in the interval $S\cup\overline{S}$ we have

(3.10) $$B(p,m)=\int_0^R ypJ_m(py)\overline{\psi}(y,m)\,dy+\int_R^\infty ypJ_m(py)\overline{f_2}(y,m)\,dy$$

Substituting (3.10) into (3.7), then interchanging the order of integration, we get a Fredholm integral equation of the second kind for determination the unknown function $\overline{\psi}(r,m)$

(3.11) $$\overline{\psi}(r,m)+\alpha\int_0^R\overline{\psi}(y,m)\,K(r,y,m)\,dy=\alpha\overline{F}(r,m)\qquad,r\in S$$

with kernel

$$(3.12)\qquad K(r,y,m)=\int_0^\infty pyJ_m(py)J_m(pr)g(p,m)\,dp$$

and free term

$$(3.13)\qquad \overline{F}(r,m)=\overline{f_2}(r,m)+\int_R^\infty\int_0^\infty p\,y\,\overline{f_2}(y,m)\,J_m(pr)J_m(py)\,g(p,m)\,dp\,dy$$

Integral equation (3.11) should be solved with the use of numerical methods for some choices of $f_1, f_2, R, m, m=0,1,2,....$ by using some software packages such *mathematica* or *matlab*. The kernel given in (3.12) continuous and quadratic integrable in the square $\{\Omega : 0<r,y<R\}$, for some certain numerical values of R furthermore, the free term (3.13) is integrable and bounded in the interval $0<y<R$ [2].Finally the general solution $\overline{T}(r,z,m,\tau)$ in the Fourier transform domain is given by the expression

$$\overline{T}(r,z,m,\tau)=e^{-\lambda\tau}\int_0^\infty\frac{1}{\alpha h(p,m)+\beta}$$

$$(3.14)\qquad \left(\int_0^R ypJ_m(py)\psi(y,m)\,dy+\int_R^\infty ypJ_m(py)f_2(y,m)\,dy\right)h(p,m)J_m(pr)dp$$

Put the value of the general solution (3.14) into the inversion formula of the inverse Fourier transform, the general solution of the initial mixed boundary value problem (2.1)-(2.5)

$$T(r,z,\phi,\tau)=$$

$$(3.15)\qquad \frac{1}{\pi}\left[\frac{\overline{T}(r,z,\tau,0)}{2}+\sum_{m=1}^{\infty}(\overline{T}(r,z,\tau,2m)\cos m\phi+\overline{T}(r,z,\tau,2m-1)\sin m\phi)\right]$$

If $m=0$, the solution (3.15) reduced to the solution of an axial symmetry heat equation with mixed conditions[6],moreover, if $h(p,m)=p$, the above solution is reduced to the solution of the Laplace's' equation with mixed conditions in different engineering and physical applications[3-7].

Theory mentioned above of the applications of a integral transforms involving exact solution of the mixed initial boundary value problem can be used widely to solve various mixed boundary problems for a non-stationary heat equation in an infinite or finite cylinder, unsymmetrical cylindrical coordinates for unbounded plate, spherical coordinates and other mixed problems.

4 Conclusion

Finally the above technique involving application of integral transforms for solving mixed boundary value problems can be used for investigating several homogeneous problems (heat equation, Helmholtz equation and Laplace equation) in different coordinate system and many areas of applications in technical and physical sciences under mixed boundary conditions of the first, the second and of the third kinds.

5 Open Problem

We consider an initial mixed boundary value problem (2.1) (2.2) for an infinite plate of high h, in cylindrical coordinates with non axially symmetry, subject to discontinuous inhomogeneous mixed boundary conditions of the second and of the third kind on a surface $z = 0$.

(3.16) $\mu\, \partial T\,(r,0,\phi,\tau)/\partial z = -f_1(r,\phi,\tau)\ , r \in S$,

(3.17) $\alpha\, \partial T\,(r,0,\phi,\tau)/\partial z - \beta T\,(r,0,\phi,\tau) = -f_2(r,\phi,\tau),\ \ r \in \overline{S}$

On a level surface $z = h$, located a third kind linear inhomogeneous boundary conditions

(3.18) $\lambda \partial T(r,h,\phi,\tau)/\partial z + \gamma T(r,h,\phi,\tau) = f_3(r,\phi,\tau)$

Where $f_i\,, i = 1,2,3$ known functions, $S = (0,R)$, $\overline{S} = (R,\infty)$, $\mu,\alpha,\beta\ \ ,\lambda,\gamma$ constants.

The above mixed problem (3.16)-(3.18), introduced to some type of dual integral equations, however no one in the world solve this problem, since the boundary condition (3.18) complicates solution of the given problem furthermore, known methods concerning dual integral equations may be difficult to use.

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