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Certain differential sandwich theorem using an extended generalized Sălăgean operator and extended Ruscheweyh operator

Andrei Loriana

Department of Mathematics and Computer Science, University of Oradea,
str. Universitatii nr. 1, 410087 Oradea, Romania
Department of Mathematics, University of Pitesti,
str. Targul din Vale nr. 1, 110040 Pitesti, Romania
e-mail:lori_andrei@yahoo.com

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Abstract

The purpose of this paper is to introduce sufficient conditions for strong differential subordination and strong differential superordination involving the extended operator $DR_\lambda^{m,n}$ ***and also to obtain a sandwich-type result.***

Keywords: *analytic functions, extended differential operator, strong differential subordination, strong differential superordination.*
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1 Introduction

Denote by U the unit disc of the complex plane $U = \{z \in \mathbb{C} : \ |z| < 1\}$, $\overline{U} = \{z \in \mathbb{C} : \ |z| \leq 1\}$ the closed unit disc of the complex plane and $\mathcal{H}(U \times \overline{U})$ the class of analytic functions in $U \times \overline{U}$.

Let

$$\mathcal{A}^*_{n\zeta} = \{f \in \mathcal{H}(U \times \overline{U}), \ f(z,\zeta) = z + a_{n+1}(\zeta)\, z^{n+1} + \ldots, \ z \in U, \ \zeta \in \overline{U}\},$$

with $\mathcal{A}^*_{1\zeta} = \mathcal{A}^*_{\zeta}$, where $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq 2$, and

$\mathcal{H}^*[a,n,\zeta] = \{f \in \mathcal{H}(U\times\overline{U}),\ f(z,\zeta) = a + a_n(\zeta)\, z^n + a_{n+1}(\zeta)\, z^{n+1} + \ldots,\ z \in U,$ $\zeta \in \overline{U}\}$, for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, $a_k(\zeta)$ are holomorphic functions in $\overline{U}$ for $k \geq n$.

Generalizing the notion of differential subordinations, J.A. Antonino and S. Romaguera have introduced in [16] the notion of strong differential subordinations, which was developed by G.I. Oros and Gh. Oros in [17].

Definition 1.1. *[17] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly subordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z,\zeta) = H(w(z),\zeta)$ for all $\zeta \in \overline{U}$. In such a case we write $f(z,\zeta) \prec\prec H(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.2. *[17] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.1 is equivalent to $f(0,\zeta) = H(0,\zeta)$, for all $\zeta \in \overline{U}$, and $f\left(U \times \overline{U}\right) \subset H\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong subordination becomes the usual notion of subordination.

As a dual notion of strong differential subordination G.I. Oros has introduced and developed the notion of strong differential superordinations in [18].

Definition 1.3. *[18] Let $f(z,\zeta)$, $H(z,\zeta)$ analytic in $U \times \overline{U}$. The function $f(z,\zeta)$ is said to be strongly superordinate to $H(z,\zeta)$ if there exists a function w analytic in U, with $w(0) = 0$ and $|w(z)| < 1$, such that $H(z,\zeta) = f(w(z),\zeta)$, for all $\zeta \in \overline{U}$. In such a case we write $H(z,\zeta) \prec\prec f(z,\zeta)$, $z \in U$, $\zeta \in \overline{U}$.*

Remark 1.4. *[18] (i) Since $f(z,\zeta)$ is analytic in $U \times \overline{U}$, for all $\zeta \in \overline{U}$, and univalent in U, for all $\zeta \in \overline{U}$, Definition 1.3 is equivalent to $H(0,\zeta) = f(0,\zeta)$, for all $\zeta \in \overline{U}$, and $H\left(U \times \overline{U}\right) \subset f\left(U \times \overline{U}\right)$.*

(ii) If $H(z,\zeta) \equiv H(z)$ and $f(z,\zeta) \equiv f(z)$, the strong superordination becomes the usual notion of superordination.

Definition 1.5. *We denote by Q^* the set of functions that are analytic and injective on $\overline{U} \times \overline{U} \backslash E(f,\zeta)$, where $E(f,\zeta) = \{y \in \partial U : \lim\limits_{z\to y} f(z,\zeta) = \infty\}$, and are such that $f'_z(y,\zeta) \neq 0$ for $y \in \partial U \times \overline{U} \backslash E(f,\zeta)$. The subclass of Q^* for which $f(0,\zeta) = a$ is denoted by $Q^*(a)$.*

For two functions $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta)\, z^j$ and $g(z,\zeta) = z + \sum_{j=2}^{\infty} b_j(\zeta)\, z^j$ analytic in $U \times \overline{U}$, the Hadamard product (or convolution) of $f(z,\zeta)$ and $g(z,\zeta)$, written as $(f * g)(z,\zeta)$ is defined by

$$f(z,\zeta) * g(z,\zeta) = (f * g)(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta)\, b_j(\zeta)\, z^j.$$

Definition 1.6. *([1]) For* $f \in \mathcal{A}_{\zeta}^{*}$, $\lambda \geq 0$ *and* $m \in \mathbb{N}$, *the extended generalized Sălăgean operator* D_{λ}^{m} *is defined by* $D_{\lambda}^{m} : \mathcal{A}_{\zeta}^{*} \to \mathcal{A}_{\zeta}^{*}$,

$$\begin{aligned} D_{\lambda}^{0} f(z,\zeta) &= f(z,\zeta) \\ D_{\lambda}^{1} f(z,\zeta) &= (1-\lambda) f(z,\zeta) + \lambda z f_{z}'(z,\zeta) = D_{\lambda} f(z,\zeta) \\ &\dots \\ D_{\lambda}^{m+1} f(z,\zeta) &= (1-\lambda) D_{\lambda}^{m} f(z,\zeta) + \lambda z \left(D_{\lambda}^{m} f(z,\zeta)\right)_{z}' = D_{\lambda}\left(D_{\lambda}^{m} f(z,\zeta)\right), \end{aligned}$$

for $z \in U$, $\zeta \in \overline{U}$.

Remark 1.7. *If* $f \in \mathcal{A}_{\zeta}^{*}$ *and* $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then* $D_{\lambda}^{m} f(z,\zeta) = z + \sum_{j=2}^{\infty} [1+(j-1)\lambda]^{m} a_j(\zeta) z^j$, *for* $z \in U$, $\zeta \in \overline{U}$.

Definition 1.8. *([2]) For* $f \in \mathcal{A}_{\zeta}^{*}$, $m \in \mathbb{N}$, *the extended Ruscheweyh derivative* R^m *is defined by* $R^m : \mathcal{A}_{\zeta}^{*} \to \mathcal{A}_{\zeta}^{*}$,

$$\begin{aligned} R^{0} f(z,\zeta) &= f(z,\zeta) \\ R^{1} f(z,\zeta) &= z f_{z}'(z,\zeta) \\ &\dots \\ (m+1) R^{m+1} f(z,\zeta) &= z\left(R^{m} f(z,\zeta)\right)_{z}' + m R^{m} f(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}. \end{aligned}$$

Remark 1.9. *If* $f \in \mathcal{A}_{\zeta}^{*}$, $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then* $R^m f(z,\zeta) = z + \sum_{j=2}^{\infty} \frac{(m+j-1)!}{m!(j-1)!} a_j(\zeta) z^j$, $z \in U$, $\zeta \in \overline{U}$.

Extending the results from [10] to the class $\mathcal{A}_{\zeta}^{*}$ we obtain:

Definition 1.10. *([11]) Let* $\lambda \geq 0$ *and* $n, m \in \mathbb{N}$. *Denote by* $DR_{\lambda}^{m,n} : \mathcal{A}_{\zeta}^{*} \to \mathcal{A}_{\zeta}^{*}$ *the operator given by the Hadamard product of the extended generalized Sălăgean operator* D_{λ}^{m} *and the extended Ruscheweyh operator* R^n,

$$DR_{\lambda}^{m,n} f(z,\zeta) = \left(D_{\lambda}^{m} * R^{n}\right) f(z,\zeta),$$

for any $z \in U$, $\zeta \in \overline{U}$, *and each nonnegative integers* m, n.

Remark 1.11. *If* $f \in \mathcal{A}_{\zeta}^{*}$ *and* $f(z,\zeta) = z + \sum_{j=2}^{\infty} a_j(\zeta) z^j$, *then*
$DR_{\lambda}^{m,n} f(z,\zeta) = z + \sum_{j=2}^{\infty} [1+(j-1)\lambda]^{m} \frac{(n+j-1)!}{n!(j-1)!} a_j^2(\zeta) z^j$, *for* $z \in U$, $\zeta \in \overline{U}$.

Remark 1.12. *For* $m = n$ *we obtain the operator* DR_{λ}^{m} *studied in [12], [13], [14], [15], [3], [4], [5].*

For $\lambda = 1$, $m = n$, *we obtain the Hadamard product* SR^n *[6] of the Sălăgean operator* S^n *and Ruscheweyh derivative* R^n, *which was studied in [7], [8], [9].*

Using simple computation one obtains the next result.

Proposition 1.13. *For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have For $m, n \in \mathbb{N}$ and $\lambda \geq 0$ we have*

$$DR_\lambda^{m+1,n} f(z,\zeta) = (1-\lambda) DR_\lambda^{m,n} f(z,\zeta) + \lambda z \left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z \tag{1}$$

and

$$z \left(DR_\lambda^{m,n} f(z,\zeta)\right)'_z = (n+1) DR_\lambda^{m,n+1} f(z,\zeta) - n DR_\lambda^{m,n} f(z,\zeta). \tag{2}$$

The main object of the present paper is to find sufficient condition for certain normalized analytic functions to satisfy

$$q_1(z,\zeta) \prec\prec \frac{z DR_\lambda^{m+1,n} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^2} \prec\prec q_2(z,\zeta),$$

where q_1 and q_2 are given convex and univalent functions in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\xi \in \overline{U}$.

In order to prove our strong differential subordination and strong differential superordination results, we make use of the following known results.

Lemma 1.14. *Let the function q be univalent in $U \times \overline{U}$ and θ and ϕ be analytic in a domain D containing $q\left(U \times \overline{U}\right)$ with $\phi(w) \neq 0$ when $w \in q\left(U \times \overline{U}\right)$. Set $Q(z,\zeta) = zq'_z(z,\zeta)\phi(q(z,\zeta))$ and $h(z,\zeta) = \theta(q(z,\zeta)) + Q(z,\zeta)$. Suppose that*

1. Q is starlike univalent in $U \times \overline{U}$ and

2. $Re\left(\frac{zh'_z(z,\zeta)}{Q(z,\zeta)}\right) > 0$ for $z \in U$, $\zeta \in \overline{U}$.

If p is analytic with $p(0,\zeta) = q(0,\zeta)$, $p\left(U \times \overline{U}\right) \subseteq D$ and

$$\theta(p(z,\zeta)) + zp'_z(z,\zeta)\phi(p(z,\zeta)) \prec\prec \theta(q(z,\zeta)) + zq'_z(z,\zeta)\phi(q(z,\zeta)),$$

then $p(z,\zeta) \prec\prec q(z,\zeta)$ and q is the best dominant.

Lemma 1.15. *Let the function q be convex univalent in $U \times \overline{U}$ and ν and ϕ be analytic in a domain D containing $q\left(U \times \overline{U}\right)$. Suppose that*

1. $Re\left(\frac{\nu'_z(q(z,\zeta))}{\phi(q(z,\zeta))}\right) > 0$ for $z \in U$, $\zeta \in \overline{U}$ and

2. $\psi(z,\zeta) = zq'_z(z,\zeta)\phi(q(z,\zeta))$ is starlike univalent in $U \times \overline{U}$.

If $p(z,\zeta) \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$, with $p\left(U \times \overline{U}\right) \subseteq D$ and $\nu(p(z,\zeta)) + zp'_z(z)\phi(p(z,\zeta))$ is univalent in $U \times \overline{U}$ and

$$\nu(q(z,\zeta)) + zq'_z(z,\zeta)\phi(q(z,\zeta)) \prec\prec \nu(p(z,\zeta)) + zp'_z(z,\zeta)\phi(p(z,\zeta)),$$

then $q(z,\zeta) \prec\prec p(z,\zeta)$ and q is the best subordinant.

2 Main results

We begin with the following

Theorem 2.1. *Let* $\frac{zDR_{\lambda}^{m+1,n}f(z,\zeta)}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)^2} \in \mathcal{H}\left(U \times \overline{U}\right),\ z \in U,\ \xi \in \overline{U},\ f \in \mathcal{A}_{\xi}^{*}$, $m,n \in \mathbb{N},\ \lambda \geq 0$ *and let the function* $q(z,\xi)$ *be convex and univalent in* $U \times \overline{U}$ *such that* $q(0,\xi) = 1$. *Assume that*

$$Re\left(\frac{\alpha+\beta}{\beta} + \frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right) > 0, \quad z \in U,\ \zeta \in \overline{U}, \tag{3}$$

for $\alpha, \beta, \in \mathbb{C},\ \beta \neq 0$ *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta) := \left(\alpha + \beta\frac{2-\lambda(n+1)}{\lambda}\right)\frac{zDR_{\lambda}^{m+1,n}f(z,\zeta)}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)^2} +$$

$$\lambda\beta(n+1)(n+2)\frac{zDR_{\lambda}^{m,n+2}f(z,\zeta)}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)^2} \tag{4}$$

$$+\beta(n+1)\left[1-\lambda(n+2)\right]\frac{zDR_{\lambda}^{m,n+1}f(z,\zeta)}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)^2} - \frac{2\beta}{\lambda}\frac{z\left(DR_{\lambda}^{m+1,n}f(z,\zeta)\right)^2}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)^3}.$$

If q *satisfies the following strong differential subordination*

$$\psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta) \prec\prec \alpha q(z,\zeta) + \beta z q'_z(z,\zeta), \tag{5}$$

for $\alpha, \beta \in \mathbb{C}, \beta \neq 0$ *then*

$$\frac{zDR_{\lambda}^{m+1,n}f(z,\zeta)}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)^2} \prec\prec q(z,\zeta), \quad z \in U,\ \zeta \in \overline{U}, \tag{6}$$

and q *is the best dominant.*

Proof Let the function p be defined by $p(z,\zeta) = \frac{zDR_{\lambda}^{m+1,n}f(z,\zeta)}{\left(DR_{\lambda}^{m,n}f(z,\zeta)\right)^2}$, $z \in U$, $\zeta \in \overline{U}$, $z \neq 0$, $f \in \mathcal{A}_{\zeta}^{*}$. The function p is analytic in $U \times \overline{U}$ and $p(0,\zeta) = 1$.

Differentiating this function, with respect to z,we get

$$zp'_z(z,\zeta) = \frac{1}{DR_{\lambda}^{m,n}f(z,\zeta)}\frac{1}{\lambda}\cdot$$

$$\left[\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)} + \frac{DR_{\lambda}^{m+2,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)} - 2\left(\frac{DR_{\lambda}^{m+1,n}f(z,\zeta)}{DR_{\lambda}^{m,n}f(z,\zeta)}\right)^2\right]$$

By using the identity (1) and (2), we obtain

$$zp'_z(z,\zeta)=\frac{z}{DR_\lambda^{m,n}f(z,\zeta)}\cdot$$
$$\left[\frac{2-\lambda(n+1)}{\lambda}\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}+\lambda(n+1)(n+2)\frac{DR_\lambda^{m,n+2}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right.$$
$$\left.+(n+1)\left[1-\lambda(n+2)\right]\frac{DR_\lambda^{m,n+1}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}-\frac{2}{\lambda}\left(\frac{DR_\lambda^{m+1,n}f(z,\zeta)}{DR_\lambda^{m,n}f(z,\zeta)}\right)^2\right]=$$
$$\frac{2-\lambda(n+1)}{\lambda}\frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}+\lambda(n+1)(n+2)\frac{zDR_\lambda^{m,n+2}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}+$$
$$(n+1)\left[1-\lambda(n+2)\right]\frac{zDR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}-\frac{2}{\lambda}\frac{z\left(DR_\lambda^{m+1,n}f(z,\zeta)\right)^2}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^3}. \tag{7}$$

By setting

$$\theta(w):=\alpha w \quad\text{and}\quad \phi(w):=\beta, \qquad \alpha,\beta\in\mathbb{C},\ \beta\neq 0,$$

it can be easily verified that θ is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi(w)\neq 0$, $w\in\mathbb{C}\backslash\{0\}$.

Also, by letting $Q(z,\zeta)=zq'_z(z,\zeta)\phi(q(z,\zeta))=\beta zq'_z(z,\zeta)$ we find that $Q(z,\zeta)$ is starlike univalent in $U\times\overline{U}$.

Let $h(z,\zeta)=\theta(q(z,\zeta))+Q(z,\zeta)=\alpha q(z,\zeta)+\beta zq'_z(z,\zeta)$.

If we derive the function Q, with respect to z, perform calculations, we have $Re\left(\frac{zh'_z(z,\zeta)}{Q(z,\zeta)}\right)=Re\left(\frac{\alpha+\beta}{\beta}+\frac{zq''_{z^2}(z,\zeta)}{q'_z(z,\zeta)}\right)>0$.

By using (7), we obtain $\alpha p(z,\zeta)+\beta zp'_z(z,\zeta)=\alpha\frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}+$
$\beta\left[\frac{2-\lambda(n+1)}{\lambda}\frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}\lambda(n+1)(n+2)\frac{zDR_\lambda^{m,n+2}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}+\right.$
$\left.(n+1)\left[1-\lambda(n+2)\right]\frac{zDR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}-\frac{2}{\lambda}\frac{z\left(DR_\lambda^{m+1,n}f(z,\zeta)\right)^2}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^3}\right]=$
$\left(\alpha+\beta\frac{2-\lambda(n+1)}{\lambda}\right)\frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}+\lambda\beta(n+1)(n+2)\frac{zDR_\lambda^{m,n+2}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}+$
$\beta(n+1)\left[1-\lambda(n+2)\right]\frac{zDR_\lambda^{m,n+1}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}-\frac{2\beta}{\lambda}\frac{z\left(DR_\lambda^{m+1,n}f(z,\zeta)\right)^2}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^3}$.

By using (5), we have

$$\alpha p(z,\zeta)+\beta zp'_z(z,\zeta)\prec\prec\alpha q(z,\zeta)+\beta zq'_z(z,\zeta).$$

Therefore, the conditions of Lemma 1.14 are met, so we have

$$p(z,\zeta)\prec\prec q(z,\zeta),\ z\in U,\ \zeta\in\overline{U}, \quad \text{i.e.}$$
$$\frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}\prec\prec q(z,\zeta),\quad z\in U,\ \zeta\in\overline{U},$$

and q is the best dominant.

Corollary 2.2. *Let* $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\le B<A\le 1$, $m,n\in\mathbb{N}$, $\lambda\ge 0$, $z\in U$. *Assume that (3) holds. If* $f\in\mathcal{A}_{\xi}^{*}$ *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta)\prec\prec\alpha\frac{\zeta+Az}{\zeta+Bz}+\beta\frac{(A-B)\,\zeta z}{(\zeta+Bz)^{2}},$$

for $\alpha,\beta\in\mathbb{C}$, $\beta\ne 0$, $-1\le B<A\le 1$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (4), then*

$$\frac{zDR_{\lambda}^{m+1,n}f(z,\zeta)}{(DR_{\lambda}^{m,n}f(z,\zeta))^{2}}\prec\prec\frac{\zeta+Az}{\zeta+Bz}$$

and $\frac{\zeta+Az}{\zeta+Bz}$ *is the best dominant.*

Proof For $q(z,\zeta)=\frac{\zeta+Az}{\zeta+Bz}$, $-1\le B<A\le 1$, in Theorem 2.1 we get the corollary.

Corollary 2.3. *Let* $q(z,\zeta)=\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$, $m,n\in\mathbb{N}$, $\lambda\ge 0$, $z\in U$, $\zeta\in\overline{U}$. *Assume that (3) holds. If* $f\in\mathcal{A}_{\zeta}^{*}$ *and*

$$\psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta)\prec\prec\alpha\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}+\beta\frac{2\gamma\zeta z}{(\zeta-z)^{2}}\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma-1},$$

for $\alpha,\beta\in\mathbb{C}$, $0<\gamma\le 1$,$\beta\ne 0$, *where* $\psi_{\lambda}^{m,n}$ *is defined in (4), then*

$$\frac{zDR_{\lambda}^{m+1,n}f(z,\zeta)}{(DR_{\lambda}^{m,n}f(z,\zeta))^{2}}\prec\prec\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma},$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$ *is the best dominant.*

Proof Corollary follows by using Theorem 2.1 for $q(z,\zeta)=\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$, $0<\gamma\le 1$.

Theorem 2.4. *Let* q *be convex and univalent in* $U\times\overline{U}$ *such that* $q(0,\zeta)=1$, $m,n\in\mathbb{N}$, $\lambda\ge 0$. *Assume that*

$$Re\left(\frac{\alpha}{\beta}q_{z}'(z,\zeta)\right)>0,\ \text{for } \alpha,\beta\in\mathbb{C},\ \beta\ne 0,\ z\in U,\ \zeta\in\overline{U}. \tag{8}$$

If $f \in \mathcal{A}^*_\zeta$, $\frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2} \in \mathcal{H}^*\left[q\left(0,\zeta\right),1,\zeta\right] \cap Q^*$ *and* $\psi_\lambda^{m,n}\left(\alpha,\beta;z,\zeta\right)$ *is univalent in* $U \times \overline{U}$, *where* $\psi_\lambda^{m,n}\left(\alpha,\beta;z,\zeta\right)$ *is as defined in (4), then*

$$\alpha q\left(z,\zeta\right) + \beta z q'_z\left(z,\zeta\right) \prec\prec \psi_\lambda^{m,n}\left(\alpha,\beta;z,\zeta\right), \quad z \in U,\ \zeta \in \overline{U}, \tag{9}$$

implies

$$q\left(z,\zeta\right) \prec\prec \frac{zDR_\lambda^{m+1,n}f\left(z,\zeta\right)}{\left(DR_\lambda^{m,n}f\left(z,\zeta\right)\right)^2}, \quad z \in U,\ \zeta \in \overline{U}, \tag{10}$$

and q *is the best subordinant.*

Proof Let the function p be defined by $p\left(z,\zeta\right) := \frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2}$, $z \in U$, $z \neq 0$, $\zeta \in \overline{U}$, $f \in \mathcal{A}^*_\zeta$.

By setting $\nu\left(w\right) := \alpha w$ and $\phi\left(w\right) := \beta$, where $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, it can be easily verified that ν is analytic in $\mathbb{C}$, ϕ is analytic in $\mathbb{C}\backslash\{0\}$ and that $\phi\left(w\right) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Since $\frac{\nu'_z(q(z,\zeta))}{\phi(q(z,\zeta))} = \frac{\alpha}{\beta} q'_z\left(z,\zeta\right)$, it follows that $Re\left(\frac{\nu'_z(q(z,\zeta))}{\phi(q(z,\zeta))}\right) = Re\left(\frac{\alpha}{\beta} q'_z\left(z,\zeta\right)\right) > 0$, for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$.

Now, by using (9) we obtain

$$\alpha q\left(z,\zeta\right) + \beta z q'_z\left(z,\zeta\right) \prec\prec \alpha p\left(z,\zeta\right) + \beta z p'_z\left(z,\zeta\right), \quad z \in U,\ \zeta \in \overline{U}.$$

From Lemma 1.15, we have

$$q\left(z,\zeta\right) \prec\prec p\left(z,\zeta\right) = \frac{zDR_\lambda^{m+1,n}f\left(z,\zeta\right)}{\left(DR_\lambda^{m,n}f\left(z,\zeta\right)\right)^2}, \quad z \in U,\ \zeta \in \overline{U},$$

and q is the best subordinant.

Corollary 2.5. *Let* $q\left(z,\zeta\right) = \frac{\zeta+Az}{\zeta+Bz}$, $-1 \leq B < A \leq 1$, $m,n \in \mathbb{N}$, $\lambda \geq 0$. *Assume that (8) holds. If* $f \in \mathcal{A}^*_\zeta$, $\frac{zDR_\lambda^{m+1,n}f(z,\zeta)}{\left(DR_\lambda^{m,n}f(z,\zeta)\right)^2} \in \mathcal{H}^*\left[q\left(0,\zeta\right),1,\zeta\right] \cap Q^*$ *and*

$$\alpha\frac{\zeta+Az}{\zeta+Bz} + \beta\frac{\left(A-B\right)\zeta z}{\left(\zeta+Bz\right)^2} \prec\prec \psi_\lambda^{m,n}\left(\alpha,\beta;z,\zeta\right),$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B < A \leq 1$, *where* $\psi_\lambda^{m,n}$ *is defined in (4), then*

$$\frac{\zeta+Az}{\zeta+Bz} \prec\prec \frac{zDR_\lambda^{m+1,n}f\left(z,\zeta\right)}{\left(DR_\lambda^{m,n}f\left(z,\zeta\right)\right)^2}$$

and $\frac{\zeta+Az}{\zeta+Bz}$ *is the best subordinant.*

Proof For $q(z,\zeta) = \frac{\zeta + Az}{\zeta + Bz}$, $-1 \leq B < A \leq 1$, in Theorem 2.4 we get the corollary.

Corollary 2.6. *Let $q(z,\zeta) = \left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$, $m,n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (8) holds. If $f \in \mathcal{A}^*_{\zeta}$, $\frac{zDR_{\lambda}^{m+1,n} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^2} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ and*

$$\alpha\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma} + \beta\frac{2\gamma\zeta z}{(\zeta-z)^2}\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma-1} \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta),$$

for $\alpha,\beta \in \mathbb{C}$, $0 < \gamma \leq 1$, $\beta \neq 0$, where $\psi_{\lambda}^{m,n}$ is defined in (4), then

$$\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma} \prec\prec \frac{zDR_{\lambda}^{m+1,n} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^2}$$

and $\left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$ is the best subordinant .

Proof Corollary follows by using Theorem 2.4 for $q(z,\zeta) = \left(\frac{\zeta+z}{\zeta-z}\right)^{\gamma}$, $0 < \gamma \leq 1$.

Combining Theorem 2.1 and Theorem 2.4, we state the following sandwich theorem.

Theorem 2.7. *Let q_1 and q_2 be analytic and univalent in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\zeta \in \overline{U}$, with $z(q_1)'_z(z,\zeta)$ and $z(q_2)'_z(z,\zeta)$ being starlike univalent. Suppose that q_1 satisfies (3) and q_2 satisfies (8). If $f \in \mathcal{A}^*_{\zeta}$, $\frac{zDR_{\lambda}^{m+1,n} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^2} \in \mathcal{H}^*[q(0,\zeta),1,\zeta] \cap Q^*$ and $\psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta)$ is as defined in (4) univalent in $U \times \overline{U}$, then*

$$\alpha q_1(z,\zeta) + \beta z(q_1)'_z(z,\zeta) \prec\prec \psi_{\lambda}^{m,n}(\alpha,\beta;z,\zeta) \prec\prec \alpha q_2(z,\zeta) + \beta z(q_2)'_z(z,\zeta),$$

for $\alpha,\beta \in \mathbb{C}$, $\beta \neq 0$, implies

$$q_1(z,\zeta) \prec\prec \frac{zDR_{\lambda}^{m+1,n} f(z,\zeta)}{\left(DR_{\lambda}^{m,n} f(z,\zeta)\right)^2} \prec\prec q_2(z,\zeta), \quad \delta \in \mathbb{C},\ \delta \neq 0,$$

and q_1 and q_2 are respectively the best subordinant and the best dominant.

For $q_1(z,\zeta) = \frac{\zeta+A_1 z}{\zeta+B_1 z}$, $q_2(z,\zeta) = \frac{\zeta+A_2 z}{\zeta+B_2 z}$, where $-1 \leq B_2 < B_1 < A_1 < A_2 \leq 1$, we have the following corollary.

Corollary 2.8. *Let $m, n \in \mathbb{N}$, $\lambda \geq 0$. Assume that (3) and (8) hold for $q_1(z,\zeta) = \frac{\zeta + A_1 z}{\zeta + B_1 z}$ and $q_2(z,\zeta) = \frac{\zeta + A_2 z}{\zeta + B_2 z}$, respectively. If $f \in \mathcal{A}^*_\zeta$, $\frac{zDR_\lambda^{m+1,n} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^2} \in \mathcal{H}^*[q(0,\zeta), 1, \zeta] \cap Q^*$ and*

$$\alpha\frac{\zeta + A_1 z}{\zeta + B_1 z} + \beta\frac{(A_1 - B_1)\zeta z}{(\zeta + B_1 z)^2} \prec\prec \psi_\lambda^{m,n}(\alpha, \beta; z, \zeta) \prec\prec \alpha\frac{\zeta + A_2 z}{\zeta + B_2 z} + \beta\frac{(A_2 - B_2)\zeta z}{(\zeta + B_2 z)^2},$$

for $\alpha, \beta \in \mathbb{C}$, $\beta \neq 0$, $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$, where $\psi_\lambda^{m,n}$ is defined in (4), then

$$\frac{\zeta + A_1 z}{\zeta + B_1 z} \prec\prec \frac{zDR_\lambda^{m+1,n} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^2} \prec\prec \frac{\zeta + A_2 z}{\zeta + B_2 z},$$

hence $\frac{\zeta + A_1 z}{\zeta + B_1 z}$ and $\frac{\zeta + A_2 z}{\zeta + B_2 z}$ are the best subordinant and the best dominant, respectively.

3 Open Problem

An open problem is to find sufficient conditions for certain normalized analytic functions to satisfy

$$q_1(z,\zeta) \prec\prec \frac{zDR_\lambda^{m,n+1} f(z,\zeta)}{\left(DR_\lambda^{m,n} f(z,\zeta)\right)^2} \prec\prec q_2(z,\zeta),$$

where q_1 and q_2 are given convex and univalent functions in $U \times \overline{U}$ such that $q_1(z,\zeta) \neq 0$ and $q_2(z,\zeta) \neq 0$, for all $z \in U$, $\zeta \in \overline{U}$.

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