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ELEMENTARY OPERATORS AND NEW C^*-ALGEBRAS

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Abstract

Let H be a complex Hilbert space and $B(H)$ the algebra of all bounded linear operators on H. In this paper, we study the class of pairs of operators $A, B \in B(H)$ that have the following property, $ATB = T$ implies $B^*TA^* = T$ for all $T \in C_1(H)$ (trace class operators). The main result is the equivalence between this character and the fact that the ultra-weak closure of the range of the elementary operator $\Delta_{A,B}$ defined on $B(H)$ by $\Delta_{A,B}(X) = AXB - X$ is equivalent to the generalized quasi-adjoint operators. Some new C^*-algebras generated by a pair of operators $A, B \in B(H)$ are also presented.

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1 Introduction

For $A, B \in B(H), \Delta_{A,B}$ denotes the elementary operator on $B(H)$ defined by $\Delta_{A,B}(X) = AXB - X$ (for $X \in B(H)$). When $A = B$, $(\Delta_{A,A} = \Delta_A)$. In [1] Joel Anderson *et.al* showed that if A is D-symmetric,(i.e, $\overline{ran(\delta A)} = \overline{ran(\delta_{A^*})}$), where $\overline{ran(\delta A)}$ denotes the norm closure of the range of the inner derivation

δ_A defined by $\delta_A = AX - XA\,(X \in B(H))$, then for $T \in C_1(H), AT = TA$ implies $A^*T = TA^*$.

The ideal $C_1(H)$ admits a trace class function $tr(T)$, given by $tr(T) = \Sigma_n(Te_n, r_n)$ for any complete orthonormal system e_n in H. As a Banach space $C_1(H)$ can be identified with the dual of K of compact operators by means of the linear isometry $T \to f_T$, where $fT = tr(XT)$. Moreover, $B(H)$ is the dual of $C1(H)$. The ultra-weakly continuous linear functionals on $B(H)$ are those of the form f_T for $T \in C_1(H)$ and the weakly continuous ones are those of the form f_T with T of finite rank. In this paper we present a result similar to the result given by Anderson *et.al* in [1] for elementary operator $\Delta_{A,B}$.We also initiate the study of generalized quasi-adjoint operators.

2 Preliminaries

Anderson *et.al* In [1] proved the following theorem

Theorem 2.1 *[1] If $A \in B(H)$, then the following statements are equivalent*

(i) A is D-symmetric

(ii) (a) [A], the corresponding element of the Calkin algebra, is D-symmetric and

(b) $T \in C_1(H).AT = TA$ implies $A^*T = TA^*$.

In the following definitions we will introduce a new pairs of operators in $B(H) \times B(H)$

Definition 2.2 For $A, B \in B(H)$, the pair (A, B) is called generalized quasi-adjoint if $\overline{ran(\Delta_{A,B})} = \overline{ran(\Delta_{B^*,A^*})}$ (norm closure of the ranges). The set of all such pairs is denoted $\mathcal{GS(H)}$.

Definition 2.3 For $A, B \in B(H)$, the pair (A, B) is called generalized P-symmetric if $T \in C1(H), BTA = T$implies $A^*TB^* = T$. The set of all such pairs is denoted $\mathcal{GF}_0(H)$.

Remark 2.4 Recall that the pair (A, B) are generalized quasi-adjoint if and only if $\overline{ran(\Delta_{A,B})}$ satisfies the following property:

$Z \in \overline{ran(\Delta_{A,B})}$ implies $Z^* \in \overline{ran(\Delta_{A,B})}$. This is equivalent to $Ann(ran(\Delta_{A,B}))$ is self-adjoint, i.e,

if $f \in Ann(ran(\Delta_{A,B}))$, then $f_* \in Ann(ran(\Delta_{A,B}))$, where $f_*(X) = \overline{f(X^*)}$ for all $X \in B(H)$.

Definition 2.5 *A C^*-algebra is a Banach algebra $\mathcal{A}$ over the field of complex numbers, together with the map $* : \mathcal{A} \to \mathcal{A}$ which is called an involution.*

The image of an element $x \in \mathcal{A}$, under this involution, written x^*, satisfies the following conditions:

1. $(x+y)^* = x^* + y^*$, $\forall x, y \in \mathcal{A}$.
2. $(\lambda x)^* = \overline{\lambda} x^*$, $\forall x \in \mathcal{A}$.
3. $(x^*)^* = x$, $\forall x \in \mathcal{A}$.
4. *The C^*-identity holds for all $x \in \mathcal{A}$, that is,*

$$\|x^*x\| = \ |xx^*\| = \|x\|^2$$

The Banach algebra $B(H)$ of all bounded operators defined on a complex Hilbert space H is a C^*-algebra of operators.

3 Main results

In this section we will present some properties of generalized quasi-adjoint operators and we will prove similar result to Theorem 2.1 by taking $\Delta_{A,B}$ instead of δ_A. In , [14] J.P Williams showed that if, $A \in B(H)$, then

$$Ann(ran(\delta_A)) \simeq Ann(ran(\delta_A)) \cap Ann(K(H)) \oplus ker(\delta_A) \cap C_1,$$

where $ran(\delta_A), K(H), ker(\delta_A)$ and C_1, denote respectively, the range of δ_A, the ideal of compact operators, the kernel of δ_A and the trace class operators. We start by proving a similar decomposition for $\Delta_{A,B}$. Let $\mathcal{B}$ be a Banach space and let $\mathcal{S}$ be a subspace of $\mathcal{B}$. Denote by $\mathcal{B}'$ the set of of all linear functionals, and the set $\mathcal{B}^* = \{f \in \mathcal{B}' : f$ is bounded (norm-continuous)$\}$,

$$Ann(S) = \{f \in \mathcal{B}^* : f(s) = 0 for all s \in S\}$$

Theorem 3.1 *[6] Let E, F be Banach spaces and let $B(E, F)$ be the set of all bounded linear operators $A : E \to F$. If $S \in B(E, F)$ a bounded operator, then*

$$Ann(ran(S^{**})) = Ann(ran(S^{**})) \cap Ann(F) \oplus ker(S^*)$$

Let $A, B \in B(H)$. The following theorem gives a similar result of S.Mecheri [6] concerning $\Delta_{A,B}$.

Theorem 3.2 *Let $A, B \in B(H)$, then*

$$Ann(ran(\Delta_{A,B})) = Ann(ran(\Delta_{A,B})) \cap Ann(K(H)) \oplus ker(\Delta_{B,A}) \cap C_1.$$

Proof. It suffices to take in Theorem (3.1) $E = F = K(H)$ and

$$S = \Delta_{A,B} : K(H) \to K(H),$$

where $S^* = \Delta_{B,A} : C_1 \to C_1$ using trace duality. ■

Remark 3.3 The concept of "generalized quasi-adjoint" for elements of the Calkin algebra, can be defined as in Definition 2.1. Note that Remark 2.1 remains true in this case.

Theorem 3.4 *Let $A, B \in B(H)$. Then the following assertion are equivalent:*

1. The pair (A, B) is generalized quasi-adjoint.

2. (i) The pair $([A], [B])$, the corresponding pair of elements in the Calkin Algebra $B(H)/K(H)$ is generalized quasi-adjoint.

(ii) $BTA = T$ implies $A^*TB^* = T$ for all $T \in C_1(H)$.

3. (i) $\overline{ran(\Delta_{A,B})}^{w^*} = \overline{ran(\Delta_{B^*,A^*})}^{w^*}$

(ii) The pair $([A], [B])$ is generalized quasi-adjoint.

Proof. The proof is the same as that of Theorem(3.2) in [6]. ■

Remark 3.5 It is known from [13] that if A, B are contrations, then $ATB = T, T \in C_1$ implies $A^*TB^* - T = 0$. Thus the pair (A, B) is generalized quasi-adjoint if and only if the pair $([A], [B])$ is generalized quasi-adjoint.

In the following theorem we will present an example of quasi-adjoint operator.

Theorem 3.6 *Let $A \in B(H)$ be isometric. Then A is quasi-adjoint.*

Proof. Let $A \in B(H)$ be isometric and consider the operator P defined by $P = I - VV^*$. Then $\Delta_{A^*}(X) = \Delta_A(-A^*XA) - PX$ for all $X \in B(H)$. To prove that A is quasi-adjoint, it suffices to show that $PB(H) \subseteq \overline{R(\Delta_A)}$. Define the sequence of operators $(S_n)_{m\geq 1}$ by $S_n = \Sigma_{k=0}^{n-1} \frac{k-n}{n} A^k PXV^k$. It follows that $\Delta_A(S_n) - PX = -\frac{1}{n}\Sigma_{k=1}^{n} A^k PXA^k$. It easy to see that: $(A^kPx, A^jPy) = 0$ for all $x, y \in H$ and for all positive integers $k, j (k \neq j)$. By this we get,

$$\|A^kPXA^kx\|^2 = \sum_{i=1}^{n} \|A^kPXA^kx\|^2 \leq n\|PX\|^2\|x\|^2$$

for all $x \in H$. It follows that

$$\|\Delta_A(S_n) - PX\| \leq \frac{i}{\sqrt{n}}\|PX\|$$

for all $X \in B(H)$. This implies that $PX \in \overline{R(\Delta_A)}$. Thus A is quasi-adjoint. ■

Theorem 3.7 *Let $A, B \in B(H)$. Then $(A, B) \in \mathcal{GF}_0 \Leftrightarrow \overline{ran(\Delta_{A,B})}^{w*}$ is self-adjoint.*

Proof. The w^*-topology is generated by all f_T with $T \in C_1$ and so $\overline{ran(\Delta_{A,B})}^{w*}$ is the intersection

$$\cap\{ker f_T : f_T(\sum_{i=1}^{n} A_iXB_i - X) = 0 \forall X \in B(H)\}.$$

Since

$$f_T(\sum_{i=1}^{n} A_iXB_i - X) = tr(T(\sum_{i=1}^{n} A_iXB_i - X)) = tr((\sum_{i=1}^{n} A_iTB_i - T)X)$$

, this intersection is

$$ker\Delta_{B.A} \cap C_1(H).$$

If $(A, B) \in \mathcal{GF}_o$, Then

$$ker\Delta_{B,A} \cap C_1(H) = ker\Delta_{A^*,B^*} \cap C_1(H)$$

and so the weak *-closure of

$$(ran(\Delta_{B^*,A^*})) = (ran(\Delta_{A,B}))^*.$$

Conversely, if $\overline{ran(\Delta_{A,B})}^{w^*}$ is self-adjoint. The set of $T \in C_1(H)$ for which f_T vanishes on $ran(\Delta_{A,B})$ must be self-adjoint ($Y \in ran(\Delta_{A,B})$ implies $0 = f_T(Y^*) = tr(TY^*) = \overline{tr(T^*Y)}$). Hence

$$ker\Delta_{B,A} \cap C_1(H) = ker\Delta_{A^*,B^*} \cap C_1(H),$$

and $(A, B) \in \mathcal{GF}_0$. ■ Now consider the following sets:

$$\mathcal{T}_0(A, B) = \{(C, D) \in B(H) \times B(H) : CB(H)D + B(H\} \subset \overline{ran(\Delta_{A,B})}^{w^*}.$$

$$\mathcal{I}_0(A, B) = \{(C, D) \in B(H) \times B(H) : Cran(\Delta_{A,B})D + ran(\Delta_{A,B})\} \subset \overline{ran(\Delta_{A,B})}^{w^*}.$$

$$\mathcal{B}_0(A, B) = \{(C, D) \in \mathcal{B}(H) \times \mathcal{B}(H) : ran(\Delta_{C,D})\} \subset \overline{ran(\Delta_{A,B})}^{w^*}.$$

Theorem 3.8 *Let $A, B \in B(H)$. If the pair (A, B) is generalized quasi-adjoint, then we have*

(i) $\mathcal{T}_0(A, B)$, $\mathcal{I}_0(A, B)$ and $\mathcal{B}_0(A, B)$ are C^*-algebras w^*-closed in $B(H) \times B(H)$.

(ii) $\mathcal{T}_0(A, B)$ is a bilateral ideal of $\mathcal{I}_0(A, B)$.

(ii) $ran(\Delta_{C,D}) \subset \overline{ran(\Delta_{A,B})}^{w*}$ for all $C, D \in C^*(A, B)$, the C^*-algebra generated by the pair $(A, B) \in \mathcal{GF}_0(H)$.

Proof. (i) Let $(C,D) \in \mathcal{T}_0(A,B)$. This implies

$$C\mathcal{L}(\mathcal{H})D - \mathcal{L}(\mathcal{H}) \in \overline{ran(\Delta_{A,B})}^{w*},$$

that is,

$$CXD - Y \in \overline{ran(\Delta_{A,B})}^{w*} \forall X, Y \in B(H$$

. So if we let $X = 0$ it follows that: $Y \in \overline{ran(\Delta_{A,B})}^{w*}$, $\forall Y \in B(H)$. As a consequence C^* and D^* are in $\overline{ran(\Delta_{A,B})}^{w*}$.

Moreover, since $(A,B) \in \mathcal{GF}_0, (D^*, C^*) \in \mathcal{T}_0$, we conclude that:
$C^*(D^*XC^*)D^* - D^*XC^* \in \overline{ran(\Delta_{A,B})}^{w*}$

$$(DYC - Y) \in \overline{ran(\Delta_{A,B})}^{w*}.$$

Hence

$$(D,C) \in \mathcal{T}_0(A,B),$$

therefore $(C^*, D^*) = (D,C)^* \in \mathcal{T}_0(A,B)$.

Similarly, we can show that $\mathcal{I}_0$ is also a C^*-algebra. Just note that if $(C,D) \in \mathcal{I}_0$, then

$$(CXD - Y) \in \overline{ran(\Delta_{A,B})}^{w*}, \forall X, Y \in ran(\Delta_{C,D}) \subseteq B(H),$$

i.e., $(C,D) \in \mathcal{T}_0$. This gives $(C^*, D^*) \in \mathcal{T}_0 \subseteq \mathcal{I}_0$.

Finally, $\mathcal{B}_0$ is a C^*-algebra, because $ran(\Delta_{C,D}) \subseteq ran(\Delta_{A,B})$, but $ran(\Delta_{C,D}) \subseteq \mathcal{I}_0$. Thus $(C^*, D^*) \in \mathcal{B}_0$. Now, we want to show that the upper sets are w^*-closed. Recall that

$$(A,B) \in \mathcal{GF}_0(H) \Leftrightarrow Ann(ran(\Delta_{A,B})) \cap \mathcal{L}(\mathcal{H})'^{w*}$$

is self-adjoint if and only if $Ann(ran(\Delta_{A,B})) \cap \mathcal{L}(\mathcal{H})'^{w^*} \cong ker(\Delta_{A,B}) \cap C_1(H)$, where $\mathcal{L}(\mathcal{H})'^{w^*}$ is the set of all ultra-weakly continuous linear functionals in $B(H)$.

Let$C, D \in ker(\delta_{A,B}) \cap C_1(H) \Rightarrow C = BCA$, $D = BDA$. Thus, there exists a linear functional $f_{C,D}$ given by $f_{C,D}(X) = tr(CXD), X \in B(H)$. Consider
$f_{C,D}(\Delta_{A,B}) = tr(C(AXB)D - CXD)$
$= tr((CA)X(BD)) - tr(CXD) = tr((AC))X(BD)) - tr(DXC)$
$= tr(BDA(CX)) - tr(DXC) = tr(DCX - DXC) = 0$, hence $f_{C,D} \in Ann(ran(\Delta_{A,B})$. Consequently $\mathcal{T}_0(A,B)$ is $w*$-closed in $B(H) \times B(H)$.

(ii) Clearly $\mathcal{T}_0$ is a sub-algebra of $\mathcal{I}_0$.

Let $(C,D) \in \mathcal{I}_0(A,B)$ and
$(E,F) \in \mathcal{T}_0(A,B) \to C$, D, E and $F \in \overline{ran(\Delta_{A,B})}^{w*}$, then for all $X \in ran(\Delta_{A,B})$ we conclude that CEX and XDF are in $\overline{ran(\Delta_{A,B})}^{w*}$. Hence

$(CEXDF - X) \in \overline{ran(\Delta_{A,B})}^{w*}$. This shows that $\mathcal{T}_0(A,B)$ is a right ideal. Since $\mathcal{T}_0(A,B)$ is a C^*-algebra, it follows that $\mathcal{T}_0(A,B)$ is a bilateral ideal of $\mathcal{I}_0(A,B)$.

(iii)Note that $(A,B) \in \mathcal{B}_0(H)$, Since $ran(\Delta_{A,B}) \subseteq \overline{ran(\Delta_{A,B})}^{w*}$. Thus $\mathcal{B}_0(H)$ is a C^*-algebra containing the pair (A,B) and obviously (I,I), hence it contains $C^*(A,B)$. ■

Theorem 3.9 *Let $A,B \in B(H)$. If (A,B) is generalized quasi-adjoint, then*

$$B^* ran(\Delta_{A,B}) + ran(\Delta_{A,B})A^* \subset \overline{ran(\Delta_{A,B})}^{w^*}$$

Proof. Assume that the pair (A,B) is generalized quasi-adjoint. Then it follows from Theorem 1 that:

$\overline{ran(\Delta_{A,B})}^{w^*} = \overline{ran(\Delta_{B^*,A^*})}^{w^*}$. But since

$B^*\Delta_{B^*,A^*}(X) = \Delta_{B^*,A^*}(B^*X)$ and $\Delta_{B^*,A^*}(X)A^* = \Delta_{B^*,A^*}(XA^*)$, we deduce that

$B^* ran(\Delta_{A,B}) \subset B^*\overline{ran(\Delta_{A,B})}^{w^*} = B^*\overline{ran(\Delta_{B^*,A^*})}^{w^*} \subseteq \overline{ran(\Delta_{B^*,A^*})}^{w^*} = \overline{ran(\Delta_{A,B})}^{w^*}$.

By the same arguments shown above:

Since $\Delta_{B^*,A^*}(X)A^* = \Delta_{B^*,A^*}(XA^*)$, we deduce that

$ran(\Delta_{A,B})A^* \subset \overline{ran(\Delta_{A,B})}^{w^*} A^* = \overline{ran(\Delta_{B^*,A^*})}^{w^*} A^* \subseteq \overline{ran(\Delta_{B^*,A^*})}^{w^*} = \overline{ran(\Delta_{A,B})}^{w^*}$. This completes the proof. ■

4 Open Problem

How to extend all results in this paper to the elementary operator $AXB-CXD$?

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