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Maximum terms, maximum moduli related and slowly changing functions based growth properties of composite entire functions

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Abstract

In the paper we prove some growth properties related to the maximum terms and maximum moduli of composite entire functions using generalised L^*-order and generalised L^*-type as compared to the growths of their corresponding left and right factors.

Keywords: *Entire function, maximum term, maximum modulus, composition, growth, generalised L^*-order, generalised L^*-type, slowly changing function.*

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1 Introduction, Definitions and Notations.

Let $\mathbb{C}$ be the set of all finite complex numbers and f be entire defined in the open complex plane $\mathbb{C}$. The maximum term $\mu(r,f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $\mu(r,f) = \max(|a_n| r^n)$ and the maximum modulus $M(r,f)$ of $f = \sum_{n=0}^{\infty} a_n z^n$ on $|z| = r$ is defined by $M(r,f) = \max_{|z|=r} |f(z)|$ .We use the standard notations and definitions in the theory of entire functions which are available in [11]. In the sequel the following notation is used :

$\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ for $k = 1, 2, 3, ...$ and $\log^{[0]} x = x$.

To start our paper we just recall the following definition :

Definition 1. *The order ρ_f and lower order λ_f of an entire function f are defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r} \quad \textit{and} \quad \lambda_f = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log r}.$$

Definition 2. *The type σ_f of an entire function f is defined as*

$$\sigma_f = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho_f}}, \quad 0 < \rho_f < \infty.$$

Sato [6] defined the generalised order and generalised lower order of an entire function as follows :

Definition 3. *[6]Let m be an integer ≥ 2. The generalised order $\rho_f^{[m]}$ and generalised lower order $\lambda_f^{[m]}$ of an entire function f are defined as*

$$\rho_f^{[m]} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log r} \quad \textit{and}$$
$$\lambda_f^{[m]} = \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log r} \quad \textit{respectively.}$$

For $m = 2$, Definition 2 reduces to Definition 1.
If $\rho_f < \infty$ then f is of finite order. Also $\rho_f = 0$ means that f is of order zero. In this connection Datta and Biswas [2] gave the following definition :

Definition 4. *[2] Let f be an entire function of order zero. The quantities ρ_f^{**} and λ_f^{**} of f are defined by:*

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\log r} \quad \textit{and} \quad \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r}.$$

Let $L \equiv L(r)$ be a positive continuous function increasing slowly *i.e.,* $L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant a. *Singh and Barker* [7] defined it in the following way:

Definition 5. *[7]A positive continuous function $L(r)$ is called a slowly changing function if for $\varepsilon\,(>0)$,*

$$\frac{1}{k^{\varepsilon}} \leq \frac{L(kr)}{L(r)} \leq k^{\varepsilon} \ for\ r \geq r(\varepsilon) \ \ and$$

uniformly for $k\,(\geq 1)$.
If further, $L(r)$ is differentiable, the above condition is equivalent to

$$\lim_{r\to\infty} \frac{rL'(r)}{L(r)} = 0 \ .$$

Somasundaram and Thamizharasi [8] introduced the notions of L-order and L-type for entire function where $L \equiv L(r)$ is a positive continuous function increasing slowly i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant 'a'. The more generalised concept for L-order and L-type for entire function are L^*-order and L^*-type. Their definitions are as follows:

Definition 6. *[8]The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of an entire function f are defined as*

$$\rho_f^{L^*} = \limsup_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]} \ \ and\ \lambda_f^{L^*} = \liminf_{r\to\infty} \frac{\log^{[2]} M(r,f)}{\log\left[re^{L(r)}\right]}.$$

Definition 7. *The L^*-type $\sigma_f^{L^*}$ of an entire function f is defined as*

$$\sigma_f^{L^*} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}\ ,\ \ 0 < \rho_f^{L^*} < \infty\ .$$

In the line of Sato [6], Datta and Biswas [2] one can define the generalised L^*-order $\rho_f^{[m]L^*}$ and generalised L^*-lower order$\lambda_f^{[m]L^*}$ of an entire function f in the following manner :

Definition 8. *Let m be an integer ≥ 1. The generalised L^*-order $\rho_f^{[m]L^*}$ and generalised L^*-lower order $\lambda_f^{[m]L^*}$ of an entire function f are defined as*

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log\left[re^{L(r)}\right]} \ \ and\ \lambda_f^{[m]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\log\left[re^{L(r)}\right]} \ \ respectively.$$

Datta, Biswas and Hoque [3] reformulated Definition 8 in terms of the maximum terms of entire functions in the following way:

Definition 9. *[3] The growth indicators* $\rho_f^{[m]L^*}$ *and* $\lambda_f^{[m]L^*}$ *for an entire function* f *are defined as*

$$\rho_f^{[m]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} \mu(r, f)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{[m]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} \mu(r, f)}{\log\left[re^{L(r)}\right]}$$

respectively where m *be an integer* ≥ 1.

Similarly, in the line of Somasundaram and Thamizharasi [8] for any positive integer $m \geq 2$ one may define the generalised L^*-type $\sigma_f^{[m-1]L^*}$ in the following manner:

Definition 10. *The generalised* L^**-type* $\sigma_f^{[m-1]L^*}$ *for* $m \geq 2$ *of an entire function* f *is defined as follows:*

$$\sigma_f^{[m-1]L^*} = \limsup_{r\to\infty} \frac{\log^{[m]} M(r, f)}{\left[re^{L(r)}\right]^{\rho_f^{[m]L^*}}}, \ 0 < \rho_f^{[m]L^*} < \infty .$$

Lakshminarasimhan [4] introduced the idea of the functions of L-bounded index. Later Lahiri and Bhattacharjee [5] worked on entire functions of L-bounded index and of non uniform L-bounded index. In the paper we study some growth properties related to the maximum terms and maximum moduli of composite entire functions using generalised L^*-order and generalised L^*-type as compared to the growths of their corresponding left and right factors.

2 Lemmas.

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[9] Let* f *and* g *be any two entire functions with* $g(0) = 0$. *Then for all sufficiently large values of* r,

$$\mu(r, f \circ g) \geq \frac{1}{2}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right).$$

Lemma 2. *[1] If* f *and* g *are two entire functions then for all sufficiently large values of* r,

$$M\left(\frac{1}{8}M\left(\frac{r}{2}, g\right) - |g(0)|, f\right) \leq M(r, f \circ g) \leq M(M(r, g), f).$$

3 Theorems.

In this section we present the main results of the paper.

Theorem 3. *Let f and g be any two entire functions such that $\rho_f^{[m]L^*}$ and $\rho_g^{L^*}$ are both finite and positive where $m \geq 1$. Then for each $\alpha \in (-\infty, \infty)$,*

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} \mu\left(r, f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)} = 0 \text{ and}$$

$$\liminf_{r\to\infty} \frac{\left\{\log^{[m]} \mu\left(r, f\circ g\right)\right\}^{1+\alpha}}{\log^{[2]} \mu\left(\exp\left(r^{\beta}\right), g\right)} = 0 \quad \text{where } \beta > (1+\alpha)\,\rho_g^{L^*}.$$

Proof. If $1+\alpha < 0$, then the theorem is trivial. So we take $1+\alpha > 0$.Now in view of Lemma 2 and the inequality $\mu\left(r, f\right) \leq M\left(r, f\right)$ $\{cf.$ [10] $\}$, we have for all sufficiently large values of r that

$$\mu(r, f\circ g) \leq M(r, f\circ g) \leq M\left(M\left(r, g\right), f\right)$$
$$i.e.,\ \log^{[m]} \mu(r, f\circ g) \leq \log^{[m]} M\left(M\left(r, g\right), f\right)$$
$$i.e.,\ \log^{[m]} \mu(r, f\circ g) \leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\left[\log M\left(r, g\right) e^{L(M(r,g))}\right]$$

$$i.e.,\ \log^{[m]} \mu(r, f\circ g)$$
$$\leq \left(\rho_f^{[m]L^*} + \varepsilon\right)\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)} + \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M\left(r, g\right)\right)$$

$$i.e.,\ \left\{\log^{[m]} \mu\left(r, f\circ g\right)\right\}^{1+\alpha}$$
$$\leq \left[\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[m]L^*} + \varepsilon\right) + \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M\left(r, g\right)\right)\right]^{1+\alpha}. \tag{1}$$

Again we get for a sequence of r tending to infinity and for $\varepsilon(> 0)$ that

$$\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right) \geq \left(\rho_f^{[m]L^*} - \varepsilon\right)\log\left[\exp\left(r^{\beta}\right)\exp\left\{L\left(\exp\left(r^{\beta}\right)\right)\right\}\right]$$
$$i.e.,\ \log^{[m]} \mu\left(\exp\left(r^{\beta}\right)\right), f) \geq \left(\rho_f^{[m]L^*} - \varepsilon\right)\left[r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right]. \tag{2}$$

So from (1) and (2) we obtain for a sequence of r tending to infinity that

$$\frac{\left\{\log^{[m]} \mu\left(r, f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]} \mu\left(\exp\left(r^{\beta}\right), f\right)}$$
$$\leq \frac{\left[\left[re^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[m]L^*} + \varepsilon\right) + \left(\rho_f^{[m]L^*} + \varepsilon\right) L\left(M\left(r, g\right)\right)\right]^{1+\alpha}}{\left(\rho_f^{[m]L^*} - \varepsilon\right)\left[r^{\beta} + L\left(\exp\left(r^{\beta}\right)\right)\right]}. \tag{3}$$

Let

$$\begin{aligned}\left[e^{L(r)}\right]^{\left(\rho_g^{L^*}+\varepsilon\right)}\left(\rho_f^{[m]L^*}+\varepsilon\right) &= k_1, \left(\rho_f^{[m]L^*}+\varepsilon\right)L\left(M\left(r,g\right)\right)=k_2,\\ \left(\rho_f^{[m]L^*}-\varepsilon\right) &= k_3, \left(\rho_f^{[m]L^*}-\varepsilon\right)L\left(\exp\left(r^{\beta}\right)\right)=k_4.\end{aligned}$$

Then from (3) we obtain for a sequence of r tending to infinity that

$$\begin{aligned}\frac{\left\{\log^{[m]}\mu\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]}\mu\left(\exp\left(r^{\beta}\right),f\right)} &\leq \frac{\left[r^{\left(\rho_g^{L^*}+\varepsilon\right)}k_1+k_2\right]^{1+\alpha}}{k_3r^{\beta}+k_4}\\ i.e., \ \frac{\left\{\log^{[m]}\mu\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]}\mu\left(\exp\left(r^{\beta}\right),f\right)} &\leq \frac{r^{\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha)}\left[k_1+\frac{k_2}{r^{\left(\rho_g^{L^*}+\varepsilon\right)}}\right]^{1+\alpha}}{k_3r^{\beta}+k_4}\end{aligned}$$

where k_1, k_2,k_3 and k_4 are finite.
Since$\left(\rho_g^{L^*}+\varepsilon\right)(1+\alpha)<\beta$, therefore

$$\liminf_{r\to\infty}\frac{\left\{\log^{[m]}\mu\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]}\mu\left(\exp\left(r^{\beta}\right),f\right)}=0$$

where we choose $\varepsilon(>0)$ such that

$$0<\varepsilon<\min\left\{\rho_f^{[m]L^*},\frac{\beta}{1+\alpha}-\rho_g^{L^*}\right\},$$

which proves the first part of the theorem.
Similarly, the second part of the theorem follows from the following inequality in place of (2)

$$i.e., \ \log^{[2]}\mu\left(\exp\left(r^{\beta}\right),g\right)\geq\left(\rho_g^{L^*}-\varepsilon\right)\left[r^{\beta}+L\left(\exp\left(r^{\beta}\right)\right)\right]$$

for a sequence of values of r tending to infinity.
This proves the theorem. □

Remark 1. *In Theorem 3 if we take the condition "* $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ *and* $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ *" in place of "* $\rho_f^{[m]L^*}$ *and* $\rho_g^{L^*}$ *are both finite and positive " the theorem remains true with "* $\lim$ *" replaced by "* $\liminf$ *".*

In the line of Theorem 3, the following theorem can be proved:

Theorem 4. *Let f and g be any two entire functions with finite and positive* $\rho_f^{[m]L^*}$ *and* $\rho_g^{L^*}$ *where* $m\geq1$*.Then for each* $\alpha\in(-\infty,\infty)$,

$$\liminf_{r\to\infty}\frac{\left\{\log^{[m]}M\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[m]}M\left(\exp\left(r^{\beta}\right),f\right)}=0 \ \textit{and}$$

$$\liminf_{r\to\infty}\frac{\left\{\log^{[m]}M\left(r,f\circ g\right)\right\}^{1+\alpha}}{\log^{[2]}M\left(\exp\left(r^{\beta}\right),g\right)}=0\quad where\ \beta>\left(1+\alpha\right)\rho_g^{L^*}.$$

Remark 2. *Also in Theorem 4 if we take the condition* " $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ *and* $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$ " *inplace of* " $\rho_f^{[m]L^*}$ *and* $\rho_g^{L^*}$ *are both finite and positive* " *the theorem remains true with* " $\lim$ " *replaced by* "$\liminf$".

Theorem 5. *Let f and g be any two entire functions with $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ where m is any positive integer and $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[m+1]}\mu\left(r,f\circ g\right)}{\log^{[m]}\mu\left(r,f\right)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-\left|g(0)\right|\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}\ .$$

Proof. In view of Lemma 1, we have for all sufficiently large values of r

$$\log^{[m]}\mu\left(r,f\circ g\right)\geq o\left(1\right)+\log^{[m]}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-\left|g(0)\right|,f\right). \tag{4}$$

$$i.e.,\ \log^{[m]}\mu\left(r,f\circ g\right)\geq o\left(1\right)+\left(\lambda_f^{[m]L^*}-\varepsilon\right)\left[\log\left\{\frac{1}{8}\mu\left(\frac{r}{4},g\right)-\left|g(0)\right|\right\}\right.$$
$$\left.+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-\left|g(0)\right|\right)\right]$$

$$i.e.,\ \log^{[m]}\mu\left(r,f\circ g\right)\geq o\left(1\right)+\left(\lambda_f^{[m]L^*}-\varepsilon\right)$$
$$\left[\log\left\{\frac{1}{8}\mu\left(\frac{r}{4},g\right)\left(1-\frac{\left|g(0)\right|}{\frac{1}{8}\mu\left(\frac{r}{4},g\right)}\right)\right\}+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-\left|g(0)\right|\right)\right]$$

$$i.e.,\ \log^{[m]}\mu\left(r,f\circ g\right)\geq\left(\lambda_f^{[m]L^*}-\varepsilon\right)\log\mu\left(\frac{r}{4},g\right).$$
$$\left\{\frac{\log\mu\left(\frac{r}{4},g\right)+\log\left(1-\frac{\left|g(0)\right|}{\frac{1}{8}\mu\left(\frac{r}{4},g\right)}\right)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-\left|g(0)\right|\right)}{\log\mu\left(\frac{r}{4},g\right)}\right\}$$

$$\begin{aligned} i.e., \;\; \log^{[m+1]} \mu(r, f \circ g) \geq & \log^{[2]} \mu\left(\frac{r}{4}, g\right) \\ & + \left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[m]L^*} + \varepsilon}\right) L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right) \\ & - \log\left[\exp\left\{\left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[m]L^*} + \varepsilon}\right) L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right)\right\}\right] \\ & + \log\left\{\frac{\left[\log \mu\left(\frac{r}{4}, g\right) + L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right)\right] + o(1)}{\log \mu\left(\frac{r}{4}, g\right)}\right\} \end{aligned}$$

$$\begin{aligned} i.e., \;\; \log^{[m+1]} \mu(r, f \circ g) \geq & \log^{[2]} \mu\left(\frac{r}{4}, g\right) \\ & + \left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[m]L^*} + \varepsilon}\right) L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right) \\ & + \log\left\{\frac{\left[\log \mu\left(\frac{r}{4}, g\right) + L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right)\right] + o(1)}{\exp\left\{\left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[m]L^*} + \varepsilon}\right) L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right)\right\} \log \mu\left(\frac{r}{4}, g\right)}\right\} \end{aligned}$$

$$\begin{aligned} & i.e., \;\; \log^{[m+1]} \mu(r, f \circ g) \\ \geq & \;\; \log^{[2]} \mu\left(\frac{r}{4}, g\right) + \left(\frac{\lambda_g^{L^*} - \varepsilon}{\rho_f^{[m]L^*} + \varepsilon}\right) L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right). \end{aligned} \tag{5}$$

Now from (5) it follows for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[m+1]} \mu(r, f \circ g) \geq & \left(\rho_g^{L^*} - \varepsilon\right) \log\left\{\frac{r}{4} e^{L\left(\frac{r}{4}\right)}\right\} \\ & + \left(\frac{\rho_g^{L^*} - \varepsilon}{\rho_f^{[m]L^*} + \varepsilon}\right) L\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|\right). \end{aligned} \tag{6}$$

Now we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[m]} \mu(r, f) & \leq \left(\rho_f^{[m]L^*} + \varepsilon\right) \log\left\{r e^{L(r)}\right\} \\ i.e., \;\; \log^{[m]} \mu(r, f) & \leq \left(\rho_f^{[m]L^*} + \varepsilon\right) \log\left\{\frac{r}{4} e^{L\left(\frac{r}{4}\right)}\right\} + \log 4. \end{aligned} \tag{7}$$

Hence from (6) and (7) it follows for all sufficiently large values of r that

$$\begin{aligned}\log^{[m+1]}\mu(r,f\circ g) \;&\geq\; \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\left(\log^{[m]}\mu(r,f)-\log 4\right)\\ &\quad+\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)\end{aligned}$$

$$\begin{aligned}&i.e.,\;\; \log^{[m+1]}\mu(r,f\circ g)\\ &\geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\left[\log^{[m]}\mu(r,f)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)\right]\\ &\quad-\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\log 4\end{aligned}$$

$$\begin{aligned}&i.e.,\;\; \frac{\log^{[m+1]}\mu(r,f\circ g)}{\log^{[m]}\mu(r,f)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)}\\ &\geq \left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)-\frac{\left(\frac{\rho_g^{L^*}-\varepsilon}{\rho_f^{[m]L^*}+\varepsilon}\right)\log 4}{\log^{[m]}\mu(r,f)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)}. \qquad (8)\end{aligned}$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from (8) that

$$\limsup_{r\to\infty}\frac{\log^{[m+1]}\mu(r,f\circ g)}{\log^{[m]}\mu(r,f)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}\;.$$

Thus the theorem is established. □

In the line of Theorem 5, the following theorem can be proved:

Theorem 6. *Let f and g be any two entire functions with $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ where $m\geq 1$ and $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$. Then*

$$\liminf_{r\to\infty}\frac{\log^{[m+1]}\mu(r,f\circ g)}{\log^{[m]}\mu(r,f)+L\left(\frac{1}{8}\mu\left(\frac{r}{4},g\right)-|g(0)|\right)}\geq\frac{\lambda_g^{L^*}}{\rho_f^{[m]L^*}}\;.$$

The proof is omitted.

Theorem 7. *Let f and g be any two entire functions with $0<\lambda_f^{[m]L^*}\leq\rho_f^{[m]L^*}<\infty$ where m is any positive integer and $0<\lambda_g^{L^*}\leq\rho_g^{L^*}<\infty$. Then*

$$\limsup_{r\to\infty}\frac{\log^{[m+1]}M(r,f\circ g)}{\log^{[m]}M(r,f)+L\left(\frac{1}{8}M\left(\frac{r}{2},g\right)-|g(0)|\right)}\geq\frac{\rho_g^{L^*}}{\rho_f^{[m]L^*}}\;.$$

Theorem 8. *Let f and g be any two entire functions such that $0 < \lambda_f^{[m]L^*} \leq \rho_f^{[m]L^*} < \infty$ where $m \geq 1$ and $0 < \lambda_g^{L^*} \leq \rho_g^{L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m+1]} M(r, f\circ g)}{\log^{[m]} M(r,f) + L\left(\frac{1}{8}M\left(\frac{r}{2},g\right) - |g(0)|\right)} \geq \frac{\lambda_g^{L^*}}{\rho_f^{[m]L^*}} .$$

We omit the proofs of Theorem 7 and Theorem 8 because those can be carried out in the line of Theorem 5 and Theorem 6 respectively and with the help of Lemma 2.

Theorem 9. *Let f and g be any two entire functions with $\rho_f^{[n]L^*} < \infty$ and $\lambda_{f\circ g}^{[m]L^*} = \infty$ where m and n are positive integers. Then*

$$\lim_{r\to\infty} \frac{\log^{[m]} \mu(r, f\circ g)}{\log^{[n]} \mu(r,f)} = \infty.$$

Proof. Let us suppose that the conclusion of the theorem do not hold.Then we can find a constant $\beta > 0$ such that for a sequence of values of r tending to infinity

$$\log^{[m]} \mu(r, f\circ g) \leq \beta \log^{[n]} \mu(r,f). \tag{9}$$

Again from the definition of $\rho_f^{[n]L^*}$, it follows that for all sufficiently large values of r that

$$\begin{aligned} \log^{[n]} \mu(r,f) &\leq \left(\rho_f^{[n]L^*} + \varepsilon\right) \log\left(re^{L(r)}\right) \\ i.e.,\ \log^{[n]} \mu(r,f) &\leq \left(\rho_f^{[n]L^*} + \varepsilon\right) \log\left(re^{L(r)}\right). \end{aligned} \tag{10}$$

Thus from (9) and (10) we have for a sequence of values of r tending to infinity that

$$\begin{aligned} \log^{[m]} \mu(r, f\circ g) &\leq \beta\left(\rho_f^{[n]L^*} + \varepsilon\right) \log\left(re^{L(r)}\right) \\ i.e.,\ \frac{\log^{[m]} \mu(r, f\circ g)}{\log\left(re^{L(r)}\right)} &\leq \frac{\beta\left(\rho_f^{[n]L^*} + \varepsilon\right) \log\left(re^{L(r)}\right)}{\log\left(re^{L(r)}\right)} \\ i.e.,\ \liminf_{r\to\infty} \frac{\log^{[m]} \mu(r, f\circ g)}{\log\left(re^{L(r)}\right)} &= \lambda_{f\circ g}^{[m]L^*} < \infty. \end{aligned}$$

This is a contradiction.
Thus the theorem follows. □

In the line of Theorem 9, the following theorem may also be proved:

Remark 3. *Theorem 9 is also valid with "limit superior" instead of "limit" if* $\lambda_{f\circ g}^{[m]L^*}=\infty$ *is replaced by* $\rho_{f\circ g}^{[m]L^*}=\infty$ *and the other conditions remaining the same.*

Theorem 10. *Let f and g be any two entire functions with* $\rho_f^{[n]L^*}<\infty$ *and* $\lambda_{f\circ g}^{[m]L^*}=\infty$ *where m and n are positive integers. Then*

$$\lim_{r\to\infty}\frac{\log^{[m]} M(r,f\circ g)}{\log^{[n]} M(r,f)}=\infty.$$

Further if $\rho_{f\circ g}^{[m]L^*}=\infty$ *instead of* $\lambda_{f\circ g}^{[m]L^*}=\infty$ *then*

$$\liminf_{r\to\infty}\frac{\log^{[m]} M(r,f\circ g)}{\log^{[n]} M(r,f)}=\infty.$$

Corollary 11. *Under the assumptions of Theorem 9 or Remark 3 and Theorem 10,*

$$\lim_{r\to\infty}\frac{\log^{[m-1]} \mu(r,f\circ g)}{\log^{[n-1]} \mu(r,f)}=\infty$$

and

$$\lim_{r\to\infty}\frac{\log^{[m-1]} M(r,f\circ g)}{\log^{[n-1]} M(r,f)}=\infty.$$

Proof. By Theorem 9 or Remark 3 we obtain for all sufficiently large values of r and for $K>1$ that

$$\begin{aligned}\log^{[m]}\mu(r,f\circ g) &> K\log^{[n]}\mu(r,f)\\ i.e.,\ \log^{[m-1]}\mu(r,f\circ g) &> \log^{[n-1]}\{\mu(r,f)\}^K,\end{aligned}$$

from which the first part of the corollary follows.
Similary, from Theorem 10 the second part of the corollary is established. □

Remark 4. *The condition* $\lambda_{f\circ g}^{[m]L^*}=\infty$ *is necessary in Theorem 9, Theorem 10 and Corollary 11 which is evident from the following example :*

Example 1. *Let $f=\exp z$, $g=z$, $m=n=2$ and $L(r)=\frac{1}{p}\exp\left(\frac{1}{r}\right)$ where p is any positive real number.*
Also

$$\rho_f^{L^*}=1<\infty \text{ and } \lambda_{f\circ g}^{L^*}=1<\infty.$$

Now taking $R=2r$ in the inequality $\mu(r,f)\le M(r,f)\le\frac{R}{R-r}\mu(R,f)$ $\{cf.$ [10] $\}$ we get that

$$\mu(r,f\circ g)\le M(r,f\circ g)=\exp r$$

and

$$\mu(r,f) \leq M(r,f) = \exp r.$$

Also

$$\mu(r,f\circ g) \geq \frac{1}{2}M\left(\frac{r}{2}, f\circ g\right) = \frac{1}{2}\exp\left(\frac{r}{2}\right)$$

and

$$\mu(r,f) \geq \frac{1}{2}M\left(\frac{r}{2}, f\right) = \frac{1}{2}\exp\left(\frac{r}{2}\right).$$

Therefore

$$\lim_{r\to\infty}\frac{\log^{[2]}\mu(r,f\circ g)}{\log^{[2]}\mu(r,f)} = \lim_{r\to\infty}\frac{\log^{[2]}M(r,f\circ g)}{\log^{[2]}M(r,f)} = 1.$$

Also

$$\lim_{r\to\infty}\frac{\log M(r,f\circ g)}{\log M(r,f)} = 1 \text{ and}$$

$$\frac{1}{2} \leq \liminf_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(r,f)} \leq \limsup_{r\to\infty}\frac{\log\mu(r,f\circ g)}{\log\mu(r,f)} \leq 2.$$

Remark 5. *Considering $f = \exp z$, $g = z$, $m = n = 2$ and $L(r) = \frac{1}{p}\exp\left(\frac{1}{r}\right)$ for any positive real number p, one can also verify that the condition $\rho_{f\circ g}^{[m]L^*} = \infty$ in Theorem 10, Remark 3 and Corollary 11 is essential.*

Theorem 12. *If f and g be any two entire functions such that (i) $0 < \rho_f^{[n]L^*} < \infty$, (ii) $0 < \sigma_f^{[n-1]L^*} < \infty$, (iii) $\rho_{f\circ g}^{[m]L^*} = \rho_f^{[n]L^*}$ and (iv) $\sigma_{f\circ g}^{[m-1]L^*} < \infty$. Then for any $\beta > 1$,*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log^{[m-1]}\mu(r,f\circ g)}{\log^{[n-1]}\mu(r,f)} &\leq \frac{\beta^{\rho_f^{[n]L^*}}\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_f^{[n-1]L^*}} \text{ and}\\ \frac{\sigma_{f\circ g}^{[m-1]L^*}}{\beta^{\rho_{f\circ g}^{[m]L^*}}.\sigma_f^{[n-1]L^*}} &\leq \limsup_{r\to\infty}\frac{\log^{[m-1]}\mu(r,f\circ g)}{\log^{[n-1]}\mu(r,f)}.\end{aligned}$$

Proof. From the definition of generalised L^*-type and in view of the inequality $\mu(r,f) \leq M(r,f)$ $\{cf.$ [10] $\}$, we obtain for all sufficiently large values of r that

$$\begin{aligned}\log^{[m-1]}\mu(r,f\circ g) &\leq \log^{[m-1]}M(r,f\circ g)\\ &\leq \left(\sigma_{f\circ g}^{[m-1]L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}} \qquad (11)\end{aligned}$$

and

$$\log^{[n-1]}\mu(r,f) \leq \left(\sigma_f^{[n-1]L^*}+\varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_f^{[n]L^*}}. \qquad (12)$$

Also taking $R = \beta r$ in the inequality $M(r,f) \leq \frac{R}{R-r}\mu(R,f)$ $\{cf.$ [10] $\}$ we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} & \log^{[m-1]}\mu(r,f\circ g) \\ \geq \; & \log^{[m-1]} M\left(\frac{r}{\beta}, f\circ g\right) + O(1) \geq \left(\sigma_{f\circ g}^{[m-1]L^*} - \varepsilon\right)\left\{\left(\frac{r}{\beta}\right)e^{L\left(\frac{r}{\beta}\right)}\right\}^{\rho_{f\circ g}^{[m]L^*}} \end{aligned}$$

$$i.e., \;\; \log^{[m-1]}\mu(r,f\circ g) \geq \frac{\left(\sigma_{f\circ g}^{[m-1]L^*} - \varepsilon\right)}{\beta^{\rho_{f\circ g}^{[m]L^*}}}\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}} + O(1) \tag{13}$$

and

$$\begin{aligned} & \log^{[n-1]}\mu(r,f) \\ \geq \; & \log^{[n-1]} M\left(\frac{r}{\beta}, f\right) + O(1) \geq \left(\sigma_{f}^{[n-1]L^*} - \varepsilon\right)\left\{\left(\frac{r}{\beta}\right)e^{L\left(\frac{r}{\beta}\right)}\right\}^{\rho_{f}^{[n]L^*}} \end{aligned}$$

$$i.e., \;\; \log^{[n-1]}\mu(r,f) \geq \frac{\left(\sigma_{f}^{[n-1]L^*} - \varepsilon\right)}{\beta^{\rho_{f}^{[n]L^*}}}\left\{re^{L(r)}\right\}^{\rho_{f}^{[n]L^*}} + O(1). \tag{14}$$

Now from (11) and (14) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]}\mu(r,f\circ g)}{\log^{[n-1]}\mu(r,f)} \leq \frac{\beta^{\rho_{f}^{[n]L^*}}\left(\sigma_{f\circ g}^{[m-1]L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}}}{\left(\sigma_{f}^{[n-1]L^*} - \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f}^{[n]L^*}} + O(1)} \; . \tag{15}$$

In view of the condition (iii) we get from (15) that

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}\mu(r,f\circ g)}{\log^{[n-1]}\mu(r,f)} \leq \frac{\beta^{\rho_{f}^{[n]L^*}}\left(\sigma_{f\circ g}^{[m-1]L^*} + \varepsilon\right)}{\left(\sigma_{f}^{[n-1]L^*} - \varepsilon\right)} \; .$$

As $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\liminf_{r\to\infty}\frac{\log^{[m-1]}\mu(r,f\circ g)}{\log^{[n-1]}\mu(r,f)} \leq \frac{\beta^{\rho_{f}^{[n]L^*}}\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_{f}^{[n-1]L^*}} \; . \tag{16}$$

Again from (12) and (13) we get for a sequence of values of r tending to infinity that

$$\frac{\log^{[m-1]}\mu(r,f\circ g)}{\log^{[n-1]}\mu(r,f)} \geq \frac{\left(\sigma_{f\circ g}^{[m-1]L^*} - \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f\circ g}^{[m]L^*}} + O(1)}{\beta^{\rho_{f\circ g}^{[m]L^*}}\left(\sigma_{f}^{[n-1]L^*} + \varepsilon\right)\left\{re^{L(r)}\right\}^{\rho_{f}^{[n]L^*}}} \; . \tag{17}$$

Since $\rho_{f\circ g}^{[m]L^*} = \rho_f^{[n]L^*}$, we obtain from (17) that

$$\limsup_{r\to\infty} \frac{\log^{[m-1]} \mu(r, f\circ g)}{\log^{[n-1]} \mu(r,f)} \geq \frac{\left(\sigma_{f\circ g}^{[m-1]L^*} - \varepsilon\right)}{\beta^{\rho_{f\circ g}^{[m]L^*}} \left(\sigma_f^{[n-1]L^*} + \varepsilon\right)} .$$

As $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\limsup_{r\to\infty} \frac{\log^{[m-1]} \mu(r, f\circ g)}{\log^{[n-1]} \mu(r,f)} \geq \frac{\sigma_{f\circ g}^{[m-1]L^*}}{\beta^{\rho_{f\circ g}^{[m]L^*}} .\sigma_f^{[n-1]L^*}} . \tag{18}$$

Thus the theorem follows from (16) and (18). □

In the line of Theorem 12, we may state the following theorem without proof :

Theorem 13. *If f and g be any two entire functions with (i) $0 < \rho_g^{[n]L^*} < \infty$, (ii) $0 < \sigma_g^{[n-1]L^*} < \infty$, (iii) $\rho_{f\circ g}^{[m]L^*} = \rho_g^{[n]L^*}$ and (iv) $\sigma_{f\circ g}^{[m-1]L^*} < \infty$. Then for any $\beta > 1$,*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} \mu(r, f\circ g)}{\log^{[n-1]} \mu(r,g)} \leq \frac{\beta^{\rho_g^{[n]L^*}} \sigma_{f\circ g}^{[m-1]L^*}}{\sigma_g^{[n-1]L^*}} \text{ and}$$

$$\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\beta^{\rho_{f\circ g}^{[m]L^*}} .\sigma_g^{[n-1]L^*}} \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} \mu(r, f\circ g)}{\log^{[n-1]} \mu(r,g)} .$$

Theorem 14. *If f and g be any two entire functions such that (i) $0 < \rho_f^{[n]L^*} < \infty$, (ii) $0 < \sigma_f^{[n-1]L^*} < \infty$, (iii) $\rho_{f\circ g}^{[m]L^*} = \rho_f^{[n]L^*}$ and (iv) $\sigma_{f\circ g}^{[m-1]L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g)}{\log^{[n-1]} M(r,f)} \leq \left(\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_f^{[n-1]L^*}}\right) \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g)}{\log^{[n-1]} M(r,f)} .$$

Theorem 15. *If f and g be any two entire fucntions such that (i) $0 < \rho_g^{[n]L^*} < \infty$, (ii) $0 < \sigma_g^{[n-1]L^*} < \infty$, (iii) $\rho_{f\circ g}^{[m]L^*} = \rho_g^{[n]L^*}$ and (iv) $\sigma_{f\circ g}^{[m-1]L^*} < \infty$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g)}{\log^{[n-1]} M(r,g)} \leq \left(\frac{\sigma_{f\circ g}^{[m-1]L^*}}{\sigma_g^{[n-1]L^*}}\right) \leq \limsup_{r\to\infty} \frac{\log^{[m-1]} M(r, f\circ g)}{\log^{[n-1]} M(r,g)} .$$

The proof of Theorem 14 and Theorem 15 are omitted because those can be carried out in the line of Theorem 12 and Theorem 13 respectively.

4 Open Problem

Actually this paper deals with the extension of the works on the growth properties of composite entire functions on the basis of their generalised L^*-order and generalised L^*-type. Further in order to determine the relative growth of two entire functions having same non zero finite generalised L^*-lower order, one may introduce the definition of generalised L^*-weak type denoted as $\tau_f^{[m-1]L^*}$ of entire functions having finite generalised L^*-lower order in the following way:

$$\tau_f^{[m-1]L^*} = \liminf_{r\to\infty} \frac{\log^{[m]} M(r,f)}{\left[re^{L(r)}\right]^{\lambda_f^{[m]L^*}}}\ ,\ 0 < \lambda_f^{[m]L^*} < \infty$$

and therefore using this growth indicator one may calculate the above growth rates of composite entire functions under some different conditions. In this connection, the following natural questions may arrise for the worker of this branch :

1. Can these theories be modified by the treatment of the notions of L^*-relative order (respectively generalised L^*-relative order), L^*-relative type (respectively generalised L^*-relative type) and L^*-relative weak type (respectively generalised L^*-relative weak type)?

2. Further can some extensions of the same be done for special type of linear differential polynomials viz. the wronskians, differential polynomials and differential monomials?

References

[1] J. Clunie, *The composition of entire and meromorphic functions*, Mathematical Essays dedicated to A. J. Macintyre,Ohio University Press (1970), 75-92.

[2] S.K. Datta and T. Biswas, *On the definition of a meromorphic function of order zero*, International Mathematical Forum, 4 (37) (2009), 1851-1861.

[3] S. K. Datta, T. Biswas and Md. A. Hoque, *Maxumum modulus and maxumum term based growth analysis of entire function in the light of slowly changing function*, Investigations in Mathematical Sciences, 3 (1) (2013), 113-125.

[4] T.V. Lakshminarasimhan, *A note on entire functions of bounded index*, J. Indian Math. Soc., 38 (1974), 43-49.

[5] I. Lahiri and N.R. Bhattacharjee, *Functions of L-bounded index and of non-uniform L-bounded index*, Indian J. Math., 40 (1), (1998), 43-57.

[6] D. Sato, *On the rate of growth of entire functions of fast growth*, Bull. Amer. Math. Soc., 69 (1963), 411-414.

[7] S.K. Singh and G.P. Barker, *Slowly changing functions and their applications*, Indian J. Math., 19 (1), (1977), 1-6.

[8] D. Somasundaram and R. Thamizharasi, *A note on the entire functions of L-bounded index and L-type*, Indian J. Pure Appl.Math., 19 (3), (March 1988), 284-293.

[9] A. P. Singh, *On maximum term of composition of entire functions*, Proc. Nat. Acad. Sci. India, 59(A) (Part I), (1989),103-115.

[10] A. P. Singh and M. S. Baloria, *On maximum modulus and maximum term of composition of entire functions*, Indian J. Pure Appl. Math., 22 (12), (1991), 1019-1026.

[11] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company, (1949).