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A common fixed point theorem in weak non-Archimedean intuitionistic fuzzy metric spaces

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Abstract

In this paper, we introduce weak non-Archimedean intuitionistic fuzzy metric space and study some properties of the topology induced by a weak non-Archimedean intuitionistic fuzzy metric and also we prove a common fixed theorem for generalized $\psi-\phi$-contractive mappings.

Keywords: *non-Archimedean intuitionistic fuzzy metric, $\psi-\phi$-contractive mappings, fixed points.*

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1 Introduction and preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [3] in 1965. Since that time, to use this concept in topology and analysis, many authors have expansively developed the theory of fuzzy sets and applications. Especially, Deng [13], Erceg [14], Kaleva and Seikkala [15], Kramosil and Michalek [17], Georege and Veeramani [10] have introduced the concept of fuzzy metric space in different ways. Grabiec [6] initiated the study of fixed point theory in fuzzy metric spaces, which is parallel to fixed point theory in probabalistic metric space. Many authors followed this concept by introducing and investigating the different types of fuzzy contractive mappings. For example, Gregori and Sapena [16] have introduced fuzzy contractive mappings and proved Banach contraction theorem in the meaning of George and Veeramani by using a strong condition for completeness. These result have become recently of interest for many authors. Mihet [9], who realised this strong condition, defined a new fuzzy contraction called ψ-contraction which enlarges the class of fuzzy contractive mappings of Gregori and Sapena, and proved fixed point theorems under different hypotheses in fuzzy metric space in the meaning of Kramosil and Michalek. For instance, he assumed that the space under consideration is a non-Archimedean fuzzy metric spaces and he proved a fixed point theorem for fuzzy ψ-contractive mapping in this space [8]. Then, Wang in [18] proved similar theorem in fuzzy metric space.

Recently, Vetro [1] introduced the concept of weak non-Archimedean fuzzy metric space and proved common fixed point results for a pair of generalized contractive type mappings. Also, he present that every non-Archimedean fuzzy metric space is itself a weak non-Archimedean fuzzy metric space.

On the other hand, Atanassov [7] introduced and studied the notion of intuitionistic fuzzy set by generalizing the noiton of fuzzy set. An intuitionistic fuzzy set gives both a membership degree and a nonmembership degree. Using the idea of intuitionistic fuzzy set, Park [5] defined the notion of intuitionistic fuzzy metric space with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric space due to George and Veeramani [10] and proved some known results of metric spaces for intuitionistic fuzzy metric space. Since then, many authors studied the structure of intuitionistic fuzzy metric space and fixed point theorems in those spaces [4, 11, 12].

The aim of this paper is to introduce the concept of weak non-Archimedian intuitionistic fuzzy metic space by changing triangular inequalty with similar approach [1, 5] and study some properties of the topology induced by a weak non-Archimedean intuitionistic fuzzy metric.

Also we prove a common fixed point theorem in weak non-Archimedean intuitionistic fuzzy metric space for generalized ψ-ϕ-contractive mappings. Since every non-Archimedean intuitionistic fuzzy metric is a weak non-Archimedean intuitionistic fuzzy metric, our work generalizes papers of non-Archimedean intuitionistic fuzzy metric regarding fixed point theory.

Now we begin our paper with giving some usefull definitions.

Definition 1.1 ([2]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(i) $*$ is associative and commutative,

(ii) $a * 1 = a$ for every $a \in [0,1]$,

(iii) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

If in addition, $*$ is continuous, then $*$ is called a continuous t-norm. Typical examples of a continuous t-norms are $a*b = \min\{a,b\}$, $a*b = ab/\max\{a,b,\lambda\}$ for $0 < \lambda < 1$, $a * b = ab$, $a * b = \max\{a+b-1, 0\}$.

Definition 1.2 ([2]) A binary operation $\diamond : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following conditions:

(i) $\diamond$ is associative and commutative,

(ii) $a \diamond 0 = a$ for every $a \in [0,1]$,

(iii) $a \diamond b \leq c \diamond d$ whenever $a \leq c$ and $b \leq d$, for $a, b, c, d \in [0,1]$.

If in addition, $\diamond$ is continuous, then $\diamond$ is called a continuous t-conorm. Typical examples of a continuous t-conorms are $a \diamond b = a+b-ab$, $a \diamond b = \max\{a,b\}$, $a \diamond b = \min\{a+b, 1\}$.

Remark 1.3

(a) For any $r_1, r_2 \in (0,1)$ with $r_1 > r_2$, there exist $r_3, r_4 \in (0,1)$ such that $r_1 * r_3 \geq r_2$ and $r_1 \geq r_4 \diamond r_2$.

(b) For any $r_5 \in (0,1)$, there exist $r_6, r_7 \in (0,1)$ such that $r_6 * r_6 \geq r_5$ and $r_7 \diamond r_7 \leq r_5$.

Definition 1.4 ([17]) A fuzzy metric space (in the sense of Kramosil and Michalek) is a triple $(X, M, *)$, where X is a nonempty set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times [0,\infty)$, satisfying the following conditions: for all $x, y, z \in X$, $s, t > 0$,

(KM$_1$) $M(x, y, 0) = 0$,

(KM$_2$) $M(x, y, t) = 1$ iff $x = y$,

(KM$_3$) $M(x, y, t) = M(y, x, t)$,

(KM$_4$) $M(x, y, \cdot) : [0,\infty) \to [0,1]$ is left continuous, (KM$_5$) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$.

If, in the above definition, the triangular inequality (KM$_5$) is replaced by: $\forall x,y,z \in X,\ s,t > 0$,

$$M\left(x,z,\max\{t,s\}\right) \geq M(x,y,t) * M(y,z,s)$$

or equivalently

$$M(x,z,t) \geq M(x,y,t) * M(y,z,t)$$

then the triple $(X, M, *)$ is called a non-Archimedean fuzzy metric space [19].

Definition 1.5 ([1]) If, in the Definition 1, the triangular inequality (KM$_5$) is replaced by: $\forall x,y,z \in X,\ s,t > 0$,

$$M(x,z,t) \geq \max\{M(x,y,t) * M(y,z,t/2), M(x,y,t/2) * M(y,z,t)\}$$

then the triple $(X, M, *)$ is called a weak non-Archimedean fuzzy metric space.

Remark 1.6 A weak non-Archimedean fuzzy metric space is not necessarily a fuzzy metric space. If $M(x,y,\cdot)$ is nondecreasing, then a weak non-Archimedean fuzzy metric space is a fuzzy metric space.

Example 1.7 Let $X = [0,\infty)$, $a * b = ab$ for every $a,b \in [0,1]$. Define $M(x,y,t)$ by: $M(x,y,0) = 0$, $M(x,x,t) = 1$ for all $t > 0$, $M(x,y,t) = t$ for $x \neq y$ and $0 < t \leq 1$, $M(x,y,t) = t/2$ for $x \neq y$ and $1 < t \leq 2$, $M(x,y,t) = 1$ for $x \neq y$ and $t > 2$. Then $(X, M, *)$ is a weak non-Archimedean fuzzy metric space but it is not a fuzzy metric space [1].

Definition 1.8 [5] A 5-tuple $(X, M, N, *, \diamond)$ is said to be an intuitionistic fuzzy metric space if X an arbitrary set, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X \times X \times (0,\infty)$ satisfying the following conditions: for all $x,y,z \in X,\ s,t > 0$,

(IFM$_1$) $M(x,y,t) + N(x,y,t) \leq 1$,

(IFM$_2$) $M(x,y,t) > 0$,

(IFM$_3$) $M(x,y,t) = 1$ if and only if $x = y$,

(IFM$_4$) $M(x,y,t) = M(y,x,t)$,

(IFM$_5$) $M(x,z,t+s) \geq M(x,y,t) * M(y,z,s)$,

(IFM$_6$) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous, (IFM$_7$) $N(x,y,t) > 0$,

(IFM$_8$) $N(x,y,t) = 0$ if and only if $x = y$,

(IFM$_9$) $N(x,y,t) = N(y,x,t)$,

(IFM$_{10}$) $N(x,z,t+s) \leq N(x,y,t) \diamond N(y,z,s)$,

(IFM$_{11}$) $N(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Remark 1.9 Every fuzzy metric space $(X,M,*)$ is an intuitionistic fuzzy metric space of the form $(X,M,1-M,*,\diamond)$ such that t-norm $*$ and t-conorm $\diamond$ are associated, i.e., $x \diamond y = 1 - ((1-x) * (1-y))$ for any $x, y \in X$.

Remark 1.10 In intuitionistic fuzzy metric space X, $M(x,y,\cdot)$ is non-decreasing and $N(x,y,\cdot)$ is non-increasing for all $x,y \in X$.

In the above definition, if the triangular inequality (IFM_5) and (IFM_{10}) are replaced by the following:

$$(NA) \quad \begin{array}{l} M(x,z,\max\{t,s\}) \geq M(x,y,t) * M(y,z,s) \\ N(x,z,\max\{t,s\}) \leq N(x,y,t) \diamond N(y,z,s) \end{array}$$

or equivalently

$$\begin{array}{l} M(x,z,t) \geq M(x,y,t) * M(y,z,t) \\ N(x,z,t) \leq N(x,y,t) \diamond N(y,z,t) \end{array}$$

then $(X,M,N,*,\diamond)$ is called non-Archimedean intuitionistic fuzzy metric space [4]. It is easy to check that the triangle inequality (NA) implies (IFM_5) and (IFM_{10}), that is, every non-Archimedean intuitionistic fuzzy metric space is itself an intuitionistic fuzzy metric space.

Example 1.11 Let X be a non-empty set with at least two elements. Define $M(x,y,t)$ by: If we define the intuitionistic fuzzy set (X,M,N) by $M(x,x,t) = 1$, $N(x,x,t) = 0$ for all $x \in X$ and $t > 0$, and $M(x,y,t) = 0$, $N(x,y,t) = 1$ for $x \neq y$ and $0 < t \leq 1$, and $M(x,y,t) = 1$, $N(x,y,t) = 0$ for $x \neq y$ and $t > 1$. Then $(X,M,N,*,\diamond)$ is a non-Archimedean intuitionistic fuzzy metric space with arbitrary continuous t-norm $*$ and t-conorm $\diamond$. Clearly $(X,M,N,*,\diamond)$ is also an intuitionistic fuzzy metric space.

2 Weak non-Archimedean intuitionistic fuzzy metric spaces and properties

In this section, we introduce the concept of weak non-Archimedean intuitionistic fuzzy metric space and the induced topology giving some properties of that.

Definition 2.1 In Definition 1, if the triangular inequality (IFM_5) and (IFM_{10}) are replaced by the following:

$$(WNA)\quad \begin{array}{l} M(x,z,t) \geq \max\{M(x,y,t) * M(y,z,t/2), M(x,y,t/2) * M(y,z,t)\} \\ N(x,z,t) \leq \min\{N(x,y,t) \diamond N(y,z,t/2), N(x,y,t/2) \diamond N(y,z,t)\} \end{array}$$

for all $x, y, z \in X$ and $t > 0$, then $(X, M, N, *, \diamond)$ is said to be a weak non-Archimedean intuitionistic fuzzy metric space.

Obviously every non-Archimedean intuitionistic fuzzy metric space is itself a weak non-Archimedean intuitionistic fuzzy metric space.

Condition (WNA) does not implies that $M(x, y, \cdot)$ is non decreasing and $N(x, y, \cdot)$ is non increasing. Thus a weak non-Archimedean intuitionistic fuzzy metric space is not necessarily an intuitionistic fuzzy metric space.

Example 2.2 Let $X = [0, \infty)$ and define $M(x, y, t)$, $N(x, y, t)$ by

$$M(x,y,t) = \begin{cases} 1, & x = y \\ \frac{t}{t+1}, & x \neq y \end{cases}$$

and

$$N(x,y,t) = \begin{cases} 0, & x = y \\ \frac{1}{t+1}, & x \neq y \end{cases}$$

for all $t > 0$. $(X, M, N, *, \diamond)$ is a weak non Archimedean intuitionistic fuzzy metric space with $a * b = ab$ and $a \diamond b = a + b - ab$ for every $a, b \in [0, 1]$. It is easy to check that (IFM_1)-(IFM_4), (IFM_6)-(IFM_9) and (IFM_{11}) are satisfied. With respect to (WNA), we have that

$$\begin{aligned} M(x,z,t) &= \frac{t}{t+1} \geq \frac{t}{t+1}\frac{t}{t+2} \\ &= \max\left\{\frac{t}{t+1}\frac{t}{t+2}, \frac{t}{t+2}\frac{t}{t+1}\right\} \\ &= \max\{M(x,y,t) * M(y,z,t/2), M(x,y,t/2) * M(y,z,t)\} \end{aligned}$$

and

$$\begin{aligned} N(x,z,t) &= \frac{1}{t+1} \leq \frac{1}{t+1}\left(1+\frac{2t}{t+2}\right) \\ &= \min\left\{\frac{1}{t+1}+\frac{2}{t+2}-\frac{2}{(t+1)(t+2)}, \frac{2}{t+2}+\frac{1}{t+1}-\frac{2}{(t+1)(t+2)}\right\} \\ &= \min\left\{N(x,y,t)\diamond N(y,z,t/2), N(x,y,t/2)\diamond N(y,z,t)\right\}. \end{aligned}$$

Definition 2.3 Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space, and let $r \in (0,1)$, $t > 0$ and $x \in X$. The set $B(x,r,t) = \{y \in X : M(x,y,t) > 1-r, N(x,y,t) < r\}$ is called the open ball with centre x and radius r with respect to t.

Proposition 2.4 Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space, then every open ball is an open set.

Proof Let $B(x,r,t)$ be an open ball with centre x and radius r with respect to t. Now, $y \in B(x,r,t)$ implies $r_0 = M(x,y,t) > 1-r$ and $N(x,y,t) < r$. Then, there exist $s \in (0,1)$ such that $r_0 > 1-s > 1-r$. Hence, from Remark 1 there exist $r_1, r_2 \in (0,1)$ such that $r_0 * r_1 > 1-s$ and $(1-r_0)\diamond(1-r_2) \leq s$. Put $r_3 = \max\{r_1, r_2\}$ and consider the open ball $B(y, 1-r_3, t/2)$. We claim that $B(y, 1-r_3, t/2) \subset B(x,r,t)$. Now, let $z \in B(y, 1-r_3, t/2)$. Then $M(y,z,t/2) > r_3$ and $\mathrm{N}(y,z,t/2) < 1-r_3$. Therefore

$$M(x,z,t) \geq M(x,y,t) * M(y,z,t/2) \geq r_0 * r_3 \geq r_0 * r_1 > 1-s > 1-r$$

and

$$N(x,z,t) \leq N(x,y,t)\diamond N(y,z,t/2) \leq (1-r_0)\diamond(1-r_3) \leq (1-r_0)\diamond(1-r_2) \leq s < r.$$

Thus $z \in B(x,r,t)$ and hence $B(y, 1-r_3, t/2) \subset B(x,r,t)$.

Remark 2.5 Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space, the family

$$\tau = \{A \subset X : \forall x \in A, \exists t > 0 \text{ and } r \in (0,1) \text{ such that } B(x,r,t) \subset A\}$$

is a topology on X.

Proposition 2.6 Every weak non-Archimedean intuitionistic fuzzy metric space is Hausdorff.

Proof Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space and $x, y \in X$, with $x \neq y$. Then $0 < M(x,y,t) < 1$ and $0 <$

$N(x,y,t) < 1$. Put $r_1 = M(x,y,t)$, $r_2 = N(x,y,t)$ and $r = \max\{r_1, 1-r_2\}$. For each $r_0 \in (r,1)$, from Remark 1 there exist r_3 and r_4 such that $r_3 * r_3 \geq r_0$ and $(1-r_4) \diamond (1-r_4) \leq 1-r_0$. Put $r_5 = \max\{r_3, r_4\}$ and consider the open balls $B(x, 1-r_5, t)$ and $B(y, 1-r_5, t/2)$. Clearly $B(x, 1-r_5, t) \cap B(y, 1-r_5, t/2) = \emptyset$. For if there exists $z \in B(x, 1-r_5, t) \cap B(y, 1-r_5, t/2)$, then

$$r_1 = M(x,y,t) \geq M(x,z,t) * M(z,y,t/2) \geq r_5 * r_5 \geq r_3 * r_3 \geq r_0 > r_1$$

and

$$\begin{aligned} r_2 = N(x,y,t) \leq N(x,z,t) \diamond N(z,y,t/2) \leq (1-r_5) \diamond (1-r_5) &\leq (1-r_4) \diamond (1-r_4) \\ &\leq 1-r_0 < r_2 \end{aligned}$$

which is a contradiction.

Proposition 2.7 Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space and τ be the topology on X induced by the weak non-Archimedean intuitionistic fuzzy metric. Then for a sequence $\{x_n\}$ in X, $x_n \to x$ if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$ for all $t > 0$.

Proof Fix $t > 0$. Suppose $x_n \to x$. Then for $r \in (0,1)$, there exist $n_0 \in \mathbb{N}$ such that $x_n \in B(x,r,t)$ for all $n \geq n_0$.Then $1 - M(x_n, x, t) < r$ and $N(x_n, x, t) < r$ and hence $M(x_n, x, t) \to 1$and $N(x_n, x, t) \to 0$ as $n \to \infty$.

Conversely, if for each $t > 0$, $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$, then for $r \in (0,1)$, there exist $n_0 \in \mathbb{N}$ such that $1 - M(x_n, x, t) < r$ and $N(x_n, x, t) < r$ for all $n \geq n_0$. It follows that $M(x_n, x, t) > 1 - r$ and $N(x_n, x, t) < r$ for all $n \geq n_0$. Thus $x_n \in B(x, r, t)$ for all $n \geq n_0$ and hence $x_n \to x$.

Definition 2.8 Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space. A sequence $\{x_n\}$ in X called a Cauchy sequence, if for each $r \in (0,1)$ and $t > 0$ there exist $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - r$ and $N(x_n, x_m, t) < r$ for all $m, n \geq n_0$.

Definition 2.9 The weak non-Archimedean intuitionistic fuzzy metric space $(X, M, N, *, \diamond)$ is said to be complete if every Cauchy sequence is convergent.

Lemma 2.10 Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space and $\{x_n\} \subset X$ be a sequence convergent to $x \in X$ then $\lim\limits_{n\to\infty} M(y, x_n, t) = M(y, x, t)$ and $\lim\limits_{n\to\infty} N(y, x_n, t) = N(y, x, t)$.

Proof Since

$$M(y, x_n, t) \geq M(y, x, t) * M(x, x_n, t/2)$$

$$M(y, x, t) \geq M(y, x_n, t) * M(x, x_n, t/2)$$

and

$$N(y, x_n, t) \leq N(y, x, t) \diamond N(x, x_n, t/2)$$

$$N(y, x, t) \leq N(y, x_n, t) \diamond N(x, x_n, t/2)$$

as taking $n \to \infty$, we have

$$M(y, x, t) \leq \lim_{n\to\infty} M(y, x_n, t) \leq M(y, x, t)$$

and

$$N(y, x, t) \leq \lim_{n\to\infty} N(y, x_n, t) \leq N(y, x, t).$$

Hence, the proof is finished.

3 ψ-ϕ-contractive mappings and an application to fixed point theorems

In this section, we define generalized ψ-ϕ-contractive mappings and prove a common fixed point theorem.

Let Ψ be the class of all mappings $\psi : [0, 1] \to [0, 1]$ and Φ be the class of all mappings $\phi : [0, 1] \to [0, 1]$ such that

(i) ψ is nondecreasing and continuous,

(ii) $\psi(t) > t$ for all $t \in (0, 1)$,

(iii) ϕ is nondecreasing and continuous,

(iv) $\phi(t) < t$ for all $t \in (0, 1)$.

Firstly, we give some lemmas.

Lemma 3.1 If $\psi \in \Psi$ and $\phi \in \Phi$, then $\psi(1) = 1$ and $\phi(0) = 0$.

Lemma 3.2 If $\psi \in \Psi$ and $\phi \in \Phi$, then $\lim_{n\to\infty} \psi^n(t) = 1$ and $\lim_{n\to\infty} \phi^n(t) = 0$ for all $t \in (0, 1)$.

Proof Suppose that $\lim_{n\to\infty}\psi^n(t_0) = l < 1$ for some $t_0 \in (0,1)$. By the monotonicity and continuity of ψ, we have

$$l = \lim_{n\to\infty}\psi^{n+1}(t_0) = \psi\left(\lim_{n\to\infty}\psi^n(t_0)\right) = \psi(l) > l$$

which is a contradiction.

By the same way, assume that $\lim_{n\to\infty}\phi^n(t_0) = m > 0$ for some $t_0 \in (0,1)$. By the monotonicity and continuity of ϕ, we have

$$m = \lim_{n\to\infty}\phi^{n+1}(t_0) = \phi\left(\lim_{n\to\infty}\phi^n(t_0)\right) = \phi(m) < m$$

which is a contradiction.

Definition 3.3 Let $(X, M, N, *, \diamond)$ be a weak non-Archimedean intuitionistic fuzzy metric space, $\psi \in \Psi$ and $\phi \in \Phi$. Let $f, g : X \to X$, (f, g) is a pair of generalized $\psi - \phi$-contractive mappings if the following implications hold: for every $x, y, z \in X$ and $t \in (0, \infty)$

$$\begin{array}{c} M(x,y,t) > 0 \Rightarrow M(f(x), g(y), t) \geq \psi(m(x,y,t)) \\ N(x,y,t) < 1 \Rightarrow N(f(x), g(y), t) \leq \phi(n(x,y,t)), \end{array}$$

where

$$\begin{array}{c} m(x,y,t) = \min\{M(x,y,t), M(x,f(x),t), M(y,g(y),t)\} \\ n(x,y,t) = \max\{N(x,y,t), N(x,f(x),t), N(y,g(y),t)\}. \end{array}$$

Theorem 3.4 Let $(X, M, N, *, \diamond)$ be an complete weak non-Archimedean intuitionistic fuzzy metric space and $f, g : X \to X$, (f, g) is a pair of generalized ψ-ϕ-contractive mappings. If there exist $x_0 \in X$ such that $M(x_0, f(x_0), t) > 0$, and $N(x_0, f(x_0), t) < 1$ for all $t > 0$, then f and g have a unique common fixed point.

Proof Let $x_0 \in X$ be such that $M(x_0, f(x_0), t) > 0$, and $N(x_0, f(x_0), t) < 1$ for all $t > 0$. Fix $x_0 \in X$ and define the sequence (x_n) by

$$x_1 = f(x_0), \quad x_2 = g(x_1), \ ..., \quad x_{2n+1} = f(x_{2n}), \quad x_{2n+2} = g(x_{2n+1}), ...$$

we have for all $t > 0$

$$\begin{array}{rcl} M(x_1, x_2, t) & = & M(f(x_0), g(x_1), t) \\ & \geq & \psi(m(x_0, x_1, t)) \\ & = & \psi(M(x_0, x_1, t)) > 0, \end{array}$$

$$\begin{aligned} M(x_2, x_3, t) &= M(f(x_2), g(x_1), t) \\ &\geq \psi(m(x_2, x_1, t)) \\ &= \psi(M(x_2, x_1, t)) \\ &\geq \psi^2(M(x_0, x_1, t)) > 0, \end{aligned}$$

and

$$\begin{aligned} N(x_1, x_2, t) &= N(f(x_0), g(x_1), t) \\ &\leq \phi(n(x_0, x_1, t) \\ &= \phi(N(x_0, x_1, t)) < 1, \end{aligned}$$

$$\begin{aligned} N(x_2, x_3, t) &= N(f(x_2), g(x_1), t) \\ &\leq \phi(n(x_2, x_1, t)) \\ &= \phi(N(x_2, x_1, t)) \\ &\leq \phi^2(N(x_0, x_1, t)) < 1. \end{aligned}$$

Generally, for each $n \in \mathbb{N}$, we get

$$\begin{aligned} M(x_{n+1}, x_n, t) \geq \psi^n(M(x_0, x_1, t)) \\ N(x_{n+1}, x_n, t) \leq \phi^n(N(x_0, x_1, t)). \end{aligned}$$

By Lemma 3, as $n \to \infty$, we deduce that

$$\begin{aligned} \lim_{n\to\infty} M(x_{n+1}, x_n, t) = 1 \\ \lim_{n\to\infty} N(x_{n+1}, x_n, t) = 0. \end{aligned}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. If $\{x_n\}$ is not a Cauchy, then there are $r \in (0,1)$ and $t > 0$ such that for each $k \in \mathbb{N}$ there exist $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and $M\left(x_{m(k)}, x_{n(k)}, t\right) \leq 1 - r$ and $N\left(x_{m(k)}, x_{n(k)}, t\right) \geq r$. Then we can assume that $m(k)$ are odd numbers, $n(k)$ are even numbers and set

$$\begin{aligned} p(k) = \min\left\{m(k) : M\left(x_{m(k)}, x_{n(k)}, t\right) \leq 1 - r, m(k) \text{ is odd number}\right\} \\ q(k) = \min\left\{m(k) : N\left(x_{m(k)}, x_{n(k)}, t\right) \geq r, m(k) \text{ is odd number}\right\}. \end{aligned}$$

We have

$$\begin{aligned} 1 - r &\geq M\left(x_{p(k)}, x_{n(k)}, t\right) \\ &\geq M\left(x_{p(k)-2}, x_{n(k)}, t\right) * M\left(x_{p(k)-2}, x_{p(k)}, t/2\right) \\ &\geq M\left(x_{p(k)-2}, x_{n(k)}, t\right) * M\left(x_{p(k)-2}, x_{p(k)-1}, t/2\right) * M\left(x_{p(k)-1}, x_{p(k)}, t/4\right) \\ &\geq (1 - r) * M\left(x_{p(k)-2}, x_{p(k)-1}, t/2\right) * M\left(x_{p(k)-1}, x_{p(k)}, t/4\right) \end{aligned}$$

and

$$\begin{aligned} r &\le N\left(x_{q(k)}, x_{n(k)}, t\right) \\ &\le N\left(x_{q(k)-2}, x_{n(k)}, t\right) \diamond N\left(x_{q(k)-2}, x_{q(k)}, t/2\right) \\ &\le N\left(x_{q(k)-2}, x_{n(k)}, t\right) \diamond N\left(x_{q(k)-2}, x_{q(k)-1}, t/2\right) \diamond N\left(x_{q(k)-1}, x_{q(k)}, t/4\right) \\ &\le r \diamond N\left(x_{q(k)-2}, x_{q(k)-1}, t/2\right) \diamond N\left(x_{q(k)-1}, x_{q(k)}, t/4\right). \end{aligned}$$

As $k \to \infty$, we obtain

$$\lim_{k\to\infty} M\left(x_{p(k)}, x_{n(k)}, t\right) = 1 - r$$

and

$$\lim_{k\to\infty} N\left(x_{q(k)}, x_{n(k)}, t\right) = r.$$

Now, from

$$\begin{aligned} M\left(x_{p(k)}, x_{n(k)}, t\right) &\ge M\left(x_{p(k)}, x_{n(k)+1}, t\right) * M\left(x_{n(k)+1}, x_{n(k)}, t/2\right) \\ &\ge M\left(x_{p(k)+1}, x_{n(k)+1}, t\right) * M\left(x_{p(k)}, x_{p(k)+1}, t/2\right) * M\left(x_{n(k)+1}, x_{n(k)}, t/2\right) \\ &\ge \psi\left(m\left(x_{p(k)}, x_{n(k)}, t\right)\right) * M\left(x_{p(k)}, x_{p(k)+1}, t/2\right) * M\left(x_{n(k)+1}, x_{n(k)}, t/2\right) \end{aligned}$$

and

$$\begin{aligned} N\left(x_{q(k)}, x_{n(k)}, t\right) &\le N\left(x_{q(k)}, x_{n(k)+1}, t\right) \diamond N\left(x_{n(k)+1}, x_{n(k)}, t/2\right) \\ &\le N\left(x_{q(k)+1}, x_{n(k)+1}, t\right) \diamond N\left(x_{q(k)}, x_{q(k)+1}, t/2\right) \diamond N\left(x_{n(k)+1}, x_{n(k)}, t/2\right) \\ &\le \phi\left(n\left(x_{q(k)}, x_{n(k)}, t\right)\right) \diamond N\left(x_{q(k)}, x_{q(k)+1}, t/2\right) \diamond N\left(x_{n(k)+1}, x_{n(k)}, t/2\right). \end{aligned}$$

$m\left(x_{p(k)}, x_{n(k)}, t\right) = \min\left\{M\left(x_{p(k)}, x_{n(k)}, t\right), M\left(x_{n(k)}, x_{n(k)+1}, t\right), M\left(x_{p(k)}, x_{p(k)+1}, t\right)\right\}$ and $n\left(x_{q(k)}, x_{n(k)}, t\right) = \max\left\{N\left(x_{q(k)}, x_{n(k)}, t\right), N\left(x_{n(k)}, x_{n(k)+1}, t\right), M\left(x_{q(k)}, x_{q(k)+1}, t\right)\right\}$, since ψ and ϕ continuous taking limit as $k \to \infty$, we get

$$1 - r \ge \psi(1-r) * 1 * 1 = \psi(1-r) > 1 - r$$

and

$$r \le \phi(r) \diamond 0 \diamond 0 = \phi(r) < r,$$

which are contradictions. Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exist $x \in X$ such that $\lim_{n\to\infty} x_n = x$. If $f(x) \ne x$, then there exist $t > 0$ such that $M(x, f(x), t) < 1$ and $N(x, f(x)t) > 0$. From

$$M(f(x), x_{2n}, t) = M(f(x), g(x_{2n-1}), t) \ge \psi(m(x, x_{2n-1}, t))$$

and

$$N(f(x), x_{2n}, t) = N(f(x), g(x_{2n-1}), t) \le \phi(n(x, x_{2n-1}, t))$$

by Lemma 2, as $n \to \infty$, we obtain

$$M\left(f\left(x\right), x, t\right) \geq \psi\left(M\left(f(x), x, t\right)\right) > M\left(f(x), x, t\right)$$

and

$$N\left(\left(f\left(x\right), x, t\right)\right) \leq \phi\left(N\left(f(x), x, t\right)\right) < N\left(f(x), x, t\right)$$

which are contradictions. Therefore $f(x) = x$. Analogously we obtain that $g(x) = x$ and thus x is a common fixed point of f and g. Now we prove the uniqueness of the common fixed points of f, g. Assume that $x, y \in X$ are two common fixed points of f and g. If $x \neq y$,then there exist $t > 0$ such that $M\left(x, y, t\right) < 1$ and $N\left(x, y, t\right) > 0$ and hence

$$M\left(x, y, t\right) = M\left(f(x), g(y), t\right) \geq \psi\left(M\left(x, y, t\right)\right) > M\left(x, y, t\right)$$

and

$$N\left(x, y, t\right) = N\left(f(x), g(y), t\right) \leq \phi\left(N\left(x, y, t\right)\right) < N\left(x, y, t\right)$$

which are contradictions. Therefore $x = y$.

Corollary 3.5 Let $(X, M, N, *, \diamond)$ be an complete weak non-Archimedean intuitionistic fuzzy metric space and $f : X \to X$, (f, f) is a pair of generalized ψ-ϕ-contractive mappings. If there exist $x_0 \in X$ such that $M(x_0, f(x_0), t) > 0$, and $N(x_0, f(x_0), t) < 1$ for all $t > 0$, then f have a unique fixed point.

Proof In Theorem 3, if we take $f = g$ the proof is obvious.

Corollary 3.6 Let $(X, M, *)$ be an complete weak non-Archimedean fuzzy metric space and $f, g : X \to X$, (f, g) is a pair of generalized ψ-ϕ-contractive mappings. If there exist $x_0 \in X$ such that $M(x_0, f(x_0), t) > 0$, for all $t > 0$, then f and g have a unique common fixed point.

Proof Since every weak non-Archimedean fuzzy metric space is weak non-Archimedean intuitionistic fuzzy metric space, in Theorem 3, if we take $a \diamond b = 1 - ((1 - a) * (1 - b))$ and $N(x, y, t) = 1 - M(x, y, t)$ the proof is finished.

4 Conclusion

In this paper, we introduce weak non-Archimedean intuitionistic fuzzy metric space and study some properties of the topology induced by weak non-Archimedean intuitionistic fuzzy metric. Also proved a common fixed point theorem and some corollaries. This result utilizes to prove the existence theorems of solution to differential equations intuitionistic fuzzy metric space.

5 Open Promlem

Let $(X, M, N, *, \diamond)$ be a complete intuitionistic fuzzy metric space and $f, g : X \to X$, (f, g) is a pair of generalized ψ-ϕ-contractive mappings. And also there exist $x_0 \in X$ such that $M(x_0, f(x_0), t) > 0$, and $N(x_0, f(x_0), t) < 1$ for all $t > 0$. Is there any common fixed points of f and g?

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