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Majorization for certain classes of analytic functions defined by the inverse of the Dziok-Srivastava operator

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Abstract

In the present paper, the author investigates the majorization problems for certain subclasses of p-valently analytic functions in the open unit disk $\mathcal{U}$ defined by the inverse of the Dziok-Srivastava operator. Relevant connections of the results presented in this paper with those given by earlier authors are also pointed out.

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (p \in \mathrm{N} := \{1, 2, 3, ...\}), \tag{1}$$

which are analytic and p-valent in the open unit disk $\mathcal{U} = \{z \in \mathrm{C} : |z| < 1\}$. For simplicity, we write $\mathcal{A}_1 = \mathcal{A}$. For the functions $f(z)$ given by (1) and

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{k+p} z^{k+p},$$

the Hadamard product (or convolution) of f and g, written as $f * g$ is defined as

$$(f * g)(z) = z^p + \sum_{k=1}^{\infty} a_{k+p} b_{k+p} z^{k+p} = (g * f)(z).$$

Let $f^{(\delta+q)}$ denote $(\delta + q)$ th-order ordinary differential operator. For $f \in \mathcal{A}_p$, we have

$$f^{\delta+q}(z) = \frac{p!}{(p-\delta-q)!} z^{p-\delta-q} + \sum_{k=1}^{\infty} \frac{(k+p)!}{(k+p-q-\delta)!} a_{k+p} z^{k+p-q-\delta},$$
$$(p > q + \delta,\ p \in \mathrm{N},\ q,\ \delta \in \mathrm{N}_0 := \mathrm{N} \cup \{0\};\ z \in \mathcal{U}). \tag{2}$$

Let f and g be two analytic functions in $\mathcal{U}$. Then we say that f is majorized by g in $\mathcal{U}$ (see [8]) and we write

$$f(z) \ll g(z) \quad (z \in \mathcal{U}), \tag{3}$$

if there exists an analytic function $\phi(z)$ in $\mathcal{U}$ satisfying $|\phi(z)| \leq 1$ such that

$$f(z) = \phi(z) g(z) \quad (z \in \mathcal{U}). \tag{4}$$

The majorization (3) is closely related to the concept of quasi-subordination between analytic functions in $\mathcal{U}$ (see [1]). Suppose that f and g are analytic in $\mathcal{U}$. We say $f(z)$ is subordinate to $g(z)$ if there exists an analytic function $w(z)$ in $\mathcal{U}$ satisfying $w(0) = 0$ and $|w(z)| < 1 \ \ (z \in \mathcal{U})$ such that $f(z) = g(w(z))$. We denote this subordination by

$$f(z) \prec g(z) \quad (z \in \mathcal{U}).$$

It follows from this definition that

$$f(z) \prec g(z) \Longrightarrow f(0) = g(0) \ \ \textit{and} \ \ f(\mathcal{U}) \subset g(\mathcal{U}).$$

In particular, if g is univalent in $\mathcal{U}$, then the reverse implication also holds true (see [9]).

For complex parameters $a_1, a_2, ..., a_l$ and $b_1, b_2, ..., b_m$ $(l, m \in \mathrm{N}_0, b_j \notin \mathrm{Z}_0^- := \{0, -1, -2, ...\}, j = 1, 2, 3, ..., m)$, the generalized hypergeometric function ${}_lF_m$ is defined by the following infinite series (see [15]):

$${}_lF_m(a_1, a_2, ..., a_l; b_1, b_2, ..., b_m; z) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k ... (a_l)_k}{(b_1)_k (b_2)_k .... (b_m)_k} \frac{z^k}{(1)_k} \quad (z \in \mathcal{U}) \tag{5}$$

where $(\lambda)_k$ is the Pochhammer symbol (or the shifted factorial) defined in terms of the gamma function Γ by

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1 \quad (k = 0, \lambda \in \mathrm{C}^* = \mathrm{C} \setminus \{0\}), \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+k-1) \quad (k \in \mathrm{N}, \lambda \in \mathrm{C}). \end{cases}$$

The series given by (5) is absolutely convergent for all $|z| < \infty$ if $l < m+1$ and for $|z| < 1$ if $l = m+1$. While the series is divergent for all z, $z \neq 0$ if $l > m+1$.

Given a function ${}_lF_m$, the functional equation

$$\phi^{+}_{p,\lambda}(a_1, a_2, ..., a_l; b_1, b_2, ..., b_m; z) * {}_lF_m(a_1, a_2, ..., a_l; b_1, b_2, ..., b_m; z) = \frac{1}{(1-z)^{\lambda+p}},$$

$(\lambda > -p)$ has a non-trivial solution $\phi^{+}_{p,\lambda}$ in terms of a convergent power series if and only if $l = m+1$. The solution is

$$\begin{aligned}\phi^{+}_{p,\lambda}(a_1, a_2, ..., a_{m+1}; b_1, b_2, ..., b_m; z) &= \sum_{k=0}^{\infty} \frac{(b_1)_k (b_2)_k ... (b_m)_k (\lambda+p)_k}{(a_1)_k (a_2)_k ... (a_{m+1})_k} z^k \\ &={}_{m+2} F_{m+1}(b_1, b_2, ..., b_m, \lambda+p, 1; a_1, a_2, ..., a_{m+1}; z) \\ &\quad (a_i \notin \mathrm{Z}_0^-, i = 1, 2, 3, ..., m+1; z \in \mathcal{U}). \qquad (6)\end{aligned}$$

In an analogous manner to the Dziok-Srivastava operator $\mathcal{H}_p^{l,m}$ (see [3, 4]), we introduce a new transform $\mathcal{I}_p(\lambda, a_1, a_2, ..., a_{m+1}; b_1, b_2, ..., b_m) : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ defined by

$$\mathcal{I}_p(\lambda, a_1, a_2, ..., a_{m+1}; b_1, b_2, ..., b_m) f(z) = z^p \phi^{+}_{p,\lambda}(a_1, a_2, ..., a_{m+1}; b_1, b_2, ..., b_m; z) * f(z) \qquad (7)$$

Therefore, for a function f of the form (1), we have

$$\begin{aligned}\mathcal{I}_p(\lambda, a_1, a_2, ..., a_{m+1}; b_1, b_2, ..., b_m) f(z) &= z^p + \sum_{k=1}^{\infty} \frac{(b_1)_k (b_2)_k ... (b_m)_k (\lambda+p)_k}{(a_1)_k (a_2)_k ... (a_{m+1})_k} a_{k+p} z^{k+p} \\ &= \mathcal{H}_p^{m+2,m+1}(b_1, ..., b_m, \lambda+p, 1; a_1, ..., a_{m+1}; z) * f(z)\end{aligned} \qquad (8)$$

For convenience, we write

$$\mathcal{I}_p^{\lambda}(a_1, b_1) = \mathcal{I}_p(\lambda, a_1, a_2, ..., a_{m+1}; b_1, b_2, ..., b_m).$$

Very recently, Panigrahi [12] introduced and studied the generalized differential operator $\mathcal{C}_p^{\lambda,m,n} : \mathcal{A}_p \longrightarrow \mathcal{A}_p$ as

$$\begin{aligned}\mathcal{C}_p^{\lambda,m,0}(a_1, b_1) f(z) &= \mathcal{I}_p^{\lambda,m}(a_1, b_1) f(z) \\ \mathcal{C}_p^{\lambda,m,1}(a_1, b_1) f(z) &= (1-t)\mathcal{I}_p^{\lambda,m}(a_1, b_1) f(z) + \frac{tz}{p}\left(\mathcal{I}_p^{\lambda,m}(a_1, b_1) f(z)\right)' \\ &\vdots \\ \mathcal{C}_p^{\lambda,m,n}(a_1, b_1) f(z) &= \mathcal{C}_p^{\lambda,m,1}\left(\mathcal{C}_p^{\lambda,m,n-1}(a_1, b_1) f(z)\right) \quad (m \in \mathrm{N}_0, t \geq 0).\end{aligned}$$

Thus, for $f \in \mathcal{A}_p$ we have

$$\mathcal{C}_p^{\lambda,m,n}(a_1,b_1)f(z) = z^p + \sum_{k=1}^{\infty} \left(\frac{p+kt}{p}\right)^n \frac{(b_1)_k(b_2)_k...(b_m)_k(\lambda+p)_k}{(a_1)_k(a_2)_k...(a_{m+1})_k} a_{k+p}z^{k+p}. \tag{9}$$

The operator $\mathcal{C}_p^{\lambda,m,n}(a_1,b_1)$ generalizes several previously studied familiar operators (for detail, see [12]). It is easily verified from (9) that

$$z\left(\mathcal{C}_p^{\lambda,m,n}(a_1+1,b_1)f(z)\right)' = a_1\mathcal{C}_p^{\lambda,m,n}(a_1,b_1)f(z) + (p-a_1)\mathcal{C}_p^{\lambda,m,n}(a_1+1,b_1)f(z) \tag{10}$$

Using the operator $\mathcal{C}_p^{\lambda,m,n}(a_1,b_1)$, we now define a new subclass of functions $f \in \mathcal{A}_p$ as follows:

Definition 1.1 *A function $f \in \mathcal{A}_p$ is said to be in the class $\mathcal{J}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1;\alpha,\gamma;A,B)$ $(-1 \le B < A \le 1)$ of p-valently analytic functions of complex order $\gamma \neq 0$ in $\mathcal{U}$ if and only if*

$$1 + \frac{1}{\gamma}\left[\frac{z\left(\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z)\right)'}{\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z)} - p + q + \delta + \alpha\right] \prec \frac{1+\frac{\alpha}{\gamma}+Az}{1+Bz},$$

$$(\alpha, q, \delta \in \mathrm{N}_0, p > q+\delta, \gamma \in \mathrm{C}^*, |a_1| > |\gamma(A-B)+(a_1-\alpha)B|;\ z \in \mathcal{U}), \tag{11}$$

where $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z) = \left(\mathcal{C}_p^{\lambda,m,n}(a_1,b_1)f(z)\right)^{q+\delta}$ represents $(q+\delta)$ times ordinary derivative of $\mathcal{C}_p^{\lambda,m,n}(a_1,b_1)f(z)$.

For $a_i = 1$ $(i = 1,2,3,...,m+1)$, $b_j = 1$ $(j = 1,2,3,...,m)$ and $\lambda = 1$, we have the following relationships:

- $\mathcal{J}_{1,0,0}^{1,m,0}(2,1;0,\gamma;1,-1) = \mathcal{S}(\gamma)$ $(\gamma \in \mathrm{C}^*)$,
- $\mathcal{J}_{1,1,0}^{1,m,0}(2,1;0,\gamma;1,-1) = \mathcal{K}(\gamma)$ $(\gamma \in \mathrm{C}^*)$,
- $\mathcal{J}_{1,0,0}^{1,m,0}(2,1;0,1-\alpha;1,-1) = \mathcal{S}^*(\alpha)$ $(0 \le \alpha < 1)$,

where $\mathcal{S}(\gamma)$ and $\mathcal{K}(\gamma)$ are said to be the class of starlike and convex functions of complex order $\gamma \neq 0$ in $\mathcal{U}$ respectively which were considered by Nasr and Aouf [10] and Wiatrowski [16] and $\mathcal{S}^*(\alpha)$ is the class of starlike functions of order α in $\mathcal{U}$.

A majorization problem for normalized classes of starlike functions of complex order has been investigated by Altintas et al. [2]. Recently, Goyal and Goswami [6] and Goyal et al. [7] generalized these results for classes of multivalent functions defined by fractional derivatives operator and Saitoh operator respectively.

In this paper we investigate and obtain certain results involving majorization problems for the class $\mathcal{J}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1;\alpha,\gamma;A,B)$ by applying ordinary differential operator of order $(q+\delta)$. Moreover, we point out some known consequences of our main results.

2 Majorization Problems

Unless otherwise stated, we assume throughout the sequel that

$$-1 \leq B < A \leq 1,\ p \in \mathrm{N},\ \alpha, q, \delta \in \mathrm{N}_0,\ p > q + \delta;\ \ \gamma \in \mathrm{C}^* \ \ and \ \ a_1 \in \mathrm{C} \setminus \mathrm{Z}_0^-.$$

We state and prove the following results:

Theorem 2.1 *Let the function* $f \in \mathcal{A}_p$ *and suppose that* $g \in \mathcal{J}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1; \alpha, \gamma; A, B)$. *If* $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)f$ *is majorized by* $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g$ *in* $\mathcal{U}$, *then*

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)f(z)| \leq |\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)g(z)| \quad (|z| < r_0), \tag{12}$$

where $r_0 = r_0(a_1, \alpha, \gamma; A, B)$ *is the smallest positive root of the equation*

$$|\gamma(A-B)+(a_1-\alpha)B|r^3-(2|B|+|a_1|)r^2-(|\gamma(A-B)+(a_1-\alpha)B|+2)r+|a_1| = 0. \tag{13}$$

Proof: Let

$$\theta(z) = 1 + \frac{1}{\gamma}\left[\frac{z\left(\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g(z)\right)'}{\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g(z)} - p + q + \delta + \alpha\right]. \tag{14}$$

Since, by hypothesis $g(z) \in \mathcal{J}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1; \alpha, \gamma; A, B)$, hence by Definition 1.1 we have

$$\theta(z) = \frac{1 + \frac{\alpha}{\gamma} + Aw(z)}{1 + Bw(z)} \quad (w \in \mathcal{P}) \tag{15}$$

where $w(z) = c_1 z + c_2 z^2 + ...$ and $\mathcal{P}$ denote the well-known class of bounded analytic functions in $\mathcal{U}$ (see [11]) satisfying the conditions $w(0) = 0$ and $|w(z)| \leq |z| \ \ (z \in \mathcal{U})$.

It follows from (14) and (15) that

$$\frac{z\left(\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g(z)\right)'}{\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g(z)} = \frac{p - q - \delta + [\gamma(A - B) + B(p - q - \delta - \alpha)]w(z)}{1 + Bw(z)}. \tag{16}$$

It follows from (10) that

$$z(\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1{+}1, b_1)g(z))' = a_1\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)g(z) + (p{-}q{-}\delta{-}a_1)\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1{+}1, b_1)g(z). \tag{17}$$

Making use of (17) in (16) gives

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g(z)| \leq \frac{|a_1|(1 + |B||z|)}{|a_1| - |\gamma(A - B) + (a_1 - \alpha)B||z|}|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)g(z)|. \tag{18}$$

Since $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)f(z)$ is majorized by $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z)$ in $\mathcal{U}$, hence by (4) it follows that

$$\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)f(z) = \phi(z)\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z). \tag{19}$$

Differentiating (19) with respect to z and then multiplying by z we get

$$z\left(\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)f(z)\right)' = \phi(z)z\left(\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z)\right)' + z\phi'(z)\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z). \tag{20}$$

Using (17) and (19) in (20) we get

$$\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z) = \frac{z\phi'(z)}{a_1}\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z) + \phi(z)\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)g(z) \tag{21}$$

Since the Schwarz function $\phi(z) \in \mathcal{P}$ satisfies the inequality (see [11])

$$|\phi'(z)| \leq \frac{1-|\phi(z)|^2}{1-|z|^2} \quad (z \in \mathcal{U}), \tag{22}$$

and making use of (18) and (22) in (21) we have

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z)| \leq \left[|\phi(z)| + \frac{(1+|B||z|)(1-|\phi(z)|^2)|z|}{(1-|z|^2)\left(|a_1| - |\gamma(A-B)+(a_1-\alpha)B||z|\right)}\right]|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)g(z)|. \tag{23}$$

Setting $|z| = r$ and $|\phi(z)| = \rho \ \ (0 \leq \rho \leq 1)$ in (23) leads to

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z)| \leq \frac{\psi(\rho)}{(1-r^2)(|a_1| - |\gamma(A-B)+(a_1-\alpha)B|r)}|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)g(z)|$$

where

$$\psi(\rho) = -r(1+|B|r)\rho^2 + (1-r^2)(|a_1| - |\gamma(A-B)+(a_1-\alpha)B|r)\rho + r(1+|B|r)$$

takes its maximum value at $\rho = 1$ with $r_0 = r_0(a_1,\ \alpha,\ \gamma;\ A,\ B)$ is the smallest positive root of the equation (13). Furthermore, if $0 \leq \sigma \leq r_0$, then the function $\chi(\rho)$ defined by

$$\chi(\rho) = -\sigma(1+|B|\sigma)\rho^2 + (1-\sigma^2)(|a_1| - |\gamma(A-B)+(a_1-\alpha)B|\sigma)\rho + \sigma(1+|B|\sigma) \tag{24}$$

is an increasing function on the interval $0 \leq \rho \leq 1$, so that

$$\chi(\rho) \leq \chi(1) = (1-\sigma^2)(|a_1| - |\gamma(A-B)+(a_1-\alpha)B|\sigma) \quad (0 \leq \rho \leq 1, 0 \leq \sigma \leq r_0).$$

Hence, setting $\rho = 1$ in (24) we conclude that (12) of Theorem 2.1 holds true for $|z| \leq r_0 = r_0(a_1,\ \alpha,\ \gamma;\ A,\ B)$ where r_0 is the smallest positive root of the equation (13). This completes the proof of Theorem 2.1.

Corollary 2.2 *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose that $g \in \mathcal{J}_{p,q,\delta}^{\lambda,m,n}(a_1+1, b_1; \alpha, \gamma; 1, -1)$. If $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1, b_1)f$ is majorized by $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1, b_1)g$ in $\mathcal{U}$, then*

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)f(z)| \leq |\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)g(z)| \quad (|z| \leq r_1), \tag{25}$$

where

$$r_1 = r_1(a_1, \alpha, \gamma) = \frac{\eta - \sqrt{\eta^2 - 4|2\gamma - a_1 + \alpha||a_1|}}{2|2\gamma - a_1 + \alpha|} \tag{26}$$

with $\eta = |2\gamma - a_1 + \alpha| + |a_1| + 2$.

Proof: Taking $A = 1$ and $B = -1$ in Theorem 2.1, equation (13) becomes

$$|2\gamma - a_1 + \alpha|r^3 - (2 + |a_1|)r^2 - (|2\gamma - a_1 + \alpha| + 2)r + |a_1| = 0. \tag{27}$$

Clearly, $r = -1$ is one of the root of the above equation (27) and other two roots are given by

$$|2\gamma - a_1 + \alpha|r^2 - (|2\gamma - a_1 + \alpha| + 2 + |a_1|)r + |a_1| = 0. \tag{28}$$

The smallest positive root of equation (28) is $r_1 = r_1(a_1, \alpha, \gamma)$ where r_1 is given by equation (26). This completes the proof of Corollary 2.2. Setting $q = \delta = \alpha = 0$ in Corollary 2.2, we obtain the following result:

Corollary 2.3 *Let the function $f(z)$ be in the class $\mathcal{A}_p$ and suppose $g \in \mathcal{J}_{p,0,0}^{\lambda,m,n}(a_1+1, b_1; 0, \gamma; 1, -1)$. If $\mathcal{C}_{p,0,0}^{\lambda,m,n}(a_1+1, b_1)f$ is majorized by $\mathcal{C}_{p,0,0}^{\lambda,m,n}(a_1+1, b_1)g$ in $\mathcal{U}$, then*

$$|\mathcal{C}_{p,0,0}^{\lambda,m,n}(a_1, b_1)f(z)| \leq |\mathcal{C}_{p,0,0}^{\lambda,m,n}g(z)| \quad (|z| \leq r_2)$$

$$r_2 = r_2(a_1, \gamma) = \frac{\eta_1 - \sqrt{\eta_1^2 - 4|2\gamma - a_1||a_1|}}{2|2\gamma - a_1|}$$

where

$$\eta_1 = |2\gamma - a_1| + |a_1| + 2.$$

Further, by putting $p = 1,\ n = 0,\ \lambda = 1,\ b_j = 1\ \ (j = 1, 2, 3, ..., m),\ a_i = 1\ \ (i = 1, 2, 3, ..., m+1)$ in Corollary 2.3, we obtain the following result (see [5, 6, 13, 14]).

Corollary 2.4 *Let the function $f \in \mathcal{A}$ be analytic and univalent in the open unit disk $\mathcal{U}$ and suppose that $g(z) \in \mathcal{S}(\gamma)$. If f is majorized by g in $\mathcal{U}$, then*

$$|f'(z)| \leq |g'(z)| \quad for \quad |z| \leq r_3,$$

where

$$r_3 = r_3(\gamma) = \frac{3 + |2\gamma - 1| - \sqrt{9 + 2|2\gamma - 1| + |2\gamma - 1|^2}}{2|2\gamma - 1|}.$$

Letting $\gamma = 1$ in Corollary 2.4, we obtain the following well-known results (see [5, 6, 8, 13, 14]).

Corollary 2.5 *Let the function $f \in \mathcal{A}$ be univalent in the open unit disk $\mathcal{U}$, and suppose that $g \in \mathcal{S}^*$. If f is majorized by g in $\mathcal{U}$, then*

$$|f'(z)| \leq |g'(z)| \quad for \quad |z| \leq 2 - \sqrt{3}.$$

3 Majorization problem for the class $\mathcal{R}(k, \gamma)$

Let $\mathcal{R}(k, \gamma)$ be the class of functions $h(z)$ of the form

$$h(z) = 1 - \sum_{k=1}^{\infty} c_k z^k \quad (c_k \geq 0), \tag{29}$$

that are analytic in $\mathcal{U}$ satisfying the inequality

$$|h(z) + kzh'(z) - 1| < |\gamma| \quad (z \in \mathcal{U};\ \Re(k) \geq 0,\ \gamma \in \mathrm{C}^*). \tag{30}$$

For $\gamma = 1 - \beta$ $(0 \leq \beta < 1)$, the class $\Re(k, \gamma) = \Re(k, 1 - \beta)$ was considered by Altintas and Owa [1].

The following lemma is useful for our further investigation:

Lemma 3.1 *(see [2]) If the function $h(z)$ defined by (29) is in the class $\mathcal{R}(k, \gamma)$, then*

$$1 - \frac{|\gamma|}{1 + \Re(k)}|z| \leq |h(z)| \leq 1 + \frac{|\gamma|}{1 + \Re(k)}|z| \quad (z \in \mathcal{U}). \tag{31}$$

Theorem 3.2 *Let the function $f(z) \in \mathcal{A}_p$ and $g(z) \in \mathcal{R}(k, \gamma)$ be analytic in $\mathcal{U}$ and suppose that the function $g(z)$ is so normalized that it also satisfies the following inclusion property*

$$\frac{\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)g(z)}{\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g(z)} \in \mathcal{R}(k, \gamma). \tag{32}$$

If $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)f(z)$ is majorized by $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1 + 1, b_1)g(z)$ in $\mathcal{U}$, then

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)f(z)| \leq |\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)g(z)| \quad (|z| < r_4) \tag{33}$$

where $r_4 = r_4(a_1, k, \gamma)$ is the smallest positive root of the cubic equation

$$|\gamma|r^3 - [1 + \Re(k)]r^2 - [2 + |a_1||\gamma| + 2\Re(k)]r + [1 + \Re(k)]|a_1| = 0. \tag{34}$$

Proof: For appropriately normalized analytic function $g(z)$ satisfying the inclusion property (32), we find from (31) of Lemma 3.1 that

$$\left|\frac{\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)g(z)}{\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z)}\right| \geq 1-\frac{|\gamma|}{1+\Re(k)}r \quad (|z|=r,\ 0<r<1) \tag{35}$$

which implies

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z)| \leq \frac{1+\Re(k)}{1+\Re(k)-|\gamma|r}|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)g(z)| \quad (|z|=r,0<r<1). \tag{36}$$

Since $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)f(z) \ll \mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z)$ $(z\in\mathcal{U})$, there exists an analytic function w with $|w(z)|<1$ such that

$$\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)f(z) = w(z)\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1+1,b_1)g(z). \tag{37}$$

Therefore, in view of (36) and proceeding as in the proof of Theorem 2.1, we have

$$|w'(z)| \leq \frac{1-|w(z)|^2}{1-|z|^2} \quad (z\in\mathcal{U}), \tag{38}$$

and

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z)| \leq \left[|w(z)|+\frac{(1-|w(z)|^2)(1+\Re(k))r}{|a_1|(1-r^2)(1+\Re(k)-|\gamma|r)}\right]|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)g(z)|. \tag{39}$$

Taking $|w(z)|=\rho$ in (39), we have

$$|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)f(z)| \leq \frac{\theta(\rho)}{|a_1|(1-r^2)(1+\Re(k)-|\gamma|r)}|\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1,b_1)g(z)|, \tag{40}$$

where

$$\theta(\rho)=|a_1|(1-r^2)(1+\Re(k)-|\gamma|r)\rho+r(1+\Re(k))-r(1+\Re(k))\rho^2 \quad (0\leq\rho\leq 1),$$

takes on its maximum value at $\rho=1$ with $r=r_4(a_1,k,\gamma)$ given by (34). Moreover, if $0\leq\eta\leq r_4(a_1,k,\gamma)$ where $r_4(a_1,k,\gamma)$ is the root of the cubic equation (34) such that $0<r_4(a_1,k,\gamma)<1$, then the function $H(\rho)$ defined by

$$H(\rho)=|a_1|(1-\eta^2)(1+\Re(k)-|\gamma|\eta)\rho+(1+\Re(k))\eta-(1+\Re(k))\eta\rho^2 \quad (0\leq\rho\leq 1) \tag{41}$$

is seen to be an increasing function on the interval $0\leq\rho\leq 1$ so that

$$H(\rho)\leq H(1)=|a_1|(1-\eta^2)(1+\Re(k)-|\gamma|\eta) \quad (0\leq\rho\leq 1, 0\leq\eta\leq r_4(a_1,k,\gamma)). \tag{42}$$

Therefore, upon setting $\rho=1$ in (40), we complete the proof of Theorem 3.2.

4 Open Problem

In this paper, we have investigated the majorization problem for the class of multivalent analytic functions. If we define a class $f \in \sum_p$ such that

$$f(z) = \frac{1}{z^p} + \sum_{k=1}^{\infty} a_{k+p} z^{k+p} \quad (z \in \mathcal{U}^* = \mathcal{U} \setminus \{0\}), \tag{43}$$

then we need to modify the generalized operator $\mathcal{C}_{p,q,\delta}^{\lambda,m,n}(a_1, b_1)$ for the class of multivalent meromorphic function. Further using this modified operator we have to find the new majorization conditions for the modified operator.

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References

[1] O. Altintas and S. Owa, Majorizations and quasi-subordinations for certain analytic functions, *Proc. Japan Acad.*, **68**, (1992), 181-185.

[2] O. Altintas, O. Ozkan and H. M. Srivastava, Majorization by starlike functions of complex order, *Complex Var. Theory Appl.*, **46**, (2001), 207-218.

[3] J. Dziok and H. M. Srivastava, Classes of analytic functions associated with the generalized hypergeometric function, *Appl. Math. Comput.*, 103, (1999), 1-13.

[4] J. Dziok and H. M. Srivastava, Certain subclasses of analytic functions associated with the generalized hypergeometric function, *Integral Transform. Spec. Funct.*, 14, (2003), 7-18.

[5] P. Goswami, B. Sharma and T. Bulboacă, Majorization for certain classes of analytic functions using multiplier transformation, *Applied Math. Lett.*, **23**, (2010), 633-637.

[6] S. P. Goyal and P. Goswami, Majorization for certain classes of analytic functions defined by fractional derivatives , *Appl. Math. Lett.*, **22**, (2009), 1855-1858.

[7] S. P. Goyal, S. K. Bansal and P. Goswami, Majorization for the subclass of analytic functions defined by linear operator using differential subordination, *J. Appl. Math. Stat. Informatics*, **6**, (2010), 45-50.

[8] T. H. MacGregor, Majorization by univalent functions, *Duke Math. J.*, **34**, (1967), 95-102.

[9] S. S. Miller and P. T. Mocanu, Differential Subordinations: Theory and Applications, in: *Monographs and Textbooks in Pure and Applied Mathematics, 225, Marcel Dekker, New York*, 2000.

[10] M. A. Nasr and M. K. Aouf, Starlike function of complex order, *J. Natur. Sci. Math.*, **25**, (1985), 1-12.

[11] Z. Nehari, Conformal Mappings, *McGraw-Hill Book Company*, New York, Toronto, London, 1952.

[12] T. Panigrahi, On some families of analytic functions defined through subordination and hypergeometric functions, *Ph. D Thesis, Berhampur University*, (2011).

[13] C. Selvaraj and K. A. Selvakumaran, Majorization for certain classes of analytic functions defined by a generalized operator, *European J. of Pure and Applied Math.*, **3**, (2010), 1048-1054.

[14] C. Selvaraj and K. A. Selvakumaran, Majorization problems for certain classes of analytic functions, *Int. Math. Forum*, **6**, (2011), 289-294.

[15] H. M. Srivastava and P. W. Karlson, Multiple Gaussian Hypergeometric Series, *Halsted Press (Ellis Horwood limited, Chichester), John Wiley and Sons, New York*, (1985).

[16] P. Wiatrowski, On the coefficients of some family of holomorphic functions, *Zeszyty Nauk. Uniw. Lòdz. Nauk. Mat.-Przyrod. Ser. II*, **39**, (1970), 75-85.