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Certain Subclasses Of Multivalent Analytic Functions Involving Linear Operator

A.A.AL-DOHIMAN

Al-Jouf University,
Department of Mathematics, Faculty of Science
P.O. Box:762, Al-jouf, Kingdom of Saudi Arabia
a_aldohiman@ju.edu.sa

Abstract

By making use of the principle of subordination between analytic functions and the linear operator, we introduce and investigate some new subclasses of multivalent analytic functions. Such results as inclusion relationships and integral – preserving properties involving these subclasses are proved. Several subordination and superordination results associated with this operator are also derived.

Keywords: Analytic functions; multivalent functions; Hadamard product (or convolution); subordination and superordination between analytic functions.

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1 Introduction

Let $\mathcal{A}_p$ denote the class of functions of the form:

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} (p \in \mathbb{N} := \{1, 2, 3...\}), \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \ and \ |z| < 1\}.$$

For simplicity, we write

$$\mathcal{A}_1 = \mathcal{A}.$$

Let $f, g \in \mathcal{A}_p$, where f is given by (1.1) and g is defined by

$$g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k} .$$

Then the Hadamard product (or convolution) $f * g$ of the functions f and g is defined by

$$(f * g)(z) := z^p + \sum_{k=1}^{\infty} a_{p+k} b_{p+k} z^{p+k} =: (g * f)(z). \tag{1.2}$$

Let $\mathcal{P}$ denote the class of functions of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} p_k z^k ,$$

which are analytic and convex in $\mathbb{U}$ and satisfy the condition

$$Re\big(p(z)\big) > 0 \quad (z \in \mathbb{U}).$$

For two functions of f and g, analytic in $\mathbb{U}$, we say that the function f is subordinate to g in $\mathbb{U}$, and write

$$f(z) \prec g(z) \quad (z \in \mathbb{U}),$$

if there exists a Schwarz function ω, which is analytic in $\mathbb{U}$ with

$$\omega(0) = 0 \quad and \quad |\omega(z)| < 1 \quad (z \in \mathbb{U})$$

such that

$$f(z) = g\big(\omega(z)\big) \quad (z \in \mathbb{U}).$$

Indeed, it is known that

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Rightarrow f(0) = g(0) \quad and \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec g(z) \quad (z \in \mathbb{U}) \Leftrightarrow f(0) = g(0) \quad and \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

For real or complex numbers a, b, c other than $0, -1, -2, ...$, the hypergeometric series is defined by

$$ {}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k (1)_k} z^k, \tag{1.3} $$

and $(x)_k$ is Pochhammer symbol defined by

$$ (x)_k = \frac{\Gamma(x+k)}{\Gamma(x)} = x(x+1)...(x+k-1) \; for \; k = 1, 2, 3...x \in \mathbb{C}, (x)_0 = 1. \tag{1.4} $$

We note that the series (1.3) converges absolutely for all $z \in \mathbb{U}$ so that it represents an analytic function in $\mathbb{U}$. Also an incomplete beta function $\phi(a, c; z)$ is related to Gauss hypergeometric function ${}_2F_1(a, b; c; z)$ as

$$ \phi(a, c; z) = z_2F_1(a, b; c; z)\,, \tag{1.5} $$

and we note that $\phi(a, c; z) = z/(1-z)^a$, where $\phi(2, 1; z)$ is Koebe function. Using $\phi(a, c; z)$, a convolution operator [2],was defined by Carlson and Shaferr. Furthermore, Hohlov [8] introduced a convolution operator using $z_2F_1(a, b; c; z)$.

K. Al-Shaqsi and M. Darus [1] introduced and investigated the linear operator

$$ I_\mu^\lambda : \mathcal{A} \to \mathcal{A} $$

defined, in terms of the Hadamard product (or convolution), by

$$ I_\mu^\lambda(a, b, c) f(z) = \big(f_\mu(a, b, c)(z)\big)^{(-1)} * f(z) \quad (\mu \geq 0, \lambda > -1), \tag{1.6} $$

where

$$ \big(f_\mu(a, b, c)(z)\big)^{(-1)} = \sum_{k=0}^{\infty} \frac{(\lambda+1)_k (c)_k}{(\mu k + 1)(a)_k (b)_k} z^{1+k} \quad (z \in \mathbb{U}). \tag{1.7} $$

The operator $I_\mu^\lambda(a, b, c)$ is an extension of the K. I. Noor operator [13].

By setting

$$f_{\mu,p}(a,b,c) = (1-\mu)z^p{}_2F_1(a,b;c;z) + \mu z^p\big(z{}_2F_1(a,b;c;z)\big)' \qquad (\mu \geq 0) \quad (1.8)$$

we define a new function $(f_{\mu,p})^{-1}$ in terms of the hadamard product (or convolution)

$$f_{\mu,p}(a,b,c)(z) * \big(f_{\mu,p}(a,b,c)(z)\big)^{-1} = \frac{z^p}{(1-z)^{p+\lambda}} \qquad (z \in \mathbb{U}; \lambda > -p)\ . \quad (1.9)$$

It is well known that for $(\lambda > -p)$

$$\frac{z^p}{(1-z)^{p+\lambda}} = \sum_{k=0}^{\infty} \frac{(p+\lambda)_k}{k!} z^{p+\lambda}\ . \quad (1.10)$$

We now define the linear operator $I^{\lambda}_{\mu,p}(a,b,c)$ for $\lambda > -p$ as follows:

$$I^{\lambda}_{\mu,p} : \mathcal{A}_p \to \mathcal{A}_p\ ,$$

$$I^{\lambda}_{\mu,p}(a,b,c)f(z) = \big(f_{\mu,p}(a,b,c)(z)\big)^{(-1)} * f(z) \quad (1.11)$$

$$(\mu \geq 0, \lambda > -p, z \in \mathbb{U}; f \in \mathcal{A}_p).$$

We can easily find from (1.8), (1.9) and (1.11) that

$$I^{\lambda}_{\mu,p}(a,b,c)f(z) = z^p + \sum_{k=1}^{\infty} \frac{(p+\lambda)_k (c)_k}{(1+\mu k)(a)_k (b)_k} a_{p+k} z^{p+k} \qquad (z \in \mathbb{U}). \quad (1.12)$$

From (1.12), we note that

$$I^{\lambda}_{0,p}(a,\lambda+p,a)f(z) = f(z) \ \ \textit{and} \ \ I^{1}_{0,p}(a,p,a)f(z) = \frac{zf'(z)}{p}. \quad (1.13)$$

Also it can easily by verified that

$$z\big(I^{\lambda}_{\mu,p}(a,b,c)f(z)\big)' = (\lambda+p)I^{\lambda+1}_{\mu,p}(a,b,c)f(z) - \lambda I^{\lambda}_{\mu,p}(a,b,c)f(z), \quad (1.14)$$

and

$$z\big(I^{\lambda}_{\mu,p}(a+1,b,c)f(z)\big)' = aI^{\lambda}_{\mu,p}(a,b,c)f(z) - (a-p)I^{\lambda}_{\mu,p}(a+1,b,c)f(z). \quad (1.15)$$

By making use of the principle of subordination between analytic functions, we introduce the subclasses

$S_p^*(\eta;\phi), K_p(\eta;\phi), C_p(\eta,\delta;\phi,\psi)$ and $\mathcal{Q}C_p(\eta,\delta;\phi,\psi)$ of the class $\mathcal{A}_p$ (see [3]), which are defined by

$$S_p^*(\eta;\phi)=\left\{f\in\mathcal{A}_p:\frac{1}{p-\eta}\left(\frac{zf'(z)}{f(z)}-\eta\right)\prec\phi(z)(\phi\in\mathcal{P};0\leq\eta<p;z\in\mathbb{U})\right\},$$

$$K_p^*(\eta;\phi)=\left\{f\in\mathcal{A}_p:\frac{1}{p-\eta}\left(1+\frac{zf''(z)}{f'(z)}-\eta\right)\prec\phi(z)(\phi\in\mathcal{P};0\leq\eta<p;z\in\mathbb{U})\right\},$$

$C_p(\eta,\delta;\phi,\psi)=\{f\in\mathcal{A}_p:\exists g\in S_p^*(\eta;\phi)$ such that

$$\frac{1}{p-\delta}\left(\frac{zf'(z)}{g(z)}-\delta\right)\prec\psi(z)(\phi,\psi\in\mathcal{P};0\leq\eta,\delta<p;z\in\mathbb{U})\},$$

and
$\mathcal{Q}C_p(\eta,\delta;\phi,\psi)=\{f\in\mathcal{A}_p:\exists g\in K_p(\eta;\phi)$ such that

$$\frac{1}{p-\delta}\left(\frac{\left(zf'(z)\right)'}{g'(z)}-\delta\right)\prec\psi(z)(\phi,\psi\in\mathcal{P};0\leq\eta,\delta<p;z\in\mathbb{U})\}.$$

Indeed, the above-mentioned function classes are generalizations of the general starlike, convex, close - to - convex and quasi - convex functions in analytic function theory [5,7,9,11,12,14,15-18].

Next, by using the operator defined by (1.12), we define the following $K_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$, $C_{\mu,p}^{\lambda}(a,b,c;\eta,\delta;\phi,\psi)$ and subclasses $S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$, $\mathcal{Q}C_{\mu,p}^{\lambda}(a,b,c;\eta,\delta;\phi,\psi)$ of the class $\mathcal{A}_p$:

$$S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)=\{f\in\mathcal{A}_p:I_{\mu,p}^{\lambda}(a,b,c)f(z)\in S_p^*(\eta;\phi)\},$$
$$K_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)=\{f\in\mathcal{A}_p:I_{\mu,p}^{\lambda}(a,b,c)f(z)\in K_p(\eta;\phi)\},$$
$$C_{\mu,p}^{\lambda}(a,b,c;\eta,\delta;\phi,\psi)=\{f\in\mathcal{A}_p:I_{\mu,p}^{\lambda}(a,b,c)f(z)\in C_p(\eta,\delta;\phi,\psi)\},$$

and

$$\mathcal{Q}C_{\mu,p}^{\lambda}(a,b,c;\eta,\delta;\phi,\psi)=\{f\in\mathcal{A}_p:I_{\mu,p}^{\lambda}(a,b,c)f(z)\in\mathcal{Q}C_p(\eta,\delta;\phi,\psi)\}.$$

Clearly, we know that

$$f \in K^{\lambda}_{\mu,p}(a,b,c;\eta,\phi) \Leftrightarrow \frac{zf'}{p} \in S^{\lambda}_{\mu,p}(a,b,c;\eta,\phi), \tag{1.16}$$

and

$$f \in \mathcal{Q}C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi) \Leftrightarrow \frac{zf'}{p} \in C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi). \tag{1.17}$$

The main purpose of this paper is to investigate some inclusion relationships and integral - preserving properties of certain subclasses of multivalent analytic functions involving the linear operator $I^{\lambda}_{\mu,p}(a,b,c)$. Several subordination and superordination results involving this operator are also derived.

2 Preliminary results

In order to prove our main results, we need the following lemmas.

Lemma 1. *([6]): Let $k, \vartheta \in \mathbb{C}$. Suppose also that m is convex and univalent in $\mathbb{U}$ with*

$$m(0) = 1 \quad and \quad Re\big(km(z) + \vartheta\big) > 0 \qquad (z \in \mathbb{U}).$$

If u is analytic in $\mathbb{U}$ with $u(0) = 1$, then the following subordination

$$u(z) + \frac{zu'(z)}{ku(z) + \vartheta} \prec m(z) \qquad (z \in \mathbb{U})$$

implies that

$$u(z) \prec m(z) \quad (z \in \mathbb{U}).$$

Lemma 2. *([10]): Let h be convex univalent in $\mathbb{U}$ and ζ be analytic in $\mathbb{U}$ with*

$$Re\big(\zeta(z)\big) \geq 0 \quad (z \in \mathbb{U}).$$

If q is analytic in $\mathbb{U}$ and $q(0) = h(0)$, then the subordination

$$q(z) + \zeta(z)zq'(z) \prec h(z) \quad (z \in \mathbb{U}).$$

implies that

$$q(z) \prec h(z) \quad (z \in \mathbb{U}).$$

3 The main inclusion relationships

In this section, we give several inclusion relationships for multivalent analytic function classes defined in the first section, which are associated with the linear operator $I_{\mu,p}^{\lambda}(a,b,c)$.

Theorem 1. *Let* $0 \leq \eta < p$ *and* $\phi \in \mathcal{P}$ *with*

$$Re\big(\phi(z)\big) > \max\left\{0, -\frac{\lambda+\eta}{p-\eta}, -\frac{Re(b)+\eta}{p-\eta}\right\} \quad (z \in \mathbb{U}) \ . \tag{3.1}$$

Then

$$S_{\mu,p}^{\lambda+1}(a,b,c;\eta,\phi) \subset S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi) \subset S_{\mu,p}^{\lambda}(a+1,b,c;\eta,\phi). \tag{3.2}$$

Proof. First, we prove that

$$S_{\mu,p}^{\lambda+1}(a,b,c;\eta,\phi) \subset S_{\mu,p}^{\lambda}(a,b,c:\eta,\phi)$$

Let $f \in S_{\mu,p}^{\lambda+1}(a,b,c;\eta,\phi)$, and suppose that

$$h(z) = \frac{1}{p-\eta}\left(\frac{z\big(I_{\mu,p}^{\lambda}(a,b,c)f(z)\big)'}{I_{\mu,p}^{\lambda}(a,b,c)f(z)} - \eta\right), \tag{3.3}$$

where h is analytic in $\mathbb{U}$ with $h(0) = 1$. Combining (1.14) and (3.3), we find that:

$$(\lambda+p)\frac{I_{\mu,p}^{\lambda+1}(a,b,c)f(z)}{I_{\mu,p}^{\lambda}(a,b,c)f(z)} = (p-\eta)h(z) + \eta + \lambda. \tag{3.4}$$

Taking the Logarithmical differentiation on the both sides of (3.4) and multiplying the resulting equation by z, we get

$$\frac{1}{(p-\eta)}\left(\frac{z\big(I_{\mu,p}^{\lambda+1}(a,b,c)f(z)\big)'}{I_{\mu,p}^{\lambda+1}(a,b,c)f(z)} - \eta\right) = h(z) + \frac{z\acute{h}(z)}{(p-\eta)h(z)+\eta+\lambda} \prec \phi(z). \tag{3.5}$$

By virtue of (3.1), an application of lemma 2.1 to (3.5) yields $h \prec \phi$, that is $f \in S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$. Thus, the first part of Theorem 3.1 holds true.

To prove the second part of Theorem 3.1, we assume that $f \in S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$ and suppose that

$$g(z) = \frac{1}{(p-\eta)}\left(\frac{z\big(I_{\mu,p}^{\lambda}(a+1,b,c)f(z)\big)'}{I_{\mu,p}^{\lambda}(a+1,b,c)f(z)} - \eta\right), \tag{3.6}$$

where g is analytic in $\mathbb{U}$ with $g(0) = 1$. By combining (1.15), (3.1) and (3.6) and applying the similar method of proof of the first part, we deduce that $g \prec \phi$, that is $f \in S^{\lambda}_{\mu,p}(a+1, b, c; \eta, \phi)$. Therefore, the second part of Theorem 3.1 also holds true. The proof of Theorem 3.1 is evidently completed. $\square$

Theorem 2. *Let $0 \leq \eta < p$ and $\phi \in \mathcal{P}$ with (3.1) holds. Then*

$$K^{\lambda+1}_{\mu,p}(a, b, c; \eta, \phi) \subset K^{\lambda}_{\mu,p}(a, b, c; \eta, \phi) \subset K^{\lambda}_{\mu,p}(a+1, b, c; \eta, \phi) \ . \tag{3.7}$$

Proof. In view of (1.16) and Theorem 3.1, we find that

$$\begin{aligned}
&f(z) \in K^{\lambda+1}_{\mu,p}(a, b, c; \eta, \phi) \\
&\Leftrightarrow I^{\lambda+1}_{\mu,p}(a, b, c)f(z) \in K_p(\eta; \phi) \Leftrightarrow \frac{z\big(I^{\lambda+1}_{\mu,p}(a, b, c)f(z)\big)'}{p} \in S^*_p(\eta; \phi) \\
&\Leftrightarrow I^{\lambda+1}_{\mu,p}(a, b, c)\Big(\frac{zf'(z)}{p}\Big) \in S^*_p(\eta; \phi) \Leftrightarrow \frac{zf'(z)}{p} \in S^{\lambda+1}_{\mu,p}(a+1, b, c; \eta, \phi)
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
&\Rightarrow \frac{zf'(z)}{p} \in S^{\lambda}_{\mu,p}(a, b, c; \eta, \phi) \Leftrightarrow I^{\lambda}_{\mu,p}(a, b, c)\Big(\frac{zf'(z)}{p}\Big) \in S^*_p(\eta; \phi) \\
&\Leftrightarrow \frac{z\big(I^{\lambda}_{\mu,p}(a, b, c)f(z)\big)'}{p} \in S^*_p(\eta; \phi) \Leftrightarrow I^{\lambda}_{\mu,p}(a, b, c)f(z) \in K_p(\eta; \phi) \\
&\Leftrightarrow f(z) \in K^{\lambda}_{\mu,p}(a, b, c; \eta, \phi),
\end{aligned}$$

and

$$\begin{aligned}
&f(z) \in K^{\lambda}_{\mu,p}(a, b, c; \eta, \phi) \\
&\Leftrightarrow \frac{zf'(z)}{p} \in S^{\lambda}_{\mu,p}(a, b, c; \eta, \phi) \Rightarrow \frac{zf'(z)}{p} \in S^{\lambda}_{\mu,p}(a+1, b, c; \eta, \phi),
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\Leftrightarrow \frac{z\Big(I^{\lambda}_{\mu,p}(a+1, b, c)f(z)\Big)'}{p} \in S^*_p(\eta; \phi) \Leftrightarrow I^{\lambda}_{\mu,p}(a+1, b, c)f(z) \in K_p(\eta; \phi) \\
&\Leftrightarrow f(z) \in K^{\lambda}_{\mu,p}(a+1, b, c; \eta; \phi).
\end{aligned}$$

Combining (3.8) and (3.9), we deduce that the assertion (3.7) of Theorem 3.2 holds. $\square$

Theorem 3. *Let $0 \le \eta < p$, $0 \le \delta < p$ and $\varphi, \psi \in \mathcal{P}$ with (3.1) holds. Then*

$$C_{\mu,p}^{\lambda+1}(a,b,c;\eta,\delta;\phi,\psi) \subset C_{\mu,p}^{\lambda}(a,b,c;\eta,\delta;\phi,\psi) \subset C_{\mu,p}^{\lambda}(a+1,b,c;\eta,\delta;\phi,\psi) \tag{3.10}$$

Proof. Let $f \in C_{\mu,p}^{\lambda+1}(a,b,c;\eta,\delta;\phi,\psi)$. Then, by definition, we know that

$$\frac{1}{p-\delta}\left(\frac{z\big(I_{\mu,p}^{\lambda+1}(a,b,c)f(z)\big)^{'}}{I_{\mu,p}^{\lambda+1}(a,b,c)g(z)} - \delta\right) \prec \psi(z) \quad (z \in U) \tag{3.11}$$

with $g(z) \in S_{\mu,p}^{\lambda+1}(a,b,c;\eta,\phi)$, Moreover, by Theorem 3.1, we know that $g \in S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$, which implies that

$$q(z) = \frac{1}{p-\eta}\left(\frac{z\big(I_{\mu,p}^{\lambda}(a,b,c)g(z)\big)^{'}}{I_{\mu,p}^{\lambda}(a,b,c)g(z)} - \eta\right) \prec \phi(z) \quad (z \in \mathbb{U}). \tag{3.12}$$

We now suppose that

$$p(z) = \frac{1}{p-\delta}\left(\frac{z\big(I_{\mu,p}^{\lambda}(a,b,c)f(z)\big)^{'}}{I_{\mu,p}^{\lambda}(a,b,c)f(z)} - \delta\right) \quad (z \in \mathbb{U}), \tag{3.13}$$

where p is analytic in $\mathbb{U}$ with $p(0) = 1$. Combining (1.14) and (3.13), we find that

$$[p(z)(p-\delta)+\delta]I_{\mu,p}^{\lambda}(a,b,c)g(z) = (\lambda+p)I_{\mu,p}^{\lambda+1}(a,b,c)f(z) - \lambda I_{\mu,p}^{\lambda}(a,b,c)f(z). \tag{3.14}$$

Differentiating both sides of (3.14) with respect to z and multiplying the resulting equation by z, we get

$$[p(z)(p-\delta)+\delta][q(z)(p-\eta)+\eta+\lambda] + z(p-\eta)p^{'}(z) = (\lambda+p)\frac{z\big(I_{\mu,p}^{\lambda+1}(a,b,c)f(z)\big)^{'}}{I_{\mu,p}^{\lambda}(a,b,c)g(z)}. \tag{3.15}$$

In view of (1.14), (3.12) and (3.15), we deduce that

$$\frac{1}{p-\delta}\left(\frac{z\big(I_{\mu,p}^{\lambda+1}(a,b,c)f(z)\big)^{'}}{I_{\mu,p}^{\lambda+1}(a,b,c)g(z)} - \delta\right) = p(z) + \frac{zp(\acute{z})}{(p-\eta)q(z)+\eta+\lambda} \prec \psi(z)(z \in U). \tag{3.16}$$

By noting that (3.1) holds and

$$q(z) \prec \phi(z) \quad (z \in \mathbb{U}),$$

we know that

$$Re\big((p-\eta)q(z)+\eta+\lambda\big) > 0 \quad (z \in \mathbb{U}).$$

Thus, an application of lemma 2.2 to (3.16) yields

$$p(z) \prec \psi(z) \quad (z \in \mathbb{U}),$$

that is $f \in C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$, which implies that

$$C^{\lambda+1}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi) \subset C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi). \tag{3.17}$$

By means of (1.15) and (3.1), and using the similar arguments of the details above, we get

$$C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi) \subset C^{\lambda}_{\mu,p}(a+1,b,c;\eta,\delta;\phi,\psi). \tag{3.18}$$

Combining (3.17) and (3.18), we readily deduce that the assertion (3.10) of Theorem 3.3 holds. □

Theorem 4. *Let $0 \le \eta < p$, $0 \le \delta < p$ and $\phi, \psi \in \mathcal{P}$ with (3.1) holds. Then*

$$\mathcal{Q}C^{\lambda+1}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi) \subset \mathcal{Q}C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi) \subset \mathcal{Q}C^{\lambda}_{\mu,p}(a+1,b,c;\eta,\delta;\phi,\psi).$$

Proof. By virtue of (1.17) and Theorem 3.3, and by similarly applying the method of proof Theorem 3.2, we conclude that the assertion of Theorem 3.4 holds. □

4 A set of integral – preserving properties

In this section, we derive some integral - preserving properties involving certain integral operators.

Theorem 5. *Let $f \in S^{\lambda}_{\mu,p}(a,b,c;\eta,\phi)$ with $\phi \in \mathcal{P}$ and*

$$Re\big(\sigma+\eta\xi+(p-\eta)\xi\phi(z)\big) > 0 \quad (z \in \mathbb{U};\sigma \in \mathbb{C};\xi \in \mathbb{C};\xi \in \mathbb{C}/\{0\}). \tag{4.1}$$

Then the function $\mathcal{K}_\xi^\sigma(f) \in \mathcal{A}_p$ *defined by*

$$\begin{aligned} I_{\mu,p}^{\lambda}(a,b,c)\mathcal{K}_\xi^\sigma(f) &= I_{\mu,p}^{\lambda}(a,b,c)\mathcal{K}_\xi^\sigma(f)(z) \\ &= \left(\frac{\sigma+p\xi}{z^\sigma}\int_0^z t^{\sigma-1}\big(I_{\mu,p}^{\lambda}(a,b,c)f(t)\big)^\xi dt\right)^{\frac{1}{\xi}} \quad (z\in\mathbb{U};\xi\neq 0) \end{aligned} \tag{4.2}$$

belongs the class $S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$ *.*

Proof. Let $f \in S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$ and suppose that

$$\mathbb{M}(z) = \frac{1}{p-\eta}\left(\frac{z\big(I_{\mu,p}^{\lambda}(a,b,c)\mathcal{K}_\xi^\sigma(f)\big)'(z)}{I_{\mu,p}^{\lambda}(a,b,c)\mathcal{K}_\xi^\sigma(f)(z)} - \eta\right) \quad (z\in\mathbb{U}). \tag{4.3}$$

Combining (4.2) and (4.3), we have

$$\sigma+\eta\xi+(p-\eta)\xi\mathbb{M}(z) = (\sigma+p\xi)\left(\frac{I_{\mu,p}^{\lambda}(a,b,c)f(z)}{I_{\mu,p}^{\lambda}(a,b,c)\mathcal{K}_\xi^\sigma(f)}\right)^\xi. \tag{4.4}$$

Making use of (4.2), (4.3) and (4.4), we get

$$\begin{aligned} \mathbb{M}(z) + \frac{z\mathbb{M}'(z)}{\sigma+\eta\xi+(p-\eta)\xi\mathbb{M}(z)} &= \\ \frac{1}{p-\eta}\left(\frac{z\big(I_{\mu,p}^{\lambda}(a,b,c)f(z)\big)'}{I_{\mu,p}^{\lambda}(a,b,c)f(z)} - \eta\right) &\prec \phi(z) \quad (z\in\mathbb{U}). \end{aligned} \tag{4.5}$$

Since (4.1) holds, an application of lemma 2.1 to (4.5) yields

$$\mathbb{M}(z) \prec \phi(z) \quad (z\in\mathbb{U}),$$

that is $\mathcal{K}_\xi^\sigma(f) \in S_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$. This completes the proof of Theorem 4.1. □

Theorem 6. *Let* $f \in K_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$ *with* $\phi \in \mathcal{P}$ *and (4.1) holds. Then the function* $\mathcal{K}_\xi^\sigma(f) \in \mathcal{A}_p$ *defined by (4.2) belong to the class* $K_{\mu,p}^{\lambda}(a,b,c;\eta,\phi)$*.*

Proof. By virtue of (1.16) and Theorem 4.1, we conclude that the assertion of Theorem 4.2 holds. □

Theorem 7. *Let* $f \in C_{\mu,p}^{\lambda}(a,b,c;\eta,\delta;\phi,\psi)$ *with* $\phi \in \mathcal{P}$ *and (4.1) holds. Then the function* $\mathcal{K}_\xi^\sigma(f) \in \mathcal{A}_p$ *defined by (4.2) belongs to the class* $C_{\mu,p}^{\lambda}(a,b,c;\eta,\delta;\phi,\psi)$*.*

Proof. Let $f \in C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$ Then, by definition, we know that there exists a function $g \in S^{\lambda}_{\mu,p}(a,b,c;\eta,\phi)$. such that

$$\frac{1}{p-\eta}\left(\frac{z\big(I^{\lambda}_{\mu,p}(a,b,c)f(z)\big)'}{I^{\lambda}_{\mu,p}(a,b,c)g(z)} - \eta\right) \prec \psi(z) \quad (z \in \mathbb{U})$$

holds. Since $g \in S^{\lambda}_{\mu,p}(a,b,c;\eta,\phi)$, by Theorem 4.1, we easily find that $\mathcal{K}^{\sigma}_{\xi}(g) \in S^{\lambda}_{\mu,p}(a,b,c;\eta,\phi)$, which implies that

$$\mathbb{D}(z) = \frac{1}{p-\eta}\left(\frac{z\big((I^{\lambda}_{\mu,p}(a,b,c)\mathcal{K}^{\sigma}_{\xi}(g)(z)\big)'}{I^{\lambda}_{\mu,p}(a,b,c)\mathcal{K}^{\sigma}_{\xi}g(z)} - \eta\right) \prec \phi(z). \tag{4.6}$$

We now set

$$\mathbb{K}(z) = \frac{1}{p-\delta}\left(\frac{z\big((I^{\lambda}_{\mu,p}(a,b,c)\mathcal{K}^{\sigma}_{\xi}(g)(z)\big)'}{I^{\lambda}_{\mu,p}(a,b,c)\mathcal{K}^{\sigma}_{\xi}g(z)} - \delta\right), \tag{4.7}$$

where $\mathbb{K}$ is analytic in $\mathbb{U}$ with $\mathbb{K}(0) = 1$. From (4.2) and (4.7), we get

$$\xi[(p-\delta)\mathbb{K}(z)+\delta]I^{\lambda}_{\mu,p}(a,b,c)\mathcal{K}^{\sigma}_{\xi}(g)(z) + \delta I^{\lambda}_{\mu,p}(a,b,c)\mathcal{K}^{\sigma}_{\xi}(f)(z) = (\delta+p\xi)I^{\lambda}_{\mu,p}(a,b,c)f(z). \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we find that

$$\xi(p-\delta)z\mathbb{K}'(z) + [(p-\delta)\mathbb{K}(z)+\delta][(p-\eta)\xi\mathbb{D}(z)+\delta+\eta\xi] = (\delta+p\xi)\frac{z\big(I^{\lambda}_{\mu,p}(a,b,c)f(z)\big)'}{I^{\lambda}_{\mu,p}(a,b,c)\mathcal{K}^{\sigma}_{\xi}(g)(z)}. \tag{4.9}$$

By virtue of (1.14), (4.6) and (4.9), we deduce that

$$\frac{1}{p-\delta}\left(\frac{z\big(I^{\lambda}_{\mu,p}(a,b,c)f(z)\big)'}{I^{\lambda}_{\mu,p}(a,b,c)g(z)} - \delta\right) = \mathbb{K}(z) + \frac{z\mathbb{K}'(z)}{(p-\eta)\xi\mathbb{D}(z)+\delta+\eta\xi} \prec \psi(z) \quad (z \in \mathbb{U}). \tag{4.10}$$

The remainder of the proof of Theorem 4.3 is similar to that of Theorem 3.3. We therefore choose to omit the analogous details involved. We thus find that

$$\mathbb{K}(z) \prec \psi(z) \quad (z \in \mathbb{U}),$$

which implies that $\mathcal{K}^{\sigma}_{\xi}(f) \in C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$. The proof of Theorem 4.3 is thus completed. □

Theorem 8. *Let $f \in \mathcal{QC}^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$ with $\phi \in \mathcal{P}$ and (4.1) holds. Then the function $\mathcal{K}^{g}_{\xi}(f) \in \mathcal{A}_p$ defined by (4.2) belongs to the class $\mathcal{QC}^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$.*

Proof. In view of (1.17) and Theorem 4.3, we deduce that the assertion of Theorem 4.4 holds.

By similarly applying the methods of proof of Theorems 4.1- 4.4, we get the following integral - preserving properties. □

Colorally 1. *Let $f \in S^{\lambda}_{\mu,p}(a,b,c;\eta,\phi)$ with $\phi \in \mathcal{P}$ and*

$$Re\big(\phi(z)\big) > \max\left\{0, -\frac{Re(v)+\eta}{p-\eta}\right\}\big(z \in \mathbb{U}; Re(v) > -p\big)\ . \tag{4.11}$$

Then the integral operator $F_v(f)$ defined by

$$F_v(f) = F_v(f)(z) = \frac{v+p}{z^v}\int\limits_0^z t^{v-1} f(t)\ dt\ \big(z \in \mathbb{U}; Re(v) > -p\big) \tag{4.12}$$

belongs to the class $S^{\lambda}_{\mu,p}(a,b,c;\eta,\phi)$.

Colorally 2. *Let $f \in K^{\lambda}_{\mu,p}(a,b,c;\eta;\phi)$ with $\phi \in \mathcal{P}$ and (4.11) holds. Then the integral operator $F_v(f)$ defined by (4.12) belongs to the class $K^{\lambda}_{\mu,p}(a,b,c;\eta,\phi)$.*

Colorally 3. *Let $f \in C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$ with $\phi \in P$ and (4.11) holds. Then the integral operator $F_v(f)$ defined by (4.12) belongs to the class $C^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$.*

Colorally 4. *Let $f \in \mathcal{QC}^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$ with $\phi \in P$ and (4.11) holds. Then the integral operator $F_v(f)$ defined by (4.12) belongs to the class $\mathcal{QC}^{\lambda}_{\mu,p}(a,b,c;\eta,\delta;\phi,\psi)$.*

5 Subordination and Superordination results

In this section, we drive some subordination and superordination results associated with the operator $I^{\lambda}_{\mu,p}(a,b,c)$. By similarly applying the methods of proof of the results obtained by Cho and Srivastava [4], we get the following subordination and superordination results. Here, we choose to omit the details involved.

Colorally 5. *Let $f, g \in \mathcal{A}_p$ and $\lambda > -p$. If*

$$Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho\ \left(z \in \mathbb{U}; \varphi(z) = \frac{I^{\lambda+1}_{\mu,p}(a,b,c)g(z)}{z^p}\right),$$

where

$$\varrho = \frac{1+(p+\lambda)^2 - |1-(p+\lambda)^2|}{4(p+\lambda)}, \tag{5.1}$$

then the subordination relationship

$$\frac{I_{\mu,p}^{\lambda+1}(a,b,c)f(z)}{z^p} \prec \frac{I_{\mu,p}^{\lambda+1}(a,b,c)g(z)}{z^p} \quad (z \in \mathbb{U})$$

implies that

$$\frac{I_{\mu,p}^{\lambda}(a,b,c)f(z)}{z^p} \prec \frac{I_{\mu,p}^{\lambda}(a,b,c)g(z)}{z^p} \quad (z \in \mathbb{U}).$$

Furthermore, the function $\frac{I_{\mu,p}^{\lambda}(a,b,c)g(z)}{z^p}$ *is the best dominant.*

Denote by $\mathcal{Q}$ *the set of all functions* f *that are analytic and injective on* $\overline{\mathbb{U}} - E(f)$, *where*

$$E(f) = \left\{\varepsilon \in \vartheta\mathbb{U} : \lim_{z\to\varepsilon} f(z) = \infty\right\},$$

and such that $f'(\varepsilon) \neq 0$ *for* $\varepsilon \in \vartheta\mathbb{U} - E(f)$. *If* f *is subordinate to* $\mathcal{F}$, *then* $\mathcal{F}$ *is superordinate to* f.

We now derive the following superordination results.

Colorally 6. *Let* $f, g \in \mathcal{A}_p$ *and* $\lambda > -p$. *If*

$$Re\left(1 + \frac{z\varphi''(z)}{\varphi'(z)}\right) > -\varrho\left(z \in \mathbb{U}; \varphi(z) = \frac{I_{\mu,p}^{\lambda+1}(a,b,c)g(z)}{z^p}\right),$$

where ϱ *is given by (5.1), also let the function* $I_{\mu,p}^{\lambda+1}(a,b,c)f(z)/z^p$ *be univalent in* $\mathbb{U}$ *and* $I_{\mu,p}^{\lambda}(a,b,c)f(z)/z^p \in \mathcal{Q}$, *then the subordination relationship*

$$\frac{I_{\mu,p}^{\lambda+1}(a,b,c)g(z)}{z^p} \prec \frac{I_{\mu,p}^{\lambda+1}(a,b,c)f(z)}{z^p} \quad (z \in \mathbb{U})$$

implies that

$$\frac{I_{\mu,p}^{\lambda}(a,b,c)g(z)}{z^p} \prec \frac{I_{\mu,p}^{\lambda}(a,b,c)f(z)}{z^p} \quad (z \in \mathbb{U}).$$

Furthermore, the function $I^{\lambda}_{\mu,p}(a,b,c)g(z)/z^p$ *is the best subordinant.*

Combining the above - mentioned subordination and superordination results involving the operator $I^{\lambda}_{\mu,p}(a,b,c)$ *we get following' sandwich - type results.*

Colorally 7. *Let* $f, g_k \in \mathcal{A}_p$ $(k = 1,2)$ *and* $\lambda > -p$. *If*

$$Re\left(1+\frac{z\varphi_k^{''}(z)}{\varphi_k^{'}(z)}\right) > -\varrho \quad \left(z\in\mathbb{U}; \varphi_k(z) = \frac{I^{\lambda+1}_{\mu,p}(a,b,c)g_k(z)}{z^p} \quad (k=1,2)\right),$$

where ϱ *is given by (5.1), also let the function* $I^{\lambda+1}_{\mu,p}(a,b,c)f(z)/z^p$ *be univalent in* $\mathbb{U}$ *and* $I^{\lambda}_{\mu,p}(a,b,c)f(z)/z^p \in \mathcal{Q}$, *then the subordination chain*

$$\frac{I^{\lambda+1}_{\mu,p}(a,b,c)g_1(z)}{z^p} \prec \frac{I^{\lambda+1}_{\mu,p}(a,b,c)f(z)}{z^p} \prec \frac{I^{\lambda+1}_{\mu,p}(a,b,c)g_2(z)}{z^p} \quad (z\in\mathbb{U})$$

implies that

$$\frac{I^{\lambda}_{\mu,p}(a,b,c)g_1(z)}{z^p} \prec \frac{I^{\lambda}_{\mu,p}(a,b,c)f(z)}{z^p} \prec \frac{I^{\lambda}_{\mu,p}(a,b,c)g_2(z)}{z^p} \quad (z\in\mathbb{U}).$$

Furthermore, the function $I^{\lambda}_{\mu,p}(a,b,c)g_1(z)/z^p$ *and* $I^{\lambda}_{\mu,p}(a,b,c)g_2(z)/z^p$ *are, respectively, the best subordinant and the best dominant.*

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