Int. J. Open Problems Compt. Math., Vol. 1, No. 1, June 2014
ISSN 1998-6262; Copyright ©ICSRS Publication, 2014
www.i-csrs.org

On finite groups with perfect subgroup order subsets

Marius Tărnăuceanu

Faculty of Mathematics, "Al. I. Cuza" University, Iasi, Romania
e-mail: tarnauc@uaic.ro

Abstract

A finite group G is said to be a PSOS-group if for every subgroup H of G the cardinality of the set $\{K \leq G \mid |K| = |H|\}$ divides $|G|$. In this note a first step in the study of PSOS-groups is made.

Keywords: *finite groups, number of subgroups.*
MSC (2000) : Primary 20D60; Secondary 20D30.

1 Introduction

Let G be a finite group and $x \in G$. Then the set of all elements of G having the same order as x is called the *order subset* of G determined by x (see [3]). We say that G is a *group with perfect order subsets*, or briefly a *POS-group*, if the number of elements in each order subset of G is a divisor of $|G|$. Many recent works, such as [2], [4], [6, 7], [9] and [14], deal with the structure of POS-groups.

Now, let H be a subgroup of G. Inspired by the above notions, we call the *subgroup order subset* of G determined by H the set of all subgroups of G having the same order as H. We will say that G is a *group with perfect subgroup order subsets*, or briefly a *PSOS-group*, if the number of elements in each subgroup order subset of G is a divisor of $|G|$. The study of the class of PSOS-groups is the main goal of our paper.

Most of our notation is standard and will usually not be repeated here. Elementary concepts and results on group theory can be found in [5] and [11]. For subgroup lattice notions we refer the reader to [8], [10] and [12].

2 Main results

First of all, we observe that any finite cyclic group is a PSOS-group. Our study is facile for finite p-groups.

Theorem 1. *A finite p-group is a PSOS-group if and only if it is cyclic.*

Proof. Let G be a finite p-group of order p^n. It is well-known that the number of subgroups of order p^m, $m = 0, 1, ..., n$, is congruent to 1 modulo p. If G is a PSOS-group, then this number must be equal to 1. In other words, G has a unique subgroup of order p^m for every $m = 0, 1, ..., n$, i.e. it is cyclic. □

We remark that there are also many examples of non-cyclic PSOS-groups, such as S_3, D_{10}, $\mathbb{Z}_2 \times \mathbb{Z}_6$, A_4, A_5, ... and so on. We easily infer that the class of PSOS-groups is not closed under subgroups, homomorphic images, direct products or extensions.

The following four propositions give the intersection between the class of PSOS-groups and other important classes of finite groups.

Proposition 2. *The dihedral group D_{2n} is a PSOS-group if and only if n is odd.*

Proof. The subgroup structure of $D_{2n} = \langle x, y \mid x^n = y^2 = 1,\, yxy = x^{-1}\rangle$ is well-known: for every divisor d or n, D_{2n} possesses a subgroup isomorphic to $\mathbb{Z}_d$, namely $H_0^d = \langle x^{\frac{n}{d}}\rangle$, and $\frac{n}{d}$ subgroups isomorphic to D_{2d}, namely $H_i^d = \langle x^{\frac{n}{d}}, x^{i-1}y\rangle$, $i = 1, 2, ..., \frac{n}{d}$. If n is odd, then D_{2n} is obviously a PSOS-group. If n is even, then by taking $d = 2$ one obtains that D_{2n} has $n+1$ subgroups of order 2, and therefore it is not a PSOS-group since $n+1 \nmid 2n$. □

A result which is similar to Proposition 2 also holds for generalized quaternion groups.

Proposition 3. *The generalized quaternion group Q_{4n} is a PSOS-group if and only if n is odd.*

Proof. $Q_{4n} = \langle x, y \mid x^{2n} = 1,\, y^2 = x^n,\, yxy^{-1} = x^{-1}\rangle$ satisfies the following properties: $Z(Q_{4n}) = \langle x^n\rangle \cong \mathbb{Z}_2$ and $Q_{4n}/Z(Q_{4n}) \cong D_{2n}$. Moreover, for every subgroup H of Q_{4n} we have either $\langle x^n\rangle \subseteq H$ or $H \subseteq \langle x\rangle$. In other words, the subgroups of Q_{4n} either are contained in the lattice interval $[Q_{4n}/\langle x^n\rangle]$ or are contained in $\langle x\rangle$ and not contain $\langle x^n\rangle$. Clearly, Q_{4n} is a PSOS-group if n is odd. If n is even, then Q_{4n} has $n+1$ subgroups of order 4, namely $\langle yx^i\rangle$, $i = 0, 1, ..., n-1$, and $\langle x^{\frac{n}{2}}\rangle$. Because $n+1 \nmid 4n$ it follows that Q_{4n} is not a PSOS-group. □

We recall next the notion of P-group, according to [8, 10, 12]. Let p be a prime, $n \geq 2$ be an integer and G be a group. We say that G belongs to the class $P(n,p)$ if it is either elementary abelian of order p^n, or a semidirect product of an elementary abelian normal subgroup H of order p^{n-1} by a group of prime order $q \neq p$ which induces a nontrivial power automorphism on H. The group G is called a *P-group* if $G \in P(n,p)$ for some prime p and some $n \geq 2$. The class $P(n,2)$ consists only of the elementary abelian group of order 2^n. Also, for $p > 2$ the class $P(n,p)$ contains the elementary abelian group of order p^n and, for every prime divisor q of $p-1$, exactly one non-abelian P-group with elements of order q. This is of order $p^{n-1}q$ and will be denoted by $G_{n,p}$.

Proposition 4. *The non-abelian group $G_{n,p}$ is a PSOS-group if and only if $n = 2$.*

Proof. Obviously, $G_{2,p}$ has one subgroup of order 1, one subgroup of order p, p subgroups of order q and one subgroup of order pq. So, it is a PSOS-group.

Let $n \geq 3$ and suppose that $G_{n,p}$ is a PSOS-group. Then the number of subgroups of order pq in $G_{n,p}$ divides $p^{n-1}q$. On the other hand, it is easy to see that this number is $p^{n-2}\frac{p^{n-1}-1}{p-1}$. It follows that $\frac{p^{n-1}-1}{p-1} \mid pq$, and consequently $\frac{p^{n-1}-1}{p-1} \mid q$, a contradiction. $\square$

Finally, we determine the positive integers n for which the symmetric group S_n and the alternating group A_n are PSOS-groups (we wishes to thank Professor Derek Holt for this suggestions on MathOverflow – see [13]).

Proposition 5. *S_n is a PSOS-group if and only if $n \leq 3$, while A_n is a PSOS-group if and only if $n \leq 5$.*

Proof. We already have seen that S_n and A_n are PSOS-groups for $n \leq 3$ and $n \leq 5$, respectively.

Assume first that $n \geq 4$. Then S_n contains an elementary abelian 2-subgroup $\langle (1,2),(3,4),...\rangle$ of order $2^{[\frac{n}{2}]}$, and this has more than $2^{[\frac{n}{4}][\frac{n+2}{4}]}$ subgroups of order $2^{[\frac{n}{4}]}$. For $n \geq 82$ we have

$$2^{[\frac{n}{4}][\frac{n+2}{4}]} > n!$$

and therefore the number of subgroups of order $2^{[\frac{n}{4}]}$ in S_n cannot be a divisor of $n!$. On the other hand, the number a_n of subgroups of order 2 in S_n satisfies

$$a_1 = 0, a_2 = 1 \text{ and } a_{n+1} = a_n + na_{n-1} + n,\ \forall n \geq 2$$

(see the sequence A000085) and we can easily check by computer that $a_n \nmid n!$ for all $4 \leq n \leq 81$. Hence S_n is not a PSOS-group for $n \geq 4$.

Assume now that $n \geq 4$. Then A_n contains an elementary abelian 2-subgroup of order $2^{[\frac{n}{2}]-1}$, and this has more than $2^{[\frac{n}{4}][\frac{n-2}{4}]}$ subgroups of order $2^{[\frac{n-2}{4}]}$. For $n \geq 86$ we have

$$2^{[\frac{n}{4}][\frac{n-2}{4}]} > \frac{n!}{2},$$

which implies that the number of subgroups of order $2^{[\frac{n-2}{4}]}$ in A_n is not a divisor of $\frac{n!}{2}$, i.e. A_n is not a PSOS-group. By using a computer algebra program the same conclusion is also obtained for $6 \leq n \leq 85$ (e.g. the number of subgroups of order 4 in A_6 is 75 and $75 \nmid 360 = |A_6|$). Hence A_n is not a PSOS-group for $n \geq 6$. $\square$

As we have observed above, the class of PSOS-groups is not closed under subgroups. It therefore makes sense to study finite groups whose all subgroups are PSOS-groups.

Theorem 6. *All subgroups of a finite group G are PSOS-groups if and only if G is a* ZM*-group.*

Proof. If all subgroup of G are PSOS-groups, then so are the Sylow subgroups of G. By Theorem 1 we infer that they are cyclic, and thus G is a ZM-group.

Conversely, suppose that G is a ZM-group. Then it is of type

$$\mathrm{ZM}(m,n,r) = \langle a, b \mid a^m = b^n = 1,\ b^{-1}ab = a^r \rangle,$$

where the triple (m,n,r) satisfies the conditions

$$(m,n) = (m,r-1) = 1 \text{ and } r^n \equiv 1 \ (\mathrm{mod}\ m).$$

The subgroups of $\mathrm{ZM}(m,n,r)$ have been completely described in [1]. Set

$$L = \left\{ (m_1,n_1,s) \in \mathbb{N}^3 \ \mid \ m_1|m,\ n_1|n,\ s < m_1,\ m_1|s\frac{r^n-1}{r^{n_1}-1} \right\}.$$

Then there is a bijection between L and the subgroup lattice $L(\mathrm{ZM}(m,n,r))$ of $\mathrm{ZM}(m,n,r)$, namely the function that maps a triple $(m_1,n_1,s) \in L$ into the subgroup $H_{(m_1,n_1,s)}$ defined by

$$H_{(m_1,n_1,s)} = \bigcup_{k=1}^{\frac{n}{n_1}} \alpha(n_1,s)^k \langle a^{m_1} \rangle = \langle a^{m_1}, \alpha(n_1,s) \rangle,$$

where $\alpha(x,y) = b^x a^y$, for all $0 \leq x < n$ and $0 \leq y < m$. Remark that $|H_{(m_1,n_1,s)}| = \frac{mn}{m_1n_1}$, for any s satisfying $(m_1,n_1,s) \in L$, and that there are $\gcd(m_1, \frac{r^n-1}{r^{n_1}-1})$ such triples. Consequently, for every $m_1|m$ and $n_1|n$, $\mathrm{ZM}(m,n,r)$ possesses $\gcd(m_1, \frac{r^n-1}{r^{n_1}-1})$ subgroups of order $\frac{mn}{m_1n_1}$, showing that it is a PSOS-group. Since all subgroups of a ZM-group are ZM-groups, it follows that all subgroups of G are also PSOS-groups, as desired. $\square$

Next, let G be a finite group of order n. For every prime p dividing n we denote $n_p(G) = |\{H \leq G \,|\, |H| = p\}|$. Then, by using the subgroup structure of a direct product of two groups (see (4.19) of [11], I), we infer that

$$n_p(G \times G) = 2n_p(G) + (p-1)n_p(G)^2.$$

This equality shows that if p is odd, then $n_p(G \times G)$ is even. So, we have proved the following theorem.

Theorem 7. *If G is a finite group of odd order, then $G \times G$ is not a PSOS-group. More generally, if G_1 and G_2 are two finite groups of odd orders such that* $\gcd(|G_1|, |G_2|) \neq 1$*, then $G_1 \times G_2$ is not a PSOS-group.*

We remark that the above formula is also useful in the case when n is even. For example, we have

$$n_2(S_3 \times S_3) = 2n_2(s_3) + n_2(S_3)^2 = 2 \cdot 3 + 9 = 15,$$

that does not divide $36 = |S_3 \times S_3|$. In this way, $S_3 \times S_3$ is not a PSOS-group.

Inspired by these results, we came up with the following conjecture, which we have verified by computer for many groups of small order.

Conjecture 8. *For every finite group G, the direct product $G \times G$ is not a PSOS-group.*

3 Open Problem

Determine the intersection between the class of PSOS-groups and other remarkable classes of finite groups. For example, which are the finite abelian PSOS-groups?

References

[1] W.C. Calhoun, *Counting subgroups of some finite groups*, Amer. Math. Monthly **94** (1987), 54-59.

[2] A.K. Das, *On finite groups having perfect order subsets*, Int. J. Algebra **3** (2009), 629-637.

[3] C.E. Finch and L. Jones, *A curious connection between Fermat numbers and finite groups*, Amer. Math. Monthly **109** (2002), 517-524.

[4] C.E. Finch and L. Jones, *Nonabelian groups with perfect order subsets*, JP J. Algebra Number Theory Appl. **3** (2003), 13-26. See also: *Corrigendum to: "Nonabelian groups with perfect order subsets"*, JP J. Algebra Number Theory Appl. **4** (2004), 413-416.

[5] I.M. Isaacs, *Finite group theory*, Amer. Math. Soc., Providence, R.I., 2008.

[6] L. Jones and K. Toppin, *On three questions concerning groups with perfect order subsets*, Involve **4** (2011), 251-261.

[7] S. Libera and P. Tlucek, *Some perfect order subset groups*, Pi Mu Epsilon J. **11** (2003), 495-498.

[8] R. Schmidt, *Subgroup lattices of groups*, de Gruyter Expositions in Mathematics 14, de Gruyter, Berlin, 1994.

[9] R. Shen, *A note on finite groups having perfect order subsets*, Int. J. Algebra **4** (2010), 631-641.

[10] M. Suzuki, *Structure of a group and the structure of its lattice of subgroups*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Neue Folge, Heft **10**, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956.

[11] M. Suzuki, *Group theory*, I, II, Springer Verlag, Berlin, 1982, 1986.

[12] M. Tărnăuceanu, *Groups determined by posets of subgroups*, Ed. Matrix Rom, Bucureşti, 2006.

[13] M. Tărnăuceanu, *A question on the number of subgroups of symmetric groups*, MathOverflow, http://mathoverflow.net/questions//140359.

[14] N.T. Tuan and B.X. Hai, *On perfect order subsets in finite groups*, Int. J. Algebra **4** (2010), 1021-1029.