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$M^{[X]}/G/1$ Retrial Queueing system with Second Optional Service, Random break down, Set up time and Bernoulli vacation

G. Ayyappan[1] and S.Shyamala[2]

[1]Department of Mathematics
Pondicherry Engineering College,
Pondicherry University
Pondicherry, India
e-mail:ayyappanpec@hotmail.com
[2]Department of Mathematics,
Arunai Engineering College,
Thiruvannamalai,India
e-mail:subramaniyanshyamala@gmail.com

Abstract

In this paper, we investigate a single server batch arrival non-Markovian retrial queueing model with random break down and Bernoulli vacation. Customers arrive in batches accordingly Poisson process with arrival rate λ but are served one by one with first come first served basis. All customers demand the first essential service, whereas only some of them demand the second optional service. The server is assumed to be unreliable so that it may encounter break down at any time. As the server has to be repaired, the repair process start immediately. The customer, who finds the server busy upon arrival, can either join the orbit with probability p or he/she can leave the system with probability 1-p. Upon completion of a service the server may go for a vacation with probability θ or stay back in the system to serve a next customer with probability $1-\theta$, if any. After finishing a vacation, the server could not provide service as the it should go for a setup time. Assuming that, the retrial time, the service time, the repair time, the vacation completion time and the set up time of the server are all arbitrarily distributed. We obtain the transient solution and

steady solution of the model by using supplementary variable technique. Also we derive the system performance measures, reliability indices for the prescribed model.

Key words: *Batch size, Break down, Essential service, Optional service, Reliability indices, Steady solution, Setup time, Transient solution.*

1 Introduction

The study of retrial queue in queueing theory has become an active area because of its wide applicability in telephone switching systems, telecommunication networks and computer networks. Retrial queues are characterized by the feature that a customer who finds the server is busy or down or on vacation he/she may decide to join a group of blocked customers (called orbit) for repeating their demand or request after some random amount of time or leave the system immediately. Fayolle [15] introduced constant retrial policy in which the head customer of the orbit queue can request a service with exponential retrial time. Artalejo[3] found new results for the same model. For bibliographical study on retrial queues the reader can refer Yang and Templeton [34], Fallin [18], Artalejo [4]. Artalejo and Falin [7] have done a comparative analysis between standard and retrial queues. Choi et al.[16] studied the same model with generalized retrial times and Bernoulli schedule. Retrial queues with vacation have also been attracted by many authors during recent years. Artalejo [4,5] discussed retrial queues with exhaustive vacation. Boualem et al.[9] obtained stochastic inequalities of M/G/1 retrial queue with vacations by using constant retrial policy. Atencia [8] also studied single server with general retrial retrial time and Bernoulli vacation. Zhou [35] studied the same model with FCFS orbit policy.

Retrial queues with unreliable servers and repairs were also studied by many authors. Aissani[1]and Kulkarni[28] studied retrial queueing system with repeated attempts and unreliable server. Artalejo [3] found new results in retrial queueing systems with break downs. Wang et al.[32] incorporated reliability analysis on retrial queue with server breakdowns and repairs. Gautam Choudhury [14] discussed batch arrival with above mentioned models. Djellab [17] discussed M/G/1 retial queue with breakdowns. Krishnakumar et al.[26] discussed two phase retrial queue under preemptive resume. Choudhury and Deka [11] studied Unreliable server with two phases of services and repeated attempts. Jinting Wang [22] concentrated on single server non-Markovian model with multi optional service, Bernoulli server vacation and setup time and also obtained reliability indices for the model. Ke and Chang [25]discussed retrial queue with two phase heterogeneous service, subjected to starting failures and

Bernoulli vacation. The study of retrial queue with non-persistent customers have been paid attention by many research papers. Kasthuri Ramnath and Kalidass[24] studied M/G/1 retrial queue with non-persistent customers and second optional service. Krishnamoorthy et al.[27] studied retrial queue with non-persistent customer with orbital search. Kulkarni and Choi [28] studied retrial queues subject to breakdowns and repairs. Kumar and Arumuganathan [29] studied batch arrival retrial queue with general vacation time under Bernoulli schedule and two phases of heterogeneous service.

In this paper we consider a single server queueing system in which primary customers arrive according to compound Poisson stream with rate λ. Upon arrival, customer finds the sever busy or down or on vacation or the server idle, he may obtain service immediately, Otherwise, the arriving primary customer either joins a retrial queue according to an FCFS discipline with probability p or leaves the system with probability 1- p. Also the server have to go under setup time before going to serve the customer properly. The rest of the paper is organized as follows: In Section 2, we give a brief description of the mathematical model. Section 3 deals with transient analysis of the model for which probability generating function of the distribution.In section 4 steady state solution has been obtained for the model. Some important performance measures and reliability indices of this model are derived in Section 5. In section 6, some concluding remarks have been presented and in section 7, we propose a few open problems relating with our model.

2 Mathematical Description of the Model

We consider an M/G/1 retrial queue with random break downs and Bernoulli vacation. Customers arrive at the system in batches of variable size in a compound Poisson process. Let $\lambda c_i \triangle t$ (i=1,2,3,....) be the first order probability that a batch of i customers arrives at the system during a short interval of time $(t, t + \triangle t)$, where $0 \leq c_i \leq 1$ and $\sum_{i=1}^{\infty} c_i = 1$ and $\lambda > 0$ is the mean arrival rate of batches and the customers are served one-by-one on a "first come-first served" basis. Upon arrival, if a customer finds the server idle, the customer gets service immediately. Otherwise, the server is found busy or down or on vacation, the customer is obliged to join a retrial orbit according to an FCFS discipline with probability p or leaves the system with probability 1- p.

The i^{th}service times of the customers are identically independent random variables with probability distribution function $B_i(x)$, density function $b_i(x)(i = 1, 2)$, k^{th} moment $b_k(k = 1, 2)$.As soon as the first essential service is completed the customer may opt for second optional service with probability r or leave the system with probability 1-r. When the server is serving the customers, it may encounter break down at random time so that the server will be down for a short span of time. The server's life times are generated by exogenous Poisson process with rate α. As soon as the server gets break down it is sent for repair during which the server stops providing service to the customers. The repair times $R_n; n \geq 1$ of the server are identically independent random variables with distribution function R(y)and k^{th} finite moment $r_k; k \geq 1$. After the repair process is over the server is ready to start its remaining service to the customers and in this case the service times are cumulative, which we may referred to as generalized service times.

After each service completion the server may go for a vacation of random length V with probability θ or with probability $1-\theta$ he may serve the next unit; if any. The vacation time has general distribution and its distribution function,density function, the first two moments and the conditional completion rate are$V(x), v(x), (v_1, v_2), \nu(x) = v(x)/(1-V(x))$ respectively. When the vacation period is over, the server must spend some time to setup and the time of setup is arbitrarily distributed. Its distribution function, density function, the first two moments and the conditional completion rate are $S(x), s(x), (s_1, s_2), \delta(x) = s(x)/(1-S(x))$ respectively. The retrial time of the customer in the retrial queue is generally distributed with distribution function A(x), density function $a(x)$, the mean value a, and Laplace transform $\bar{a}(s)$. Assuming that retrial times begin either at the completion instants of service or setup times so that the distribution of the remaining retrial time is $A_e(x) = \frac{1}{a}\int_0^\infty (1-A(x))dx$ and $a_e(x) = \frac{(1-A(x))}{a}$. The conditional completion rate function for retrials is given by$\gamma_e(x) = \frac{a_e(x)}{(1-A(x))}$. All stochastic processes involved in the system are assumed to be independent of each other.

Now we obtain the probability generating function of the joint distribution of the state of the server and the number in the system by treating $I^0(t), B^0(t)$ are the elapsed retial time and service time of the customers at time t respectively also $D^0(t), R^0(t)$ and $V^0(t)$ are the elapsed delay time,elapsed repair time and elapsed vacation time of the server at time t,respectively as supplementary variables. Assuming that the system is empty initially. Let N(t) be the number of customers in the retrial queue at time t, and C(t) the number of customer in service at time t. To make it a Markov process, Define the state probabilities at time t as follows:

Y(t)= 0 if the server is idle at time t; 1 if the server is idle during retrial time at time t; 2 if the server is busy with first essential service at time t; 3 if the server is busy with second optional service at time t; 4 if the server is on repair at time t; 5 if the server is on vacation at time t; 6 if the server is in set up at time t.

Introducing the supplementary $Q^0(t), B_i^0(t)(i = 1, 2), R^0(t)$ and $V^0(t)$ to obtain a bivariate Markov process $Z(t) = N(t), X(t)$,

where $X(t) = 0$ if Y(t)=0,

$X(t) = Q^0(t)$ if Y(t)=1,

$X(t) = B_1^0(t)$ if Y(t)=2,

$X(t) = B_2^0(t)$ if Y(t)=3,

$X(t) = R^0(t)$ if Y(t)=4,

$X(t) = V^0(t)$ if Y(t)=5

$X(t) = S^0(t)$ if Y(t)=6

Now We define following limiting probabilities:

$Q_0(t) = PN(t) = 0, X(t) = 0;$

$Q_n(x, t)dx = PN(t) = n, X(t) = Q^0(t); x < Q^0(t) \leq x + dx; x, t > 0, n \geq 1;$

$p_n^i(x, t)dx = PN(t) = n, X(t) = P_i^0(t); x < P_i^0(t) \leq x + dx; x, t > 0, n \geq 0,$

$V_n(x, t)dx = PN(t) = n, X(t) = V^0(t); x < V^0(t) \leq x + dx; x, t > 0, n \geq 0,$

$S_n(x, t)dx = PN(t) = n, X(t) = S^0(t); x < S^0(t) \leq x + dx; x, t > 0, n \geq 0,$

and for fixed values of x and $n \geq 0$

$R_n(x, y, t)dy = PN(t) = n, X(t) = R^0(t); y < R^0(t) \leq y + dy; x, y, t > 0.$

Further it is assumed that
$Q(0) = 0, Q(\infty) = 1, B_i(0) = 0, B_i(\infty) = 1(i = 1, 2), V(0) = 0, V(\infty) = 1, S(0) = 0, S(\infty) = 1, R(0) = 0, R(\infty) = 1$

and that Q(x),$B_i(x)(i = 1, 2)$, V(x) and S(x)are continuous at x=0 and

R(y)are continuous at y=0 respectively, so that

$\eta(x)dx = \frac{dQ(x)}{1-Q(x)}; \mu_i(x)dx = \frac{dB_i(x)}{1-B_i(x)}(i = 1, 2); \nu(x)dx = \frac{dV(x)}{1-V(x)};$

$\beta(x)dy = \frac{dR(y)}{1-R(y)}; \delta(x)dx = \frac{dS(x)}{1-S(x)}$

are the first order differential (hazard rate) functions of Q(), B(),V() and S() respectively.

3 Transient Solution of the Model

The governing equations of the model

$$\frac{d}{dt}Q_0(t) = -\lambda Q_0(t) + \int_0^\infty S_0(x,t)\delta(x)dx + (1-\theta)\left\{(1-r)\int_0^\infty P_0^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_0^{(2)}(x,t)\mu_2(x)dx\right\} \tag{1}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda + \eta(x)\right)Q_n(x,t) = 0; n \geq 1 \tag{2}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \alpha + \mu_1(x)\right)P_n^{(1)}(x,t) = \lambda p\Sigma_{i=1}^n c_i P_{n-i}^{(1)}(x,t) + \int_0^\infty R_n(x,y,t)\beta(y)dy; n \geq 0 \tag{3}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \alpha + \mu_2(x)\right)P_n^{(2)}(x,t) = \lambda p\Sigma_{i=1}^n c_i P_{n-i}^{(2)}(x,t) + \int_0^\infty R_n(x,y,t)\beta(y)dy; n \geq 0 \tag{4}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \beta(y)\right)R_n(x,y,t) = \lambda p\Sigma_{i=1}^n c_i R_{n-i}(x,y,t); n \geq 0 \tag{5}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \nu(x)\right)V_n(x,t) = \lambda p\Sigma_{i=1}^n c_i V_{n-i}(x,t); n \geq 0 \tag{6}$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + p\lambda + \delta(x)\right)S_n(x,t) = \lambda p\Sigma_{i=1}^n c_i S_{n-i}(x,t); n \geq 0 \tag{7}$$

The above equations can be solved by using the following boundary conditions,

$$Q_n(0,t) = \int_0^\infty S_n(x,t)\delta(x)dx + (1-\theta)\left\{(1-r)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx\right\} \tag{8}$$

$$P_0^{(1)}(0,t) = \lambda c_1 Q_0(t) + \int_0^\infty Q_1(x,t)\eta(x)dx \tag{9}$$

$$P_n^{(1)}(0,t) = \lambda c_{n+1}Q_n(t) + \int_0^\infty Q_{n+1}(x,t)\eta(x)dx + \lambda \int_0^\infty \Sigma_{i=1}^n c_i Q_{n+1-i}(x,t)dx; n \geq 1 \tag{10}$$

$$P_n^{(2)}(0,t) = r\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx; n \geq 0 \tag{11}$$

$$R_n(x,0,t) = \alpha P_n^{(1)}(x,t) + \alpha P_n^{(2)}(x,t); n \geq 0 \tag{12}$$

$$V_n(0,t) = \theta\left\{(1-r)\int_0^\infty P_n^{(1)}(x,t)\mu_1(x)dx + \int_0^\infty P_n^{(2)}(x,t)\mu_2(x)dx\right\} \tag{13}$$

$$S_n(0,t) = \int_0^\infty V_n(x,t)\nu(x)dx \tag{14}$$

The initial conditions are given by $Q_n(0) = 1; P_n^{(j)}(x,0) = 0; j = 1,2;$

$R_n(x,y,0) = 0; V_n(x,0) = 0; n > 0, x > 0;$

We define the probability generating function

$$Q_q(x,z,t) = \sum_{n=1}^\infty z^n Q_n(x,t); Q_q(z,t) = \sum_{n=1}^\infty z^n Q_n(t); \tag{15}$$

$$P_q^{(j)}(x,z,t) = \sum_{n=0}^\infty z^n P_n^{(j)}(x,t); P_q^{(j)}(z,t) = \sum_{n=1}^\infty z^n P_n^{(j)}(t); j = 1,2; \tag{16}$$

$$R_q(x,y,z,t) = \sum_{n=0}^\infty z^n R_n(x,y,t); R_q(x,z,t) = \sum_{n=0}^\infty z^n R_n(x,t); \tag{17}$$

$$S_q(x,z,t) = \sum_{n=0}^\infty z^n S_n(x,t); S_q(z,t) = \sum_{n=0}^\infty z^n S_n(t) \tag{18}$$

$$C(z) = \sum_{n=1}^\infty c_n z^n \tag{19}$$

which are convergent inside the circle given by $|z| \leq 1$ and define the Laplace transform of a function f(t) as

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}dt \tag{20}$$

Taking Laplace transform for the equations (1) - (14)

$$\begin{aligned}(s+\lambda)\bar{Q}_0(s) &= 1 + \int_0^\infty \bar{S}_0(x,s)\delta(x)dx + \\ &= (1-\theta)\left\{(1-r)\int_0^\infty \bar{P}_0^{(1)}(x,s)\mu_1(x)dx + \int_0^\infty \bar{P}_0^{(2)}(x,s)\mu_2(x)dx\right\}\end{aligned} \tag{21}$$

$$\left(\frac{d}{dx}+s+\lambda+\eta(x)\right)\bar{Q}_n(x,s)=0;n\geq 0 \tag{22}$$

$$\left(\frac{d}{dx}+s+p\lambda+\alpha+\mu_1(x)\right)\bar{P}_n^{(1)}(x,s)=\lambda p\Sigma_{i=1}^{n}c_i\bar{P}_{n-i}^{(1)}(x,s)+\int_0^\infty \bar{R}_n(x,y,s)\beta(y)dy;n\geq 0 \tag{23}$$

$$\left(\frac{d}{dx}+s+p\lambda+\alpha+\mu_2(x)\right)\bar{P}_n^{(2)}(x,s)=\lambda p\Sigma_{i=1}^{n}c_i\bar{P}_{n-i}^{(2)}(x,s)+\int_0^\infty \bar{R}_n(x,y,s)\beta(y)dy;n\geq 0 \tag{24}$$

$$\left(\frac{d}{dx}+s+\lambda p+\beta(y)\right)\bar{R}_n(x,y,s)=\lambda p\Sigma_{i=1}^{n}c_i\bar{R}_{n-i}(x,y,s);n\geq 0 \tag{25}$$

$$\left(\frac{d}{dx}+s+\lambda p+\nu(x)\right)\bar{V}_n(x,s)=\lambda p\Sigma_{i=1}^{n}c_i\bar{V}_{n-i}(x,s);n\geq 0 \tag{26}$$

$$\left(\frac{d}{dx}+s+\lambda p+\delta(x)\right)\bar{S}_n(x,s)=\lambda p\Sigma_{i=1}^{n}c_i\bar{S}_{n-i}(x,s);n\geq 0 \tag{27}$$

$$\bar{Q}_n(0,s)=\int_0^\infty \bar{S}_n(x,s)\delta(x)dx+(1-\theta)\left\{(1-r)\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx+\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx\right\} \tag{28}$$

$$\bar{P}_0^{(1)}(0,s)=\lambda c_1\bar{Q}_0(s)+\int_0^\infty \bar{Q}_1(x,s)\eta(x)dx \tag{29}$$

$$\bar{P}_n^{(1)}(0,s)=\lambda c_{n+1}\bar{Q}_n(x,s)+\int_0^\infty \bar{Q}_{n+1}(s)\eta(x)dx+\lambda\int_0^\infty \Sigma_{i=1}^{n}c_i\bar{Q}_{n+1-i}(x,t)dx;n\geq 1 \tag{30}$$

$$\bar{P}_n^{(2)}(0,s)=r\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx;n\geq 0 \tag{31}$$

$$\bar{R}_n(x,0,s)=\alpha\bar{P}_n^{(1)}(x,s)+\alpha\bar{P}_n^{(2)}(x,s);n\geq 0 \tag{32}$$

$$\bar{V}_n(0,s)=\theta\left\{(1-r)\int_0^\infty \bar{P}_n^{(1)}(x,s)\mu_1(x)dx+\int_0^\infty \bar{P}_n^{(2)}(x,s)\mu_2(x)dx\right\} \tag{33}$$

$$\bar{S}_n(0,s)=\int_0^\infty \bar{V}_n(x,s)\delta(x)dx \tag{34}$$

Applying probability generating function for the equations (22) -(34)

$$\left(\frac{d}{dx}+s+\lambda+\eta(x)\right)\bar{Q}_q(x,z,s)=0 \tag{35}$$

$$\left(\frac{d}{dx}+s+\lambda p(1-C(z))+\alpha+\mu_1(x)\right)\bar{P}_q^{(1)}(x,z,s)=\int_0^\infty \bar{R}_q(x,y,z,s)\beta(y)dy \tag{36}$$

$$\left(\frac{d}{dx}+s+\lambda p(1-C(z))+\alpha+\mu_2(x)\right)\bar{P}_q^{(2)}(x,z,s)=\int_0^\infty \bar{R}_q(x,y,z,s)\beta(y)dy \tag{37}$$

$$\left(\frac{d}{dx}+s+\lambda p(1-C(z))+\beta(y)\right)\bar{R}_q(x,y,z,s)=0 \tag{38}$$

$$\left(\frac{d}{dx}+s+\lambda p(1-C(z))+\nu(x)\right)\bar{V}_q(x,z,s)=0 \tag{39}$$

$$\left(\frac{d}{dx}+s+\lambda p(1-C(z))+\delta(x)\right)\bar{S}_q(x,z,s)=0 \tag{40}$$

$$\bar{Q}_q(0,z,s)=1-(s+\lambda)\bar{Q}_0(s)+\int_0^\infty \bar{S}_q(x,z,s)\delta(x)dx$$
$$(1-\theta)\left\{(1-r)\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx+\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx\right\} \tag{41}$$

$$z\bar{P}_q^{(1)}(0,z,s)=\int_0^\infty \bar{Q}_q(x,z,s)\eta(x)dx+\lambda C(z)\int_0^\infty \bar{Q}_q(x,z,s)dx+\lambda C(z)\bar{Q}_0(s) \tag{42}$$

$$\bar{P}_q^{(2)}(0,z,s)=r\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx \tag{43}$$

$$\bar{R}_q(x,0,z,s)=\alpha\bar{P}_q^{(1)}(x,z,s)+\alpha\bar{P}_q^{(2)}(x,z,s) \tag{44}$$

$$\bar{V}_q(0,z,s)=\theta\left\{(1-r)\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx+\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx\right\} \tag{45}$$

$$\bar{S}_q(0,z,s)=\int_0^\infty \bar{V}_q(x,z,s)\nu(x)dx \tag{46}$$

solving equations (35)-(40)

$$\bar{Q}_q(x,z,s)=\bar{Q}_q(0,z,s)e^{-(s+\lambda)x-\int_0^x \eta(t)dt} \tag{47}$$

$$\bar{P}_q^{(j)}(x,z,s)=\bar{P}_q^{(j)}(0,z,s)e^{-\phi(z,s)x-\int_0^x \mu_j(t)dt};\, j=1,2 \tag{48}$$

where $\phi(z,s)=s+\lambda p(1-C(z))-\alpha[1-\bar{R}(s+\lambda p(1-C(z)))]$

$$\bar{R}_q(x,y,z,s)=\bar{R}_q(x,0,z,s)e^{-(s+\lambda p(1-C(z))x)-\int_0^y \beta(t)dt} \tag{49}$$

$$\bar{V}_q(x,z,s)=\bar{V}_q(0,z,s)e^{-(s+\lambda p(1-C(z))x)-\int_0^x \nu(t)dt} \tag{50}$$

$$\bar{S}_q(x,z,s)=\bar{S}_q(0,z,s)e^{-(s+\lambda p(1-C(z))x)-\int_0^x \delta(t)dt} \tag{51}$$

integrate equations (47)-(51)w.r.to x

$$\bar{Q}_q(z,s)=\bar{Q}_q(0,z,s)\left[\frac{1-\bar{Q}_e(s+\lambda)}{(s+\lambda)}\right] \tag{52}$$

$$\bar{P}_q^{(j)}(z,s)=\bar{P}_q^{(j)}(0,z,s)\left[\frac{1-\bar{B}_j(\phi(z,s))}{(\phi(z,s))}\right];j=1,2 \tag{53}$$

$$\bar{R}_q(x,z,s)=\bar{R}_q(x,0,z,s)\left[\frac{1-\bar{R}(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{54}$$

$$\bar{V}_q(z,s)=\bar{V}_q(0,z,s)\left[\frac{1-\bar{V}(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{55}$$

$$\bar{S}_q(z,s)=\bar{S}_q(0,z,s)\left[\frac{1-\bar{S}(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{56}$$

$$\int_0^\infty \bar{Q}_q(x,z,s)\eta(x)dx=\bar{Q}_q(0,z,s)\bar{Q}_e(s+\lambda) \tag{57}$$

$$\int_0^\infty \bar{P}_q^{(j)}(x,z,s)\mu_j(x)dx=\bar{P}_q^{(j)}(0,z,s)\bar{B}_j(\phi(z,s));j=1,2 \tag{58}$$

$$\int_0^\infty \bar{R}_q(x,y,z,s)\beta(y)dy=\bar{R}_q(x,0,z,s)\bar{R}(s+\lambda p(1-C(z))) \tag{59}$$

$$\int_0^\infty \bar{V}_q(x,z,s)\nu(x)dx=\bar{V}_q(0,z,s)\bar{V}(s+\lambda p(1-C(z))) \tag{60}$$

$$\int_0^\infty \bar{S}_q(x,z,s)\delta(x)dx=\bar{S}_q(0,z,s)\bar{S}(s+\lambda p(1-C(z))) \tag{61}$$

$$\begin{aligned}\bar{Q}_q(0,z,s)&=1-(s+\lambda)\bar{Q}_0(s)+\int_0^\infty \bar{S}_q(x,z,s)\delta(x)dx\\&+(1-\theta)\left\{(1-r)\int_0^\infty \bar{P}_q^{(1)}(x,z,s)\mu_1(x)dx+\int_0^\infty \bar{P}_q^{(2)}(x,z,s)\mu_2(x)dx\right\}\end{aligned} \tag{62}$$

$$\bar{P}_q^{(1)}(0,z,s)=\frac{\lambda C(z)\bar{Q}_0(s)+\left[1-(s+\lambda)\bar{Q}_0(s)\right]\left[\lambda C(z)\left(\frac{(1-\bar{Q}_e(s+\lambda))}{(s+\lambda)}\right)+\bar{Q}_e(s+\lambda)\right]}{D(z,s)} \tag{63}$$

$$D(z,s)=z-[(1-\theta)+\theta M(z,s)]H(z,s)\left[\lambda C(z)\left(\frac{(1-\bar{Q}_e(s+\lambda))}{(s+\lambda)}\right)+\bar{Q}_e(s+\lambda)\right] \tag{64}$$

where
$H(z,s)=(1-r)\bar{B}_1(\phi(z,s))+r\bar{B}_1(\phi(z,s))\bar{B}_2(\phi(z,s))$

$$M(z,s) = \bar{V}(s+\lambda p(1-C(z)))\bar{S}(s+\lambda p(1-C(z)))$$

substitute the value for $\bar{P}_q(0,z,s)$ we can obtain the probability generating function of the system at the following states

$$\bar{Q}_q(z,s) = [1-(s+\lambda)\bar{Q}_0(s)] + \left[\frac{1-\bar{Q}_e(s+\lambda)}{(s+\lambda)}\right][(1-\theta)+\theta M(z,s)]H(z,s)\bar{P}_q^{(1)}(0,z,s) \tag{65}$$

$$\bar{P}_q^{(1)}(z,s) = \bar{P}_q^{(1)}(0,z,s)\left[\frac{1-\bar{B}_1(\phi(z,s))}{(\phi(z,s))}\right] \tag{66}$$

$$\bar{P}_q^{(2)}(z,s) = r\bar{P}_q^{(1)}(0,z,s)\bar{B}_1(\phi(z,s))\left[\frac{1-\bar{B}_2(\phi(z,s))}{(\phi(z,s))}\right] \tag{67}$$

$$\bar{R}_q(x,z,s) = \alpha\bar{P}_q^{(1)}(0,z,s)[1-(1-r)\bar{B}_1(\phi(z,s))-r\bar{B}_1(\phi(z,s))\bar{B}_2(\phi(z,s))]\left[\frac{1-\bar{R}(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{68}$$

$$\bar{V}_q(z,s) = \theta\bar{P}_q^{(1)}(0,z,s)H(z,s)\left[\frac{1-\bar{V}(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{69}$$

$$\bar{S}_q(z,s) = \theta\bar{P}_q^{(1)}(0,z,s)H(z,s)\bar{V}(s+\lambda p(1-C(z)))\left[\frac{1-\bar{S}(s+\lambda p(1-C(z)))}{(s+\lambda p(1-C(z)))}\right] \tag{70}$$

4 Steady State Solution

In this section we shall derive the steady state probability distribution for our queueing model. To define the steady state probabilities, suppress the argument 't' where ever it appears in the time dependent analysis. By using well known Tauberian property

$$Lt_{s\to 0}s\bar{f}(s) = Lt_{t\to\infty}f(t) \tag{71}$$

$$Q_q(z) = \frac{Q_0[1-\bar{Q}_e(\lambda)](C(z)-1)[(1-\theta)+\theta M(z)]H(z)-D(z)}{D(z)} \tag{72}$$

$$P_q^{(1)}(z) = \frac{Q_0\bar{Q}_e(\lambda)\lambda(C(z)-1)[1-\bar{B}_1(\phi(z))]}{D(z)\phi(z)} \tag{73}$$

$$P_q^{(2)}(z) = \frac{Q_0\bar{Q}_e(\lambda)\lambda(C(z)-1)\bar{B}_1(\phi(z))[1-\bar{B}_2(\phi(z))]}{D(z)\phi(z)} \tag{74}$$

$$R_q(x,z) = \frac{\alpha Q_0\bar{Q}_e(\lambda)[1-H(z)][\bar{R}(\lambda p(1-C(z)))-1]}{pD(z)\phi(z)} \tag{75}$$

$$V_q(z) = \frac{\theta Q_0\bar{Q}_e(\lambda)H(z)[\bar{V}(\lambda p(1-C(z)))-1]}{pD(z)} \tag{76}$$

$$S_q(z) = \frac{\theta Q_0 \bar{Q}_e(\lambda) H(z) \bar{V}(\lambda p(1-C(z)))[\bar{S}(\lambda p(1-C(z))) - 1]}{pD(z)} \tag{77}$$

where

$$D(z) = z - [(1-\theta) + \theta M(z)]\,[H(z)]\,\left[\lambda C(z)(1-\bar{Q}_e(\lambda)) + \bar{Q}_e(\lambda)\right]$$

$$H(z) = (1-r)\bar{B}_1(\phi(z)) + r\bar{B}_1(\phi(z))\bar{B}_2(\phi(z))$$

$$M(z) = \bar{V}(\lambda p(1-C(z)))\bar{S}(\lambda p(1-C(z)))$$

using the normalization condition Q_0 can be obtained

$$Q_0 + Lt_{z\to 1}(Q_q(z) + P_q^{(1)}(z) + P_q^{(2)}(z) + R_q(z) + V_q(z) + S_q(z)) = 1 \tag{78}$$

$$Q_0 = \frac{[1 - C_{[1]}(1-\bar{Q}_e(\lambda)) - p\rho]}{\bar{Q}_e(\lambda)(1+\rho(1-p))} \tag{79}$$

$$\rho = \lambda C_{[1]}[(b_1 + rb_2)(1+\alpha r_1) + \theta(v_1 + s_1)] \tag{80}$$

In addition, various system state probabilities also be given from equations (81)-(88) by putting z=1

Prob [the server is idle in non-empty queue]

$$= Q_q(1) = \frac{C_{[1]}(1-\bar{Q}_e(\lambda))p\rho}{\bar{Q}_e(\lambda)(1+\rho(1-p))} \tag{81}$$

Prob [the server is busy with first stage]= $P_q^{(1)}(1) =$

$$\frac{\lambda C_{[1]} b_1}{(1+\rho(1-p))} \tag{82}$$

Prob [the server is busy with second stage]= $P_q^{(2)}(1) =$

$$\frac{r\lambda C_{[1]} b_2}{(1+\rho(1-p))} \tag{83}$$

Prob [the server is busy]= $P_q^{(1)}(1) + P_q^{(2)}(1) =$

$$\frac{\lambda C_{[1]}(b_1 + rb_2)}{(1+\rho(1-p))} \tag{84}$$

Prob [the server is on repair]=$R_q(1)$

$$= \frac{\alpha\lambda C_{[1]}(b_1 + rb_2)r_1}{(1+\rho(1-p))} \tag{85}$$

Prob [the server is on vacation] $= V_q(1)$

$$= \frac{\theta\lambda C_{[1]}v_1}{(1+\rho(1-p))} \tag{86}$$

Prob [the server is on set up time] $= S_q(1)$

$$= \frac{\theta\lambda C_{[1]}s_1}{(1+\rho(1-p))} \tag{87}$$

Blocking probability

$$= \frac{\rho}{(1+\rho(1-p))} \tag{88}$$

The necessary and sufficient condition for stability condition is given by

$1 - C_{[1]}(1 - \bar{Q}_e(\lambda)) > p\rho$

The expected number of customers in the orbit

$$\begin{aligned} E[N_0] = &\tfrac{p\rho(1-\bar{Q}_e(\lambda))C_{[1]}}{1-C_{[1]}(1-\bar{Q}_e(\lambda))-p\rho} \\ &+ \tfrac{[\lambda pC_{[1]}][\mu_2(1+\alpha r_1)^2+\alpha(b_1+rb_2)r_2]}{2[1-C_{[1]}(1-\bar{Q}_e(\lambda))-p\rho]} \\ &+ \tfrac{p\theta[\lambda C_{[1]}]^2[(v_2+s_2+2v_1s_1)+2(v_1+s_1)(b_1+rb_2)(1+\alpha r_1)]}{2[1-C_{[1]}(1-\bar{Q}_e(\lambda))-p\rho]} \\ &+ \frac{C_{[1]}(1-\bar{Q}_e(\lambda))+p\rho}{[1-C_{[1]}(1-\bar{Q}_e(\lambda))-p\rho]}C_{[R]} \end{aligned} \tag{89}$$

where $C_{[R]} = \frac{C_{[2]}}{2C_{[1]}}$ is the mean residual batch size.

After finding the expected number of units in the orbit,we can obtain the related performance measures viz mean number of units in the system, mean waiting time in the queue and mean waiting time in the system by using Little's formula

$$E[N_s] = E[N_0] + \rho \tag{90}$$

$$E[W_s] = \frac{E[N_s]}{\rho\lambda C_{[1]}} \tag{91}$$

$$E[W_0] = \frac{E[N_0]}{\rho\lambda C_{[1]}} \tag{92}$$

5 Reliability Indices

Let$A_v(t)$ be the system availability at time 't' i.e the probability that the server is either working for a customer or in an idle period such that the steady state availability of the server is given by

$$A_v = Lt_{t\to\infty} A_v(t) \tag{93}$$

$$A_v = P_{00} + Lt_{z\to 1}(P_q^{(1)}(1) + P_q^{(2)}(1)) = 1 - \frac{\alpha\lambda C_{[1]}[(b_1 + rb_2)r_1 + \theta(v_1 + s_1)]}{1 + \rho(1-p)} \tag{94}$$

The steady state failure frequency of the server

$$F = \alpha(P_q^{(1)}(1) + P_q^{(2)}(1)) = \frac{\alpha\lambda C_{[1]}(b_1 + rb_2)}{1 + \rho(1-p)} \tag{95}$$

6 Conclusion

In this paper, we obtained the probability generating function of various states of the system in transient state by using the supplementary variable technique and also discussed the steady state solution with performance measures of the system and the reliability indices like availability of the server and failure frequency of the server. The result of the prescribed model can be used in the design of computer networks.

7 Open Problem

In this paper, we have studied the transient behavior of the model with retrial times are generally distributed and also incoming arrivals are in batches in accordance with Poisson process. Also the server is assumed to take a setup time before giving a proper service to the customers and the customers are non-persistent. Although, it remains an interesting problem to consider the customer impatience behavior like balking/reneging on this service system. The concept of orbit search which is an important feature in retrial queue, could be incorporated in this paper and which leads to an another interesting problem in this area. It would be quite interesting to determine a control policy in order to get the best estimate for the probabilities and which minimizes the total cost of the service system.

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