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Valuation, weak global dimension and semihereditary in amalgamated algebra along an ideal

Moutu Abdou Salam Moutui

Department of Mathematics, Faculty of Science and Technology of Fez, Box 2202, University S.M. Ben Abdellah Fez, Morocco.
e-mail:moutu_2004@yahoo.fr

Abstract

Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. In this paper, we give a characterization of valuation, weak global dimension and semihereditary properties under a certain ring-theoretic construction called the amalgamation of A with B along J with respect to f (denoted by $A \bowtie^f J$), introduced and studied by D'Anna, Finocchiaro and Fontana in 2009. Our aim is to generated new classes of commutative rings satisfying theses properties.

Keywords: *Amalgamated algebra along an ideal, valuation ring, semihereditary ring, weak global dimension, Prüfer ring, amalgamated duplication.*

1 Introduction

All rings considered in this paper are commutative with identity elements and all modules are unital. Kaplansky defined in [26], a valuation ring as a ring in which any two elements, one divides the other. In 1932, Prüfer introduced and studied in [32] integral domains in which every finitely generated ideal is invertible. In 1936, Krull [28] named these rings after H. Prüfer and stated equivalent conditions that make a domain Prüfer. Through the years, Prüfer domains acquired a great many equivalent characterizations, each of which was extended to rings with zero-divisors in different ways. In 1969, Osofsky proved that the weak global dimension of an arithmetical ring is either ≤ 1 or infinite [31]. In their recent paper devoted to Gaussian properties, Bazzoni and Glaz have proved that a Prüfer ring satisfies any of the other four Prüfer conditions if and only if its total ring of quotients satisfies that same condition [5, Theorems 3.3 & 3.6 & 3.7 & 3.12]. In [2], the authors examined the transfer of the Prüfer conditions and obtained further evidence for the validity of Bazzoni-Glaz conjecture sustaining that "the weak global dimension of a Gaussian ring is 0, 1, or ∞" [5]. Recall that classical examples of non-semihereditary arithmetical rings stem from Jensen's 1966 result [24] as non-reduced principal rings, e.g., Z/n^2Z for any integer $n \geq 2$. At this point, we recall the following definition :

Definition 1.1 *Let R be a commutative ring.*

1. *R is called a* valuation ring *if for all $a, b \in R$, $a \in Rb$ or $b \in Ra$ (see [26]).*
2. *R is called a* semihereditary ring *if every finitely generated ideal of R is projective (see [7]).*
3. *R is said to have* weak global dimension ≤ 1 *(denoted by $wdim(R) \leq 1$) if every finitely generated ideal of R is flat (see [18]).*
4. *R is called an* arithmetical ring *if the lattice formed by its ideals is distributive (see [16]).*
5. *R is called a* Gaussian ring *if for every $f, g \in R[X]$, one has the content ideal equation $c(fg) = c(f)c(g)$ (see [33]).*
6. *R is called a* Prüfer ring *if every finitely generated regular ideal of R is invertible (See [7, 22]).*
   In [19], it is proved that each one of the above conditions implies the following next one :

   Semihereditary $\Rightarrow$ weak global dimension$\leq$ 1 $\Rightarrow$ Arithmetical $\Rightarrow$ Gaussian $\Rightarrow$ Prüfer.

Also examples are given to show that, in general, the implications cannot be reversed. Moreover, an investigation is carried out to see which conditions may be added to any of these properties in order to reverse the implications. Recall that in the domain context, the above class of Prüfer-like rings collapse to the notion of Prüfer domain. For more details on these notions, we refer to reader to [4, 5, 7, 18, 19, 29, 22, 33].

In this paper, we study the transfer of valuation, weak global dimension ≤ 1 and semihereditary properties in amalgamation of rings, introduced and studied by D'Anna, Finocchiaro and Fontana in [10, 11] and defined as follows :

Let A and B be two rings with identity elements, J be an ideal of B and let $f : A \rightarrow B$ be a ring homomorphism. In this setting, we consider the following subring of $A \times B$; $A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}$ called *the amalgamation of* A *and* B *along* J *with respect to* f. This construction is a generalization of amalgamated duplication of a ring along an ideal (introduced and studied by D'Anna and Fontana in [12, 13, 14]). Moreover, other classical constructions (such as the $A+XB[X]$, $A+XB[[X]]$, and the $D+M$ constructions) can be studied as particular cases of the amalgamation ([10, Examples 2.5 and 2.6]) and other classical constructions, such as the CPI extensions (in the sense of Boisen and Sheldon [6]) are strictly related to it ([10, Example 2.7 and Remark 2.8]). See for instance [10, 11, 13, 14].
For a ring R, we denote by :
$Spec(R) := \{P \subseteq R : P \text{ is a prime ideal of } R\}$.
$Max(R) := \{M \subseteq R : M \text{ is a maximal ideal of } R\}$.
$Nilp(R) := \{a \in R : a \text{ is a nilpotent element of } R\}$.
$Rad(R) :=$ the Jacobson radical of R.
$V(J) := \{P \in Spec(R) : P \supseteq J\}$ for each ideal J of a ring R.
$R_S := S^{-1}R$ is the localization of R by a multiplicative subset S.

2 Formulation problems

In this paper, we examine the transfer of valuation, weak global dimension ≤ 1 and semihereditary properties in amalgamated algebra along an ideal.

3 Main results

Our first result is a characterization of valuation property in amalgamated algebra along an ideal.

Proposition 3.1 *Let (A, B) be a pair of rings, $f : A \rightarrow B$ be a ring homomorphism and J be a non-zero ideal of B. Then $A \bowtie^f J$ is a valuation ring if and only if f is injective, $f(A) + J$ is a valuation ring and $f(A) \cap J = (0)$.*

Proof. Assume that $A \bowtie^f J$ is a valuation ring. We claim that f is injective. Deny. There exists some $0 \neq \alpha \in Ker(f)$. Using the fact $J \neq (0)$, there exists $0 \neq x \in J$. Clearly, $(0, x)$ and $(\alpha, 0)$ are elements of $A \bowtie^f J$. Since $A \bowtie^f J$ is a valuation ring, then $(0, x) \in A \bowtie^f J(\alpha, 0)$ or $(\alpha, 0) \in A \bowtie^f J(0, x)$. So, $(0, x) = (a, f(a) + i)(\alpha, 0)$ or $(\alpha, 0) = (b, f(b) + k)(0, x)$ for some $(a, f(a) + i), (b, f(b) + k) \in A \bowtie^f J$. It follows that $x = 0$ or $\alpha = 0$, which is a contradiction. Hence, f is injective, as desired. Now, we show that $f(A)+J$ is a valuation ring. Let $f(a)+i$ and $f(b)+j \in f(A)+J$, we show that $f(a)+i \in f(A)+J(f(b)+j)$ or $(f(b)+j) \in f(A)+J(f(a)+i)$. We have $(a, f(a)+i)$ and $(b, f(b)+j) \in A \bowtie^f J$. Since $A \bowtie^f J$ is a valuation ring, then $(a, f(a)+i) \in A \bowtie^f J(b, f(b)+j)$ or $(b, f(b)+j) \in A \bowtie^f J(a, f(a)+i)$. And so $(a, f(a)+i) = (b, f(b)+j)(c, f(c)+k)$ or $(b, f(b)+j) = (c', f(c')+k')(a, f(a)+i)$ for some $(c, f(c)+k)$, $(c', f(c')+k') \in A \bowtie^f J$. Therefore, $f(a)+i = (f(b)+j)(f(c)+k)$ or $(f(b)+j) = (f(c')+k')(f(a)+i)$ for some $f(c)+k$, $f(c')+k' \in f(A)+J$. Hence, $f(a)+i \in f(A)+J(f(b)+j)$ or $(f(b)+j) \in f(A)+J(f(a)+i)$. Thus, $f(A)+J$ is a valuation ring, as desired. Next, we claim that $f(A) \cap J = (0)$. Suppose that $f(A) \cap J \neq (0)$ and let $0 \neq f(a) \in J$. It is clear that $(a, 0)$ and $(0, f(a))$ are elements of $A \bowtie^f J$ which is a valuation ring. So, $(a, 0) \in A \bowtie^f J(0, f(a))$ or $(0, f(a)) \in A \bowtie^f J(a, 0)$. With similar arguments as previously, it follows that $a = 0$ or $f(a) = 0$, which is a contradiction since $f(a) \neq 0$. Hence, $f(A) \cap J = (0)$. Conversely, assume that f is injective, $f(A)+J$ is a valuation ring and $f(A) \cap J = (0)$. By [10, Proposition 5.1 (3)], the natural projection $p : A \bowtie^f J \to f(A)+J$ is a ring isomorphism. Thus, the conclusion is now straightforward.

Remark 3.2 *If $J = 0$, then by [10, Proposition 5.1 (3)], $A \bowtie^f J \simeq A$, and so $A \bowtie^f J$ is a valuation ring if and only if so is A.*

Proposition 3.1 gives new examples of non-valuation rings. The next corollary shows how to construct such rings.

Corollary 3.3 *Let A be a ring and I be a non-zero ideal of A. Then $A \bowtie I$ is never a valuation ring.*

Proof. It is easy to see that $A \bowtie I = A \bowtie^f J$ where $A = B$, f is the identity map of A and $J = I$. Suppose that $A \bowtie I$ is a valuation ring. By Proposition 3.1, $f(A)+J = A+I = A$ is a valuation ring and $f(A) \cap J = A \cap I = (0)$. So, $I = 0$ since I is an ideal of A, which is a contradiction. Hence, $A \bowtie I$ is never a valuation ring.

Now, we construct new examples of valuation rings.

Example 3.4 *Let $A := K$ be a field and let $B := K[[X]]$ be the ring of formal power series in an indeterminate X and with coefficient in K. Consider $f : A \hookrightarrow B$ be an injective ring homomorphism and $J := XK[[X]]$ be the maximal ideal of B. Then $A \bowtie^f J$ is a valuation ring.*

Proof. It is clear that $f(A) + J = K + XK[[X]] = K[[X]]$ which is a valuation domain. Since $f(A) \cap J = K \cap XK[[X]] = (0)$, then by Proposition 3.1, $A \bowtie^f J$ is a valuation ring.

We recall that for a ring A and an $A-$module E. The trivial ring extension of A by E (also called idealization of E over A) is the ring $R := A \propto E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + ea')$. Trivial ring extensions have been studied extensively. Considerable work, part of is summarized in Glaz's book [21] and Huckaba's book [23], has been concerned with trivial ring extension.

Example 3.5 *Let A be a valuation domain, E be a non-torsion $A-$module with $E \simeq qf(A)$ and let $B := A \propto E$ be the trivial ring extension of A by E. Consider*

$$\begin{array}{rccl} f: & A & \hookrightarrow & B \\ & a & \hookrightarrow & f(a) = (a, 0) \end{array}$$

be an injective ring homomorphism and $J := 0 \propto E$ be a proper ideal of B. Then $A \bowtie^f J$ is a valuation ring.

Proof. We have $f(A) \cap J = (A \propto 0) \cap (0 \propto E) = (0)$ and $f(A) + J = A \propto 0 + 0 \propto E = A \propto E$ which is a valuation ring by [25, Theorem 2.1]. Hence, by application to Proposition 3.1, we obtain $A \bowtie^f J$ is a valuation ring.

The following Theorem develops a result on the transfer of the weak global dimension ≤ 1 to amalgamation of rings $A \bowtie^f J$.

Theorem 3.6 *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a non-zero ideal of B. Then :*
$wdim(A \bowtie^f J) \leq 1$, if and only if the following statements hold :
(a) $wdim(A) \leq 1$ and $J \cap Nilp(B) = (0)$.
(b) $\forall m \in Max(A) \ / \ m \not\supseteq f^{-1}(J)$, A_m is a valuation domain.
(c) $\forall m \in Max(A) \ / \ m \supseteq f^{-1}(J)$, f_m is injective, $f_m(A_m) + J_S$ is a valuation domain and $f_m(A_m) \cap J_S = (0)$ with :
$f_m : A_m \to B_S$
$f_m(\frac{a}{s}) = \frac{f(a)}{f(s)}$ be a ring homomorphism and $S := f(A \backslash m) + J$ be a multiplicative subset of B.
(d) B_Q is a valuation domain $\forall Q \in Max(B) \backslash V(J)$.

The proof of this Theorem involves the following Lemmas.

Lemma 3.7 *[4, Theorem 4.8]*
Let R be a ring. The following conditions are equivalent :
(1) $wdim(R) \leq 1$.
(2) R is an arithmetical reduced ring.

It is proved in [24] that R is an arithmetical ring if and only if each localization R_m at a maximal ideal m is a valuation ring. Also, we will frequent use that a local gaussian reduced ring is a valuation domain.

Lemma 3.8 *([11, Proposition 2.6]). Let $f : A \to B$ be a ring homomorphism and J be an ideal of B. For all $P \in Spec(A)$, and $Q \in Spec(B)$, consider the set $P'^f := P \bowtie^f J := \{(p, f(p) + i)/p \in P, i \in J\}$ and the set $\overline{Q}^f := \{(a, f(a) + j)/a \in A, j \in J$ and $f(a) + j \in Q\ \}$. Then :*
(1) The prime ideals of $A \bowtie^f J$ are of the type P'^f or $\overline{Q}^f$, for all $P \in Spec(A)$ and $Q \in Spec(B)\backslash V(J)$.
(2) The maximal ideals of $A \bowtie^f J$ are of the type M'^f or $\overline{Q}^f$, for all $M \in Max(A)$ and $Q \in Max(B)\backslash V(J)$.

Lemma 3.9 *[15, Proposition 1.49]*
Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be an ideal of B. Then :
(1) For every prime ideal Q of B not containing J, the ring $(A \bowtie^f J)_{\bar{Q}^f}$ is canonically isomorphic to B_Q.
(2) Let P be a prime ideal of A. Consider the multiplicative subset $S := S_P := S_{(P,f,J)} := f(A\backslash P) + J$ of B, set $B_S := S^{-1}B$ and $J_S := S^{-1}J$. Let $f_P : A_P \to B_S$ be the ring homomorphism induced by f. Then, the ring $(A \bowtie^f J)_{P'^f}$ is isomorphic to $A_P \bowtie^{f_P} J_S$. In particular, for every prime ideal P of A not containing $f^{-1}(J)$, $(A \bowtie^f J)_{P'^f}$ is isomorphic to A_P.

Proof of Theorem 3.6 Suppose that $wdim(A \bowtie^f J) \leq 1$.
(a) By Lemma 3.7, $A \bowtie^f J$ is an arithmetical reduced ring. So, A is an arithmetical ring since the arithmetical property is stable under factor rings (here $A \cong \frac{A \bowtie^f J}{\{0\}\times\{J\}}$, from [10, Proposition 5.1 (3)]). By [10, Proposition 5.4], A is a reduced ring and $J \cap Nilp(B) = (0)$. Hence, $wdim(A) \leq 1$ and $J \cap Nilp(B) = (0)$, as desired.
(b) Let $m \in Max(A)$ / $m \not\supseteq f^{-1}(J)$. Then, by (2) of Lemma 3.9, $(A \bowtie^f J)_{m'^f} \cong A_m$. In fact of view $A \bowtie^f J$ is an arithmetical ring, it follows that $(A \bowtie^f J)_{m'^f}$ is an arithmetical local reduced ring, which is a valuation domain, making A_m a valuation domain.
(c) Let $m \in Max(A)$ / $m \supseteq f^{-1}(J)$. By (2) of Lemma 3.9, $(A \bowtie^f J)_{m'^f} \cong A_m \bowtie^{f_m} J_S$ with :
$f_m : A_m \to B_S$
$f_m(\frac{a}{s}) = \frac{f(a)}{f(s)}$ be a ring homomorphism. Using the fact $J \neq (0)$, there is some $0 \neq x \in J$ and so $\frac{x}{1} \in J_S = S^{-1}J$. Consequently, $J_S \neq (0)$. By assumption, $A \bowtie^f J$ is an arithmetical reduced ring. So, $(A \bowtie^f J)_{m'^f} \cong A_m \bowtie^{f_m} J_S$ is a valuation domain. By Proposition 3.1, f_m is injective, $f_m(A_m) + J_S$ is a valuation ring and $f(A_m) \cap J_S = (0)$. From [10, Proposition 5.2], $f_m(A_m) + J_S$

is a valuation domain since $A_m \bowtie^{f_m} J_S$ is a domain.
(d) By (1) of Lemma 3.9, $\forall Q \in Max(B)\backslash V(J)$, $(A \bowtie^f J)_{\bar{Q}^f}$ is canonically isomorphic to B_Q which is a valuation domain since $(A \bowtie^f J)_{\bar{Q}^f}$ is a valuation domain.
Conversely, assume that (a), (b), (c) and (d) hold. We claim that $wdim(A \bowtie^f J) \leq 1$. By Lemma 3.7, we only need to prove that $A \bowtie^f J$ is an arithmetical reduced ring. Indeed, let $Q \in Max(B)\backslash V(J)$. By (1) of Lemma 3.9, $(A \bowtie^f J)_{\bar{Q}^f} \cong B_Q$. So, $(A \bowtie^f J)_{\bar{Q}^f}$ is a valuation domain since B_Q is a valuation domain. Let $m \in Max(A)$. We envisage two cases :
Case 1: $m \not\supseteq f^{-1}(J)$. Then by (2) of Lemma 3.9, $(A \bowtie^f J)_{m'^f} \cong A_m$. Using the fact A_m is a valuation domain, it follows that $(A \bowtie^f J)_{m'^f}$ is a valuation domain.
Case 2: Assume that $m \supseteq f^{-1}(J)$. Then, by (2) of Lemma 3.9, $(A \bowtie^f J)_{m'^f} \cong A_m \bowtie^{f_m} J_S$. Since f_m is injective, $f_m(A_m) + J_S$ is a valuation domain and $f(A_m) \cap J_S = (0)$, then by Proposition 3.1 and [10, Proposition 5.2], we obtain $A_m \bowtie^{f_m} J_S$ is a valuation domain, making $(A \bowtie^f J)_{m'^f}$ a valuation domain. Hence, $A \bowtie^f J$ is an arithmetical ring. Since $wdim(A) \leq 1$, then A is a reduced ring and by assumption, $J \cap Nilp(B) = (0)$. So, by [10, Proposition 5.4], $A \bowtie^f J$ is a reduced ring. Finally, $wdim(A \bowtie^f J) \leq 1$, as desired.

The following corollary is a consequence of Theorem 3.6 and is [8, Theorem 4.1(1)].

Corollary 3.10 *Let A be a ring and I be an nonzero ideal of A. Then, $wdim(A \bowtie I) \leq 1$ if and only if $wdim(A) \leq 1$, for all $m \in Max(A) \supseteq I$, A_m is a valuation domain and $I_m = 0$.*

Theorem 3.6, generate new families of examples of Gaussian rings with weak global dimension >1. Recall that that a local ring is Gaussian if "for any two elements a, b in the ring, we have $< a, b >^2 = < a^2 >$ or $< b^2 >$; moreover, if $ab = 0$ and, say, $< a, b >^2 = < a^2 >$, then $b^2 = 0$" (see [5, Theorem 2.2]).

Example 3.11 *Let (A, m) be a local Gaussian ring, $f : A \to B$ be a ring homomorphism and J be a non-zero ideal of B such that $J \subseteq Rad(B)$. Assume that $J^2 = 0$, and $f(a)J = f(a)^2 J$ for all $a \in m$. Then :*
(1) $A \bowtie^f J$ is Gaussian.
(2) $wdim(A \bowtie^f J) > 1$.

Proof. (1) Our aim is to show that $A \bowtie^f J$ is Gaussian.
By [15, Proposition 1.36 (6)],$(A \bowtie^f J, m \bowtie^f J)$ is local. Let $(a, f(a) + i)$ and $(b, f(b) + j) \in A \bowtie^f J$. Then a and $b \in A$. We may assume that $a, b \in m$ and

$< a, b >^2 = < a^2 >$. Therefore, $b^2 = a^2x$ and $ab = a^2y$ for some $x, y \in A$. Moreover $ab = 0$ implies that $b^2 = 0$. By assumption, there exist $j_1, i_1, j_2, i_2, i_3 \in J$ such that $2f(b)j = f(a)^2 f(x)j_1$, $2f(a)if(x) = f(a)^2 i_1$, $f(a)j = f(a)^2 j_2$, $f(b)i = f(a)^2 f(x)i_2$ and $2f(a)if(y) = f(a)^2 i_3$. In view of the fact $J^2 = 0$, one can easily check that $(b, f(b) + j)^2 = (a, f(a) + i)^2(x, f(x) + f(x)j_1 - i_1)$ and $(a, f(a) + i)(b, f(b) + j) = (a, f(a) + i)^2(y, f(y) + f(x)i_2 + j_2 - i_3)$. Moreover, assume that $(a, f(a) + i)(b, f(b) + j) = (0, 0)$. Hence, $ab = 0$ and so $b^2 = 0$. Using the fact $J^2 = 0$ and $f(a)J = f(a)^2 J$ for all $a \in m$, it follows that $(b, f(b) + j)^2 = (0, 0)$. Finally, $A \bowtie^f J$ is Gaussian.
(2) By Theorem 3.6, $wdim(A \bowtie^f J) > 1$ since $J \subset Nilp(B)$.

Now, we construct a new example of Prüfer ring with weak global dimension$>$ 1.

Example 3.12 *Let (A_1, m_1) be a non-reduced local ring such that $m_1^2 = 0$ (for instance $(A_1, m_1) := (\mathbb{Z}/4\mathbb{Z}, 2\mathbb{Z}/4\mathbb{Z})$), E be a non-zero $\frac{A_1}{m_1}-$vector space. Consider $(A, m) := (A_1 \propto E, m_1 \propto E)$ be the trivial ring extension of A_1 by E. Let $B := A_1$, $f : A \to B$ be a surjective ring homomorphism and $J := m_1$ be a proper ideal of B. Then :*
(1) $A \bowtie^f J$ is a Prüfer ring.
(2) $wdim(A \bowtie^f J) > 1$.

Proof. (1) We claim that $A \bowtie^f J$ is a total ring of quotients. Indeed, since f is surjective, then $J \subset f(A)$. By [2, Theorem 3.1 (1)], A is a local total ring of quotient. Using the fact $J = Rad(B)$, by application to [15, Proposition 1.74], we obtain $A \bowtie^f J$ is a total ring of quotients. Hence, $A \bowtie^f J$ is a Prüfer ring.
(2) By Theorem 3.6, $wdim(A \bowtie^f J) > 1$ since $J \cap Nilp(B) = m_1 = J \neq (0)$.

Recall that by [4, Theorem 3.3], a ring R is semihereditary if and only if it is coherent and has weak global dimension at most 1. By application to the characterization of semihereditary property above, we establish the transfer of semihereditary property to $A \bowtie^f J$.

Theorem 3.13 *Let (A, B) be a pair of rings, $f : A \to B$ be a ring homomorphism and J be a nonzero ideal of B. Assume that J and $f^{-1}(J)$ are finitely generated ideals of $f(A) + J$ and A. Then :*
$A \bowtie^f J$ is semihereditary if and only if the following statements hold :
(a) A is semihereditary and $J \cap Nilp(B) = (0)$.
(b) $f(A) + J$ is coherent.
(c) $\forall m \in Max(A) \ / \ m \not\supseteq f^{-1}(J)$, A_m is a valuation domain.
(d) $\forall m \in Max(A) \ / \ m \supseteq f^{-1}(J)$, f_m is injective, $f_m(A_m) + J_S$ is a valuation domain and $f_m(A_m) \cap J_S = (0)$ with :
$f_m : A_m \to B_S$

$f_m(\frac{a}{s}) = \frac{f(a)}{f(s)}$ *be a ring homomorphism and* $S := f(A\backslash m)+J$ *be a multiplicative subset of* B.
(e) B_Q *is a valuation domain* $\forall Q \in Max(B)\backslash V(J)$.

Proof. Assume that $A \bowtie^f J$ is semihereditary. By [4, Theorem 3.3], $A \bowtie^f J$ is coherent and $wdim(A \bowtie^f J) \leq 1$. From [27, Theorem 2.2], A and $f(A)+J$ are coherent. So, (b) hold. By application to Theorem 3.6, it follows that (a), (c), (d) and (e) hold. Conversely, assume that (a), (b), (c), (d) and (e) hold. Combining Theorem 3.6 and [27, Theorem 2.2], it follows that $wdim(A \bowtie^f J) \leq 1$ and $A \bowtie^f J$ is coherent. Hence, $A \bowtie^f J$ is semihereditary, as desired.

The following corollary is a consequence of Theorem 3.13 and is [8, Theorem 4.1(2)].

Corollary 3.14 *Let* A *be a ring and* I *be a nonzero finitely generated ideal of* A. *Then,* $A \bowtie I$ *is semihereditary if and only if* A *is semihereditary, for all* $m \in Max(A) \supseteq I$, A_m *is a valuation domain and* $I_m = 0$.

4 Conclusion

These are the main results of the paper.

Proposition 4.1 *Let* (A, B) *be a pair of rings,* $f : A \to B$ *be a ring homomorphism and* J *be a non-zero ideal of* B. *Then* $A \bowtie^f J$ *is a valuation ring if and only if* f *is injective,* $f(A) + J$ *is a valuation ring and* $f(A) \cap J = (0)$.

Theorem 4.2 *Let* (A, B) *be a pair of rings,* $f : A \to B$ *be a ring homomorphism and* J *be a non-zero ideal of* B. *Then :*
$wdim(A \bowtie^f J) \leq 1$, *if and only if the following statements hold :*
(a) $wdim(A) \leq 1$ *and* $J \cap Nilp(B) = (0)$.
(b) $\forall m \in Max(A)$ / $m \not\supseteq f^{-1}(J)$, A_m *is a valuation domain.*
(c) $\forall m \in Max(A)$ / $m \supseteq f^{-1}(J)$, f_m *is injective,* $f_m(A_m) + J_S$ *is a valuation domain and* $f_m(A_m) \cap J_S = (0)$ *with :*
$f_m : A_m \to B_S$
$f_m(\frac{a}{s}) = \frac{f(a)}{f(s)}$ *be a ring homomorphism and* $S := f(A\backslash m)+J$ *be a multiplicative subset of* B.
(d) B_Q *is a valuation domain* $\forall Q \in Max(B)\backslash V(J)$.

Theorem 4.3 *Let* (A, B) *be a pair of rings,* $f : A \to B$ *be a ring homomorphism and* J *be a nonzero ideal of* B. *Assume that* J *and* $f^{-1}(J)$ *are finitely generated ideals of* $f(A) + J$ *and* A. *Then :*
$A \bowtie^f J$ *is semihereditary if and only if the following statements hold :*

(a) *A is semihereditary and $J \cap Nilp(B) = (0)$.*
(b) *$f(A) + J$ is coherent.*
(c) *$\forall m \in Max(A)$ / $m \not\supseteq f^{-1}(J)$, A_m is a valuation domain.*
(d) *$\forall m \in Max(A)$ / $m \supseteq f^{-1}(J)$, f_m is injective, $f_m(A_m) + J_S$ is a valuation domain and $f_m(A_m) \cap J_S = (0)$ with :*
$f_m : A_m \to B_S$
$f_m(\frac{a}{s}) = \frac{f(a)}{f(s)}$ *be a ring homomorphism and $S := f(A \backslash m) + J$ be a multiplicative subset of B.*
(e) *B_Q is a valuation domain $\forall Q \in Max(B) \backslash V(J)$.*

5 Open Problem

Question 1. Let (A, B) be a pair of commutative rings, $f : A \to B$ be a ring homomorphism and J be a nonzero ideal of B such that $f^{-1}(J)$ and J are not necessarily finitely generated ideals of $f(A) + J$ and A. Is Theorem 3.13 true ?

Question 2. Let (A, B) be a pair of non commutative rings, $f : A \to B$ be a ring homomorphism and J be a nonzero ideal of B. Is $wdim(A) \leq 1$ if and only if $wdim(A \bowtie^f J) \leq 1$, in general ?

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