On Certain Conditions of Multivariate Power Series Distributions

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Abstract

During the last decades, no researches have conducted in order to prove some properties of the multivariate power series distribution, as results of the present study proved that any multivariate power series distribution is determined uniquely from the mean–function of any marginal random variable. Furthermore, these results indicated also that any given function satisfying certain conditions construct a random vector with multivariate power series distribution which has a mean of the marginal random variable. A useful technique can be applied in model building when we have information about the mean–function.

Key Words: Multivariate Power Series Distributions, Defining Function, Maclaurin Expansion, Multivariate Logarithmic Distribution, Truncated Power Series Distributions.

1 Introduction

function. Ghosh et al. (1977) considered on his paper a characterization of positive and negative multinomial distributions and some properties of multivariate power series distributions. Rao and Janardan (1982) discussed a general approach to findings the moments of two classes of multivariate discrete distribution. Gupta and Das (2008) derived anew distribution called the quasi multivariate logarithmic series distribution (QMLSD) of order k from the multivariate able series distribution (MASDs) of order k. Simic (2009) calculated the moments of distributions as inflated parameter distribution has been discussed by Momeni (2011). Mahmoudi and Jafari (2012) obtained a new class of distribution contains several lifetime model by compound generalized exponential and power series distributions. Silva et al. (2013) introduced a new class of distributions which obtained by compounding the extended Weibull and power series distributions. However, the mentioned researchers have not discussed some properties of multivariate power series, such that the unique and the mean of marginal random variable. Therefore, the present study is an attempt to prove the followings:

**Firstly**: Any multivariate power series distribution is determined uniquely from the mean-function of any marginal random variable.

**Secondly**: Any given function satisfying certain conditions constructs a random vector with multivariate power series distribution which has a mean of the marginal random variable. This paper is organized as follows, the first section is the introduction the second section is the methodology then the third section is an empirical example then the forth section is an open problem, then finally is the conclusion.

## 2 Proposed Method

### 1. Methodology

Let $f(\theta_1, \theta_2, \ldots, \theta_k) = \sum_{x_1, x_2, \ldots, x_k} a_{x_1, x_2, \ldots, x_k} \theta_1^{x_1} \theta_2^{x_2} \cdots \theta_k^{x_k}, \quad x_1 x_2 \cdots x_k \geq 0$, be a convergent series for $(\theta_1, \theta_2, \ldots, \theta_k) \in \pi^{\infty}_{r_i} [0, r_i]$, $r_i$ is a real number or infinity, the intervals $[0, r_i]$ are non-intersecting or overlapping, and $x_i = n_i, n_i + 1, n_i + 2, \ldots$ where $n_i$, $i=1,2,\ldots,k$ is non-negative integer which maybe the same for all $i$ or different for each $I$, for more details and discussion refer to (Katri, 1959; Ghosh, et al., 1977).

Let $p \left( X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k \right) = a_{x_1, x_2, \ldots, x_k} \theta_1^{x_1} \theta_2^{x_2} \cdots \theta_k^{x_k} / f \left( \theta_1, \theta_2, \ldots, \theta_k \right)$.

It is easy to see that:

$p \left( X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k \right)$ is defined if $\theta_i \in (0, r_i), \ i=1,2,\ldots,k$. Also
On Certain Conditions of Multivariate Power Series Distributions

\[\lim_{\theta \to \theta_0^+} p \left( X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k \right) = \lim_{\theta \to \theta_0^+} \frac{a_{x_1 x_2 \ldots x_k} \theta_1^{x_1} \theta_2^{x_2} \cdots \theta_k^{x_k}}{f(\theta_1, \theta_2, \ldots, \theta_k)} = \lim_{\theta \to \theta_0^+} \frac{a_{x_1 x_2 \ldots x_k} \theta_1^{x_1} \theta_2^{x_2} \cdots \theta_k^{x_k}}{\sum_{x_i} \psi_{x_i} \theta_i^{x_i}}.\]

Where \( \psi_{x_i} = \sum_{x_1, \ldots, x_i, \ldots, x_k} a_{x_1 x_2 \ldots x_k} \theta_1^{x_1} \cdots \theta_i^{x_i} \cdots \theta_k^{x_k} \), is independent of \( \theta_i \).

Therefore,

\[\lim_{\theta \to \theta_0^+} p \left( X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k \right) = \lim_{\theta \to \theta_0^+} \frac{a_{x_1 x_2 \ldots x_k} \theta_1^{x_1} \cdots \theta_i^{x_i} \cdots \theta_k^{x_k}}{\psi_{x_i} \theta_i^{x_i} + \psi_{x_i+1} \theta_i^{x_i+1} + \cdots}\]

\[= \begin{cases} 1 & \text{if } x_i = n_i, \; i = 1, 2, \ldots, k \\ 0 & \text{if } x_i \neq n_i \end{cases}\]

Using this limiting value we see that \( P \left( X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k \right) \) is defined for all \( x_i \) and \( \theta_i \in (0, r_i) \), \( i = 1, 2, \ldots, k \). The above conditions insure that:

\[P \left( X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k \right) \geq 0,\] for all \( x_i \) and \( \theta_i \in (0, r_i) \).

Also,

\[\sum_{x_1 x_2 \ldots x_k} P \left( X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k \right) = \frac{1}{f(\theta_1, \theta_2, \ldots, \theta_k)} \sum_{x_1 x_2 \ldots x_k} a_{x_1 x_2 \ldots x_k} \theta_1^{x_1} \theta_2^{x_2} \cdots \theta_k^{x_k} = 1\]

Because of these properties we say that \( \left( X_1, X_2, \ldots, X_k \right) \) is a random vector with multivariate power series distribution in \( k \)-parameters.

The function \( f(\theta_1, \theta_2, \ldots, \theta_k) \) called the defining function of the distribution of the random vector \( \left( X_1, X_2, \ldots, X_k \right) \).

The mean function of the marginal random variable \( X_i \) is:
\[ \mu_1 (\theta_1, \theta_2, ..., \theta_k) = EX, \]
\[ = \sum_{x_1, x_2, ..., x_k} x_i P \left( X_1 = x_1, X_2 = x_2, ..., X_k = x_k \right) \]
\[ = \frac{1}{f(\theta_1, \theta_2, ..., \theta_k)} \sum_{x_1, x_2, ..., x_k} x_i a_{x_1, x_2, ..., x_k} \theta_1^{x_i_1} \theta_2^{x_i_2} ... \theta_k^{x_i_k} \]
\[ = \frac{\theta_1}{f(\theta_1, \theta_2, ..., \theta_k)} \sum_{x_1, x_2, ..., x_k} x_i a_{x_1, x_2, ..., x_k} \theta_1^{x_i_1} \theta_2^{x_i_2} ... \theta_k^{x_i_k} \]
\[ = \theta_1 \frac{\partial}{\partial \theta_1} f(\theta_1, \theta_2, ..., \theta_k) \]
\[ = \theta_1 \frac{\partial}{\partial \theta_1} \ln f(\theta_1, \theta_2, ..., \theta_k) \]

It is clear that \( \mu_1 (\theta_1, \theta_2, ..., \theta_k) \) and \( \mu_1 (\theta_1, \theta_2, ..., \theta_k) / \theta_i \) are non-negative and continuous functions for all \( \theta_i \in (0, r_i) \).

In fact the two functions are differentiable for all \( \theta_i \in (0, r_i) \). Finally,

\[ \lim_{\theta_i \to 0^+} \mu_1 (\theta_1, \theta_2, ..., \theta_k) \]
\[ = \lim_{\theta_i \to 0^+} \sum_{x_1, x_2, ..., x_k} x_i P \left( X_1 = x_1, X_2 = x_2, ..., X_k = x_k \right) \]
\[ = \lim_{\theta_i \to 0^+} \sum_{x_1, x_2, ..., x_k} x_i a_{x_1, x_2, ..., x_k} \theta_1^{x_i_1} \theta_2^{x_i_2} ... \theta_k^{x_i_k} \]
\[ = \lim_{\theta_i \to 0^+} \sum_{x_i} x_i \psi_{x_i} \theta_i^{x_i} \]
\[ = \lim_{\theta_i \to 0^+} \sum_{x_i} x_i \psi_{x_i} \theta_i^{x_i} \]
Theorem 1. The multivariate random vector \((X_1, X_2, ..., X_k)\) having multivariate power series distribution is uniquely determined by the mean of any one marginal random variable of multivariate random vector \((X_1, X_2, ..., X_k)\).

Proof. Suppose that the mean of the marginal random variable \(X_i\) is

\[
m_i(\theta_1, \theta_2, ..., \theta_k) = \theta_i \frac{\partial}{\partial \theta_i} \ln f(\theta_1, \theta_2, ..., \theta_k)
\]

Let \(0 < u \leq \theta_i \leq t < r_i\). Then for \(\theta_i \in (0, r_i)\) we have

\[
m_i(\theta_1, \theta_2, ..., \theta_k) = \frac{\partial}{\partial \theta_i} \ln f(\theta_1, \theta_2, ..., \theta_k)
\]

Therefore,

\[
\int_u^t \frac{m_i(\theta_1, \theta_2, ..., \theta_k)}{\theta_i} d \theta_i = \frac{\partial}{\partial \theta_i} \int_u^t \ln f(\theta_1, \theta_2, ..., \theta_k) d \theta_i
\]

Implies,

\[
M_i(\theta_1, \theta_2, ..., t, ..., \theta_k) - M_i(\theta_1, \theta_2, ..., u, ..., \theta_k)
\]

\[
= \ln \left\{ \frac{f(\theta_1, \theta_2, ..., t, ..., \theta_k)}{f(\theta_1, \theta_2, ..., u, ..., \theta_k)} \right\}
\]

Where

\[
\frac{\partial}{\partial \theta_i} M_i(\theta_1, \theta_2, ..., \theta_k) = m_i(\theta_1, \theta_2, ..., \theta_k) / \theta_i
\]

Therefore,

\[
f(\theta_1, \theta_2, ..., t, ..., \theta_k) = f(\theta_1, \theta_2, ..., u, ..., \theta_k) e^{-M_i(\theta_1, \theta_2, ..., u, ..., \theta_k)} e^{M_i(\theta_1, \theta_2, ..., t, ..., \theta_k)}
\]

\[
= R(\theta_1, ..., \theta_{i-1}, u, \theta_i+1, ..., \theta_k) e^{M_i(\theta_1, \theta_2, ..., t, ..., \theta_k)}
\]

Changing \(t\) by \(\theta_i\), we get
\( f ( \theta_1, \theta_2, ..., \theta_k ) = R ( \theta_1, ..., \theta_{i-1}, u, \theta_{i+1}, ..., \theta_k ) e^{M ( \theta_1, \theta_2, ..., \theta_k )} \) ...... (1)

Since \( f ( \theta_1, \theta_2, ..., \theta_k ) = \sum_{x_1x_2...x_k} a_{x_1x_2...x_k} \theta_1^{x_1} \theta_2^{x_2} ... \theta_k^{x_k} \)

\[
= \sum_{x_i} \psi_{x_i} \theta_i^{x_i}
\]

Therefore, \( e^{M ( \theta_1, \theta_2, ..., \theta_k )} \) has power series in \( \theta_i \), say,

\[
e^{M ( \theta_1, \theta_2, ..., \theta_k )} = \sum_{x_i} \psi_{x_i} \theta_i^{x_i}, \text{ Then,}
\]

\[
P ( X_i = x_i ) = \frac{\psi_{x_i} \theta_i^{x_i}}{f ( \theta_1, \theta_2, ..., \theta_k )} = \frac{\psi_{x_i} \theta_i^{x_i}}{R ( \theta_1, ..., u, ..., \theta_k ) e^{M ( \theta_1, ..., u, ..., \theta_k )}}
\]

\[
= \frac{\sum_{x_i} \psi_{x_i} \theta_i^{x_i}}{\sum_{x_i} \psi_{x_i} \theta_i^{x_i}}
\]

\[
= \frac{\psi_{x_i} \theta_i^{x_i}}{e^{M ( \theta_1, \theta_2, ..., \theta_k )}}
\]

Hence, without loss of generality we can assume in equation (1) that \( R ( \theta_1, ..., u, ..., \theta_k ) = 1 \) and hence equation (1) becomes

\( f ( \theta_1, \theta_2, ..., \theta_k ) = e^{M ( \theta_1, \theta_2, ..., \theta_k )} \) ...... (2)

To prove uniqueness, suppose that \((X_1, X_2, ..., X_k)\) and \((Y_1, Y_2, ..., Y_k)\) are two random vectors having multivariate power series distribution with mean \(m_i ( \theta_1, \theta_2, ..., \theta_k )\) for both. Assume that the corresponding defining functions are \(f ( \theta_1, \theta_2, ..., \theta_k )\) and \(g ( \theta_1, \theta_2, ..., \theta_k )\) respectively. Therefore,

\[
m_i ( \theta_1, \theta_2, ..., \theta_k ) = \frac{\partial}{\partial \theta_i} \ln f ( \theta_1, \theta_2, ..., \theta_k )
\]

\[
= \frac{\partial}{\partial \theta_i} \ln g ( \theta_1, \theta_2, ..., \theta_k ), \text{ for all } \theta_i \in (0, r_i)
\]

Let \(0 < u \leq \theta_i \leq t < r_i\), then,

\[
\int_u^t \frac{\partial}{\partial \theta_i} \ln f ( \theta_1, \theta_2, ..., \theta_k ) d \theta_i = \int_t^u \frac{\partial}{\partial \theta_i} \ln g ( \theta_1, \theta_2, ..., \theta_k ) d \theta_i
\]
Implies
\[ f(\theta_1, \theta_2, t, ..., \theta_k) = H(\theta_1, ..., \theta_{i-1}, u, \theta_{i+1}, ..., \theta_k) \cdot g(\theta_1, ..., t, ..., \theta_k) \]

Where
\[ H(\theta_1, ..., \theta_{i-1}, u, \theta_{i+1}, ..., \theta_k) = \frac{f(\theta_1, ..., t, ..., \theta_k)}{g(\theta_1, ..., t, ..., \theta_k)} \]

Changing \( t \) by \( \theta_i \) we get
\[ f(\theta_1, \theta_2, ..., \theta_k) = H(\theta_1, ..., \theta_{i-1}, u, \theta_{i+1}, ..., \theta_k) \cdot g(\theta_1, \theta_2, ..., \theta_k) \]  .......(3)

As before and without loss of generality we can assume that:
\[ H(\theta_1, ..., \theta_{i-1}, u, \theta_{i+1}, ..., \theta_k) = 1 \] and hence equation (3) becomes
\[ f(\theta_1, \theta_2, ..., \theta_k) = g(\theta_1, \theta_2, ..., \theta_k), \text{ for all } \theta_i \in (0, r_i) \]  .......(4)

Therefore from equation (4) it follows that the two random vectors are identical, which completes the proof of the theorem.

**Theorem 2.** Let \( m_i(\theta_1, \theta_2, ..., \theta_k) \) be non-negative and continuous function for all \( \theta_i \in (0, r_i) \). There exists a multivariate random vector \( (X_1, X_2, ..., X_k) \) having \( m_i(\theta_1, \theta_2, ..., \theta_k) \) as the mean of the marginal random variable \( X_i \) iff:

(i) There exist a function \( M_i(\theta_1, \theta_2, ..., \theta_k) \) such that
\[ \frac{\partial}{\partial \theta_i} M_i(\theta_1, \theta_2, ..., \theta_k) = m_i(\theta_1, \theta_2, ..., \theta_k) / \theta_i, \text{ for all } \theta_i \in (0, r_i), \]

(ii) \( e^{M_i(\theta_1, \theta_2, ..., \theta_k)} \) has a Maclaurin expansion in \( \theta_1, \theta_2, ..., \theta_k \) with non-negative coefficients.

**Proof.**

(A). Assume (i) and (ii) hold. Let
\[ f(\theta_1, \theta_2, ..., \theta_k) = e^{M_i(\theta_1, \theta_2, ..., \theta_k)} = \sum_{x_1 \leq x_2 \leq \cdots \leq x_k} a_{x_1...x_k} \theta_1^{x_1} \theta_2^{x_2} ... \theta_k^{x_k} \]

We can define a multivariate random vector \( (X_1, X_2, ..., X_k) \) taking values \( (x_1, x_2, ..., x_k) \) with probability function
\[ p(X_1 = x_1, X_2 = x_2, ..., X_k = x_k) = a_{x_1...x_k} \theta_1^{x_1} \theta_2^{x_2} ... \theta_k^{x_k} / f(\theta_1, \theta_2, ..., \theta_k) \]
Sadoon Abdullah Ibrahim Al-Obaidy et al.

Therefore, \( (X_1, X_2, ..., X_k) \) is a random vector with multivariate power series distribution. Furthermore, the mean of the marginal random variable \( X_i \) is:

\[
\mu_i(\theta_1, \theta_2, ..., \theta_k) = \mathbb{E}X_i = \theta_i \frac{\partial}{\partial \theta_i} \ln f(\theta_1, \theta_2, ..., \theta_k)
\]

\[
= \theta_i \frac{\partial}{\partial \theta_i} M_i(\theta_1, \theta_2, ..., \theta_k)
\]

\[
=m_i(\theta_1, \theta_2, ..., \theta_k) \quad \text{Using (i)}
\]

(B) Let \( (X_1, X_2, ..., X_k) \) be a random vector with a multivariate power series distribution and having defining function \( f(\theta_1, \theta_2, ..., \theta_k) \). Then,

\[
m_i(\theta_1, \theta_2, ..., \theta_k) = \theta_i \frac{\partial}{\partial \theta_i} \ln f(\theta_1, \theta_2, ..., \theta_k)
\]

Implies, \( \frac{m_i(\theta_1, \theta_2, ..., \theta_k)}{\theta_i} = \frac{\partial}{\partial \theta_i} \ln f(\theta_1, \theta_2, ..., \theta_k) \), for all \( \theta_i \in (0, r_i) \).

Let \( 0 < u \leq \theta \leq t < r_i \). Therefore,

\[
\int_u^t \frac{m_i(\theta_1, \theta_2, ..., \theta_k)}{\theta_i} d\theta_i = \int_u^t \frac{\partial}{\partial \theta_i} \ln f(\theta_1, \theta_2, ..., \theta_k) d\theta_i
\]

Implies \( f(\theta_1, \theta_2, ..., \theta_k) = f(\theta_1, \theta_2, \theta_i, t, ..., \theta_k) e^{-M_i(\theta_1, \theta_2, \theta_i, t, \theta_k)} \)

\[
= K(\theta_1, \theta_2, \theta_i, t, \theta_k) e^{M_i(\theta_1, \theta_2, \theta_i, t, \theta_k)}
\]

Where \( \frac{\partial}{\partial \theta_i} M_i(\theta_1, \theta_2, ..., \theta_k) = m_i(\theta_1, \theta_2, ..., \theta_k) / \theta_i, \theta_i \in (0, r_i) \).

Thus (i) is true, changing \( t \) by \( \theta \) we get

\[
f(\theta_1, \theta_2, ..., \theta_k) = K(\theta_1, \theta_2, \theta_i, t, u, \theta_k) e^{M_i(\theta_1, \theta_2, \theta_i, \theta_k)} \quad \text{.....(5)}
\]

As before and without loss of generality we can assume in equation (5) that \( K(\theta_1, ..., \theta_i-1, u, \theta_k) = 1 \), and hence equation (5) becomes

\[
f(\theta_1, \theta_2, ..., \theta_k) = e^{M_i(\theta_1, \theta_2, \theta_k)} \quad \text{.....(6)}
\]

Since \( f(\theta_1, \theta_2, ..., \theta_k) \) has a Maclaurin expansion in \( (\theta_1, \theta_2, ..., \theta_k) \) with non-negative coefficients, then from equation (6) it follows that \( e^{M_i(\theta_1, \theta_2, \theta_k)} \) has a Maclaurin expansion with the same property. Therefore (ii) is true. This completes the proof of the theorem.
3 Demonstration Example

In order to demonstrate the value of these theories, the following example is given. Example: Let

\[ m_i(\theta_1, \theta_2, ..., \theta_k) = -\frac{\theta_i}{(1 - \theta_1 - \theta_2 - ... - \theta_k) \ln(1 - \theta_1 - \theta_2 - ... - \theta_k)}, \]

where \( \theta_i \) are the parameters and \( 0 < \theta_i < 1, \text{i} = 1, 2, ..., k \). Then,

\[ M_i(\theta_1, \theta_2, ..., \theta_k) = \ln\left\{-A \ln(1 - \theta_1 - \theta_2 - ... - \theta_k)\right\} \]

where \( A \) is independent of \( \theta_i \), in general \( A = A(\theta_1, ..., \theta_{i-1}, \theta_{i+1}, ..., \theta_k) \)

Now \( e^{M(\theta, \theta_i, \ldots, \theta_k)} = -A \ln(1 - \theta_1 - \theta_2 - ... - \theta_k) \)

\[ = A \sum_{x_1=1, \ldots, x_k=1} \frac{\Gamma(x_1 + x_2 + \ldots + x_k)}{x_1!x_2!\ldots x_k!} \theta_1^{x_1} \theta_2^{x_2} \ldots \theta_k^{x_k} \]

Therefore, there exists a random vector \((X_1, X_2, ..., X_k)\) having multivariate power series distribution with:

\[ m_i(\theta_1, \theta_2, ..., \theta_k) = -\frac{\theta_i}{(1 - \theta_1 - \theta_2 - ... - \theta_k) \ln(1 - \theta_1 - \theta_2 - ... - \theta_k)} \]

as the mean of the marginal random variable \( X_i \) the defining function of this random vector \((X_1, X_2, ..., X_k)\) is:

\[ f(\theta_1, \theta_2, ..., \theta_k) = -A \ln(1 - \theta_1 - \theta_2 - ... - \theta_k) \] \hspace{1cm} (7)

Without loss of generality we can assume \( A = 1 \), and hence equation (7) becomes

\[ f(\theta_1, \theta_2, ..., \theta_k) = -\ln(1 - \theta_1 - \theta_2 - ... - \theta_k) \] \hspace{1cm} (8)

Therefore

\[ p\left(X_1 = x_1, X_2 = x_2, ..., X_k = x_k\right) = -\frac{\Gamma(x_1 + x_2 + \ldots + x_k) \theta_1^{x_1} \theta_2^{x_2} \ldots \theta_k^{x_k}}{x_1!x_2!\ldots x_k! \ln(1 - \theta_1 - \theta_2 - ... - \theta_k)} \]

This can be called the multivariate logarithmic distribution.

4 Open Problem and Future Work

As a future work in this area, the researchers can be focus in the following topics:
1- Characterization of polynomial distributions by condition distributions and restricted linear regression.

2- The compound class of general gamma power series distributions.

5 Conclusion

In this study the authors have concluded the results that any multivariate power series distribution is determined uniquely from the mean function of any marginal random variable. Furthermore these results indicated also that any given function satisfying certain conditions construct a random vector with multivariate power series distribution which has a mean of the marginal random variable.

References


