Int. J. Open Problems Compt. Math., Vol. 6, No. 3, September, 2013
ISSN 1998-6262; Copyright ©ICSRS Publication, 2013
www.i-csrs.org

Bipancyclic subgraphs in random bipartite graphs

Yilun Shang[1,2]

[1]Institute for Cyber Security,
University of Texas at San Antonio, TX 78249, USA;
[2]Singapore University of Technology and Design,
Singapore 138682, Singapore
e-mail: shylmath@hotmail.com

Abstract

A bipartite graph on $2n$ ***vertices is bipancyclic if it contains cycles of all even lengths from*** 4 ***to*** $2n$***. In this paper we prove that the random bipartite graph*** $G(n, n, p)$ ***with*** $p(n) \gg n^{-2/3}$ ***asymptotically almost surely has the following resilience property: Every Hamiltonian subgraph*** G' ***of*** $G(n, n, p)$ ***with more than*** $(1/2 + o(1))n^2p$ ***edges is bipancyclic. This result is tight in two ways. First, the range of*** p ***is essentially best possible. Second, the proportion*** $1/2$ ***of edges cannot be reduced. Our result extends a classical theorem of Mitchem and Schmeichel.***

Keywords: *Bipancyclicity, Bipartite Graph, Random Graph, Resilience.*
2010 Mathematics Subject Classification: 05C80, 05C38, 05C45, 05D40.

1 Introduction

A bipartite graph on $2n$ vertices is called bipancyclic if it contains cycles of all even lengths from 4 to $2n$. Analogously, a graph on n vertices is called pancyclic if it contains cycles of all length t for $3 \leq t \leq n$. Clearly, (bi)pancyclic graphs are Hamiltonian but the converse is not true in general. A variety of sufficient conditions for a Hamiltonian bipartite graph to be bipancyclic have been studied in the literature, including [1, 2, 12, 13] and [15]. Recall that a bipartite graph is called balanced if the two classes of bipartition have the same cardinality. In [12] Mitchem and Schmeichel proved the following theorem.

Theorem 1.1 *Let G be a Hamiltonian bipartite balanced graph with $2n$ vertices and m edges. If $m > n^2/2$, then G is bipancyclic.*

Recently, Sudakov and Vu [19] proposed the framework of resilience of graphs, in which many extremal graph-theoretic properties such as Hamiltonicity and pancyclicity can be studied (see e.g. [3, 4, 9, 10, 11]). Let $\mathcal{P}$ be a monotone increasing graph property. Define the global resilience of a graph G with respect to $\mathcal{P}$ as the minimum number r such that by deleting r edges from G, one can obtain a graph not having $\mathcal{P}$. Using this notion, the above Theorem 1.1 can be reformulated as a global resilient statement with an additional constraint: If one deletes fewer than $n^2/2$ edges from the complete bipartite graph $K_{n,n}$ while preserving Hamiltonicity, then the resulting graph is always bipancyclic.

In this paper, we study bipancyclicity of random bipartite graphs in the context of global resilience by extending Theorem 1.1. The model of random bipartite graphs $G(n, n, p)$ is the probability distribution on the set of all bipartite balanced graphs with vertex set $\{1, 2, \cdots, 2n\}$ such that each pair of vertices from different classes of bipartition forms an edge randomly and independently with probability p. Its monopartite version is the celebrated binomial random graph $G(n, p)$ (see e.g. [8]). We say that $G(n, n, p)$ (or $G(n, p)$) possesses a graph property $\mathcal{P}$ asymptotically almost surely, or a.a.s. for short, if the probability that $G(n, n, p)$ (or $G(n, p)$) possesses $\mathcal{P}$ approaches to 1 as n tends to infinity. Lee and Samotij [10] recently proved that if $p \gg n^{-1/2}$, then $G(n, p)$ a.a.s. satisfies the following: Every Hamiltonian subgraph $G' \subset G(n, p)$ with more than $(\frac{1}{2} + o(1))n^2p/2$ edges is pancyclic. Our main result is a corresponding version for $G(n, n, p)$, which is the following generalization of Theorem 1.1 (since $G(n, n, 1) = K_{n,n}$).

Theorem 1.2 *If $p \gg n^{-2/3}$, then $G(n, n, p)$ a.a.s. satisfies the following. Every Hamiltonian subgraph $G' \subset G(n, n, p)$ with more than $(1 + o(1))n^2p/2$ edges is bipancyclic.*

Theorem 1.2 is asymptotically tight in two ways. First, one cannot improve the exponent $-2/3$. To see this, assume that $p \ll n^{-2/3}$ and fix a Hamilton cycle H in $G(n, n, p)$. From each 4-cycle in $G(n, n, p)$, delete one edge which does not belong to H. Since a.a.s there are at most $n^4p^4 = o(n^2p)$ 4-cycles in the graph, only a small proportion of edges is deleted and the resulting graph does not contain any 4-cycles, hence not bipancyclic. Second, Hamilton subgraphs with fewer than $(1 + o(1))n^2p/2$ edges need not be bipancyclic. Assume that $p \gg n^{-2/3}$ and fix a Hamilton cycle H in $G(n, n, p)$. We label

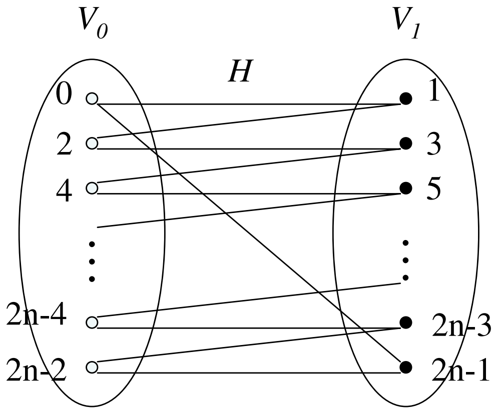

Figure 1: A bipartite balanced graph with bipartition $|V_0| = |V_1| = n$. $H = \{0, 1, 2, \cdots, 2n-1, 0\}$ is a Hamilton cycle.

the vertices as shown in Fig. 1 such that $H = \{0, 1, 2, \cdots, 2n-1, 0\}$. Delete all edges $\{0, j\}$ from $G(n, n, p)$ except two edges $\{0, 1\}$ and $\{0, 2n-1\}$. For each even i with $2 \leq i \leq 2n-2$, delete all edges $\{i, j\}$ from $G(n, n, p)$ with $j \geq i+3$. A.a.s. we will delete at most $(1 + o(1))n^2p/2$ edges, and a.a.s. the above process produces a graph G' with at least $(1 + o(1))n^2p/2$ edges. Note that $H \subset G'$ so G' is Hamiltonian. However, G' contains no 4-cycles, thus not bipancyclic.

The rest of the paper is organized as follows. In Section 2 we present some notations and preliminaries that will be needed in our development later. The proof of Theorem 1.2 comprises two parts: In Section 3 we establish the existence of short and long cycles of even lengths, while in Section 4 we establish the existence of medium ones of even lengths. An open problem is presented in Section 5.

2 Preliminaries

Let $G = (V, E)$ denote a graph with vertex set V and edge set E. Similarly, a bipartite graph G with edge set E is denoted by $G = (V_0, V_1, E)$ where V_0 and V_1 are the two classes of the bipartition. For a vertex v, we denote its neighborhood by $N(v)$, and its degree by $\deg(v) = |N(v)|$. For a set X, let $E(X)$ be the set of edges in the induced subgraph $G[X]$, and let $e(X) = |E(X)|$. When we have several graphs under consideration, we may use subscripts such as $\deg_G(v)$ to indicate the graph we are currently working with. We often omit floor and ceiling signs whenever these are not crucial. We also assume the order of the graphs is large enough throughout our derivation.

The following concentration inequality (see e.g. [8, Corollary 2.3]) will be often used in the proof of main result.

Theorem 2.1 *(Chernoff's inequality) Let* $0 < \varepsilon \leq 3/2$. *If* X *is a binomial random variable with parameter* n *and* p, *then*

$$P(|X - \mathbb{E}(X)| \geq \varepsilon\mathbb{E}(X)) \leq 2e^{-\varepsilon^2\mathbb{E}(X)/3},$$

where $\mathbb{E}$ *represents the expectation operator.*

The following results on cycles of fixed length were proved by Bondy and Simonovits [5], and Haxell et al. [7]. They yield the existence of very short cycles.

Theorem 2.2 *(i) [5] Let* k *be a positive integer, and let* G *be a graph on* n *vertices with more than* $100kn^{1+(1/k)}$ *edges. Then* G *contains a cycle of length* $2k$.

(ii) [7] For any fixed integer $l \geq 2$ *and* $\varepsilon \in (0,1)$, *there exists a constant* $C > 0$ *such that if* $p \geq Cn^{-1+1/(2l-1)}$, *then* $G(n,p)$ *a.a.s. satisfies the following. Every subgraph* $G' \subset G(n,p)$ *with at least* $(1+\varepsilon)n^2p/8$ *edges contains a cycle of length* $2l$.

As reasoned in [10], our proof of Theorem 1.2 will rely on a hypergraph construction, which fits in the general framework developed for extremal properties of random discrete structures by Schacht [14] (similar results were obtained by Conlon and Gowers [6] independently). Before introducing the general theorem, we need some definitions.

Definition 2.3 *Let* H *be a* k*-uniform hypergraph,* $\alpha \geq 0$ *and* $\varepsilon_0 \in (0,1)$. *Let* $f : (0,1) \to (0,1)$ *be a non-decreasing function. We say* H *is* $(\alpha, f, \varepsilon_0)$*-dense if the following is true.*

For every $\varepsilon \geq \varepsilon_0$ *and every* $U \subset V(H)$ *with* $|U| \geq (\alpha+\varepsilon)|V(H)|$, *we have*

$$|E(H[U])| \geq f(\varepsilon)|E(H)|.$$

For a k-uniform hypergraph H, $v \in V(H)$, $U \subset V(H)$ and $i \in \{1, 2, \cdots, k-1\}$, we define

$$\deg_i(v, U) = |\{X \in E(H) : |X \cap (U\backslash\{v\})| \geq i \text{ and } v \in X\}|.$$

For $q \in [0,1]$ and a set X, let X_q be the binomial random subset of X with survival probability q.

Definition 2.4 *Let H be a k-uniform hypergraph, $p \in (0,1)$ and $K \geq 1$. We say H is (K,p)-bounded if the following is true.*

For every $i \in \{1, 2, \cdots, k-1\}$ and $q \in [p, 1]$, we have

$$\mathbb{E}\left(\sum_{v \in V(H)} \deg_i(v, V(H)_q)^2\right) \leq Kq^{2i}\frac{|E(H)|^2}{|V(H)|}.$$

Theorem 2.5 *([14]) Suppose $(H_n)_{n\in\mathbb{N}}$ is a sequence of k-uniform hypergraphs. Let $(p_n)_{n\in\mathbb{N}}$ be a sequence of probabilities. Let $(v_n)_{n\in\mathbb{N}}$ and $(e_n)_{n\in\mathbb{N}}$ be sequences of integers satisfying $p_n v_n \to \infty$ and $p_n^k e_n \to \infty$ as $n \to \infty$. Let $\alpha \geq 0$, $K \geq 1$ and $f : (0,1) \to (0,1)$ be a non-decreasing function. For every $\varepsilon \in (0,1)$, there exist $\varepsilon_0 \in (0,1)$, $b > 0$, $C \geq 1$ and $n_0 \geq 1$ such that for every $n \geq n_0$ and every q with $n^{-1/3} \geq q \geq Cp_n$ the following holds.*

If H_n is $(\alpha, f, \varepsilon_0)$-dense and (K, p_n)-bounded satisfying $|V(H_n)| \geq v_n$ and $|E(H_n)| \geq e_n$, then with probability at least $1 - e^{-bqv_n}$, every subset $W \subset V(H_n)_q$ with $|W| \geq (\alpha + \varepsilon)|V(H_n)_q|$ contains an edge of H_n.

3 Existence of short and long cycles of even lengths

In this section, we establish the existence of short and long cycles of even lengths in the following sense.

Theorem 3.1 *For any $\varepsilon \in (0,1)$, there exist constants $C > 0$ and $\delta \in (0,1)$ such that if $p \geq Cn^{-2/3}$, then $G(n,n,p)$ a.a.s. satisfies the following. Every Hamiltonian subgraph $G' \subset G(n,n,p)$ with more than $(1+\varepsilon)n^2p/2$ edges contains a cycle of length t for all even $t \in [4, 2\delta n] \cup [2(1-\delta)n, 2n]$.*

We will follow the idea of [10] and separate the proof of Theorem 1.2 into two parts: Theorem 3.1 is responsible for short and long cycles, and Theorem 4.1 (see Section 4 below) will be responsible for intermediate cycles. Note that if we choose C in Theorem 3.1 to be large enough, the existence of 4-cycle and 6-cycle follows easily from Theorem 2.2. Hence, in what follows we will focus on cycles of length 8 and above.

Let $[2n]$ be the set of remainders modulo $2n$, namely $[2n] = \{0, 1, 2, \cdots, 2n-1\}$. The addition of the elements of $[2n]$ will be performed modulo $2n$ throughout this paper. A labeling of the vertices of the complete bipartite graph $K_{n,n} = (V_0, V_1, E)$ is called allowable if the vertices in V_0 are labeled by even numbers and those in V_1 are labeled by odd ones; c.f. Fig. 1. Fix an allowable labeling and let C_{2n} be the subgraph of $K_{n,n}$ consisting of the edges $\{i, i+1\}$ for all $i \in [2n]$. For illustration we may draw it as a circle with labels

$0, 1, 2, \cdots, 2n-1$ in the clockwise order. For each $i \in [2n]$, denote its distance from 0 on the cycle C_{2n} by $\|i\|$, namely

$$\|i\| = \min\{k \geq 0 : k \equiv i \mod 2n \text{ or } k \equiv -i \mod 2n\}.$$

For an even l with $0 \leq l \leq n$, a 4-vertex subgraph $X \subset K_{n,n} \backslash C_{2n}$ is called an l-shortcut if it is of one of the following types:

(i) There are $i_1, i_2, i_3, i_4 \in [2n]$ such that $i_1, i_1+1, i_2, i_2+1, i_3, i_3+1, i_4$ and i_4+l+1 are all distinct and lie in the clockwise order on C_{2n}, and X is composed of the edges $\{i_1, i_3\}$, $\{i_1{+}1, i_4\}$, $\{i_2, i_4{+}l{+}1\}$ and $\{i_2{+}1, i_3{+}1\}$. Moreover, i_1+1 and i_2 belong to different classes of bipartition. So do i_1 and i_2+1.

(ii) There are $i_1, i_2, i_3, i_4 \in [2n]$ such that $i_1, i_1+1, i_2, i_2+1, i_4, i_4+l+1, i_3$ and i_3+1 are all distinct and lie in the clockwise order on C_{2n}, and X is composed of the edges $\{i_1, i_3\}$, $\{i_1{+}1, i_4\}$, $\{i_2, i_4{+}l{+}1\}$ and $\{i_2{+}1, i_3{+}1\}$. Moreover, i_1+1 and i_2 belong to different classes of bipartition. So do i_1 and i_2+1.

Since our formulation is similar with that in [10], we highlight the novelties in our methodology.

- The introduction of allowable labeling make it possible to extend the analysis from monopartite graph $G(n,p)$ to bipartite graph $G(n,n,p)$.
- Additional constraints are posed on the definition of l-shortcut. These modifications imply that C_{2n} and X may be concurrent under any allowable labeling and will be critical in the proof of some technical lemmas later.

A key observation is that the graph $C_{2n} \cup X$ contains cycles of lengths $l+8$ and $2n-l$ for every even l with $0 \leq l \leq n$ and every l-shortcut X; see Fig. 2. We will prove Theorem 3.1 through a series of lemmas.

Lemma 3.2 *For any $\varepsilon_0 \in (0,1)$, there exist an $n_0 \geq 1$ such that if $\varepsilon' \geq \varepsilon_0$ and $n \geq n_0$, then every $2n$-vertex bipartite balanced graph G' with $e(G') \geq (1+\varepsilon')n^2/2$ contains at least $\varepsilon'^8 n^4/(4 \cdot 16^7)$ many of l-shortcuts for every even l with $0 \leq l \leq \varepsilon' n/8$ and every allowable labeling of the vertex set of G' with $[2n]$.*

Proof. Assume that $\varepsilon' \geq \varepsilon_0$ and $n \geq n_0 = 192/\varepsilon_0$. Fix an allowable labeling of the vertex set of G' with $[2n]$. We have

$$\sum_{i=0}^{n-1} (\deg_{G'}(2i) + \deg_{G'}(2i+1)) = 2e(G') \geq (2+\varepsilon')\frac{n^2}{2}.$$

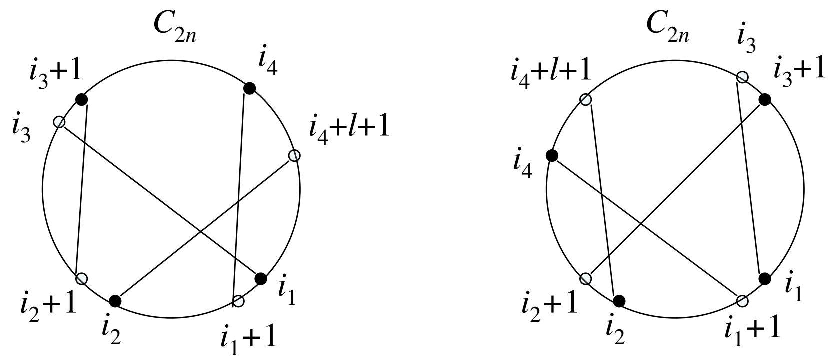

Figure 2: Two l-shortcuts of types (i) and (ii), respectively.

Define a set $I = \{i \in \{0, 1, \cdots n-1\} : \deg_{G'}(2i) + \deg_{G'}(2i+1) \geq (1+\varepsilon'/2)n\}$. Via a simple proof by contradiction we can see that $|I| \geq \varepsilon' n/2$. For every k with $0 \leq k \leq (1-\varepsilon'/4)n$, define $I(k) = \{i \in I : \deg_{G'}(2i) \in [k, k+\varepsilon' n/4)\}$. Again applying proof by contradiction we obtain that there exists some k such that

$$|I(k)| \geq \frac{|I|}{\lceil \frac{4}{\varepsilon'} \rceil} \geq \frac{\varepsilon'|I|}{8} \geq \frac{\varepsilon'^2}{16} n.$$

We define $I' = I(k)$ for any such k. Therefore, for all $i, j \in I'$ we have

$$\deg_{G'}(2i) + \deg_{G'}(2j+1) \geq \deg_{G'}(2j) + \deg_{G'}(2j+1) - \frac{\varepsilon'}{4} n \geq \left(1 + \frac{\varepsilon'}{4}\right) n. \quad (1)$$

On the other hand, it is easy to see that there exists a subset $I'' \subset I'$ satisfying $|I''| \geq \varepsilon'|I'|/32$, such that for all $i, j \in I''$, the distance between $2i$ and $2j$ on the cycle C_{2n} satisfies $\|2i - 2j\| \leq \varepsilon' n/16$. We claim

Claim 1. *For all different $i, j \in I''$ and every even l with $0 \leq l \leq \varepsilon' n/8$, there are at least $(\varepsilon' n/32)^2$ many of l-shortcuts with $\{i_1, i_2\} = \{2i, 2j\}$.*

If this is true, the total number of l-shortcut in G' is at least

$$\binom{|I''|}{2} \left(\frac{\varepsilon' n}{32}\right)^2 \geq \frac{|I''|^2 \varepsilon'^2}{4 \cdot 32^2} n^2 \geq \frac{|I'|^2 \varepsilon'^4}{4 \cdot 32^4} n^2 \geq \frac{\varepsilon'^8}{4 \cdot 16^7} n^4.$$

What remains to prove is Claim 1. Fix an even l with $0 \leq l \leq \varepsilon' n/8$. Note that $\|2i - 2j\| \leq \varepsilon' n/16$. Therefore, we may set $\{i_1, i_2\} = \{2i, 2j\}$ such that $i_2 = i_1 + k$ for some $0 < k \leq \varepsilon' n/8$. Let A be the set $\{i_2 + 2, \cdots, i_1 - 1\}$ of vertices of C_{2n} lying on the major arc connecting $i_2 + 1$ to i_1. Let $A' = \{i \in A : i + l + 1 \in A\}$. Therefore, we have

$$|A'| = |A| - (l+1) \geq \left(2 - \frac{\varepsilon'}{16}\right) n - 2 - l - 1 \geq \left(2 - \frac{3\varepsilon'}{16} - \frac{\varepsilon'}{32}\right) n.$$

Let $B = \{i \in A' : \{i_1+1, i\}, \{i_2, i+l+1\} \in E(G')\}$, $N_1 = N_{G'}(i_1+1)$ and $N_2 = \{i \in [2n] : i+l+1 \in N_{G'}(i_2)\}$. Employing (1) and the fact that $B = A' \cap N_1 \cap N_2$, we obtain

$$\begin{aligned}\left(1+\frac{\varepsilon'}{4}\right)n &\leq \deg_{G'}(i_1+1)+\deg_{G'}(i_2)\\ &= |N_1 \cup N_2| + |N_1 \cap N_2|\\ &= |N_1 \cup N_2| + |([2n]\backslash A') \cap N_1 \cap N_2| + |A' \cap N_1 \cap N_2|\\ &\leq n + 2n - |A'| + |B|,\end{aligned}$$

where the last inequality holds since l is even and i_1+1 and i_2 belong to different classes of bipartition. Hence

$$|B| \geq \left(\frac{\varepsilon'}{4} - \frac{3\varepsilon'}{16} - \frac{\varepsilon'}{32}\right)n = \frac{\varepsilon'}{32}n. \tag{2}$$

Fix some $i_4 \in B$, and let $J = \{i_4, \cdots, i_4+l+1\}$, $A'' = \{i \in A\backslash J : i+1 \in A\backslash J\}$. Therefore, we have

$$|A''| = |A| - |J| - 2 \geq \left(2 - \frac{3\varepsilon'}{16} - \frac{\varepsilon'}{32}\right)n.$$

Let $D = \{i \in A'' : \{i_1, i\}, \{i_2+1, i+1\} \in E(G')\}$. We can argue analogously as above to derive

$$|D| \geq \left(\frac{\varepsilon'}{4} - \frac{3\varepsilon'}{16} - \frac{\varepsilon'}{32}\right)n = \frac{\varepsilon'}{32}n. \tag{3}$$

Fix some $i_3 \in D$ and we readily have an l-shortcut $X \subset G'$ consisting of edges $\{i_1, i_3\}, \{i_1+1, i_4\}, \{i_2, i_4+l+1\}, \{i_2+1, i_3+1\}$. Therefore, the claim follows from (2) and (3). □

A key ingredient of the proof of Theorem 3.1 is to construct a hypergraph to which Theorem 2.5 can be applied. Inspired by the construction presented in [10], we define H_n^l be a 4-uniform hypergraph with the vertex set $V(H_n^l) = E(K_{n,n})$ for every integer n and every even l with $0 \leq l \leq n$. The hyperedges of H_n^l consist of all l-shortcuts in $K_{n,n}\backslash C_{2n}$. Therefore, we have

$$|V(H_n^l)| = |E(K_{n,n})| = n^2 \quad \text{and} \quad cn^4 \leq |E(H_n^l)| \leq 2n^4, \tag{4}$$

where $c > 0$ is an absolute constant. The following corollary is immediate from Lemma 3.2 and Definition 2.3.

Corollary 3.3 *Let $f : (0,1) \to (0,1)$ be the function defined by $f(\varepsilon') = 4\varepsilon'^8/16^6$ for all $\varepsilon' \in (0,1)$. For any $\varepsilon_0 \in (0,1)$, there exist $\delta \in (0,1)$ and $n_1 \geq 1$ such that for all $n \geq n_1$ and even l with $0 \leq l \leq 2\delta n$, the hypergraph H_n^l is $(1/2, 2f, \varepsilon_0/2)$-dense.*

Using an almost identical argument of [10, Lemma 3.6] we can establish the $(K, n^{-2/3})$-boundedness of H_n^l. We leave the proof of the following result to the reader.

Lemma 3.4 *There exists a constant $K > 0$ such that for all integer n and even l with $0 \le l \le n$, the hypergraph H_n^l is $(K, n^{-2/3})$-bounded.*

For $\delta \in (0,1)$, define a monotone increasing graph property $\mathcal{P}_\delta$ as follows. A $2n$-vertex bipartite balanced graph G satisfies $\mathcal{P}_\delta$ if and only if G contains an l-shortcut for every even l with $0 \le l \le 2\delta n$ and every allowable labeling of the vertices of G with $[2n]$.

Lemma 3.5 *For any $\varepsilon \in (0,1)$, there exist $\delta \in (0,1)$ and $C > 0$ such that if $Cn^{-2/3} \le p \le n^{-1/3}$, then $G(n,n,p)$ a.a.s. satisfies the following. Every subgraph $G' \subset G(n,n,p)$ with more than $(1/2+\varepsilon/2)e(G(n,n,p))$ edges satisfies $\mathcal{P}_\delta$.*

Proof. Set $k = 4$, $\alpha = 1/2$, $p_n = n^{-2/3}$, $v_n = n^2$ and $e_n = cn^4$ with c given in (4). Let f be the function defined in Corollary 3.3 and K be given in Lemma 3.4. We have $e_n(n^{-2/3})^4 \to \infty$ and $v_n n^{-2/3} \to \infty$, as $n \to \infty$. Furthermore, let ε_0, b, C, n_0 be the numbers satisfying the conclusion of Theorem 2.5. Let δ and n_1 be the numbers given in Corollary 3.3 by using the parameter ε_0.

Suppose that $n \ge \max\{n_0, n_1\}$ and fix an even l with $0 \le l \le 2\delta n$. Fix an allowable labeling of the vertices of $G(n,n,p)$ with $[2n]$. In view of Corollary 3.3 and Lemma 3.4 we observe that H_n^l is $(1/2, 2f, \varepsilon_0/2)$-dense and $(K, n^{-2/3})$-bounded. Let $Cn^{-2/3} \le p \le n^{-1/3}$. Since $|V(H_n^l)| = v_n$, $|E(H_n^l)| \ge e_n$ and $n \ge n_0$, Theorem 2.5 implies that with probability at least $1 - e^{-bpn^2}$, every subgraph $G' \subset G(n,n,p)$ with $e(G') \ge (1/2 + \varepsilon/2)e(G(n,n,p))$ contains a hyperedge of H_n^l, which is an l-shortcut with respect to the above fixed labeling. An application of Stirling's approximation shows that with probability at least $1 - (n!)^2 n e^{-bpn^2} = 1 - o(1)$, the random bipartite graph $G(n,n,p)$ satisfies the conclusion of Lemma 3.5. □

Completion of the proof of Theorem 3.1. Let δ and C be the numbers satisfying the conclusion of Lemma 3.5 by using the parameter $\varepsilon/4$ (instead of ε). Let $p' = Cn^{-2/3}$. An application of Theorem 2.1 shows that $e(G(n,n,p')) \le (1 + \varepsilon/8)n^2 p'$ a.a.s. Hence, by using Lemma 3.5 we obtain that a.a.s. every subgraph of $G(n,n,p')$ with more than $(1/2 + \varepsilon/4)n^2 p'$ edges satisfies $\mathcal{P}_\delta$. We claim

Claim 2. *Assume that $0 < p' \le p \le 1$ and $n^2 p' \to \infty$ as $n \to \infty$. If $G(n,n,p')$ a.a.s. has global resilience at least $(1/2 - \varepsilon/4)n^2 p'$ with respect to a monotone increasing graph property, then $G(n,n,p)$ a.a.s. has global resilience at least $(1/2 - \varepsilon/2)n^2 p$ with respect to the same property.*

This claim can be shown similarly as [9, Proposition 3.1] or [10, Proposition 2.7]. We omit the proof here.

Therefore, another application of Theorem 2.1 implies that if $p \geq Cn^{-2/3}$, then a.a.s. every subgraph $G' \subset G(n,n,p)$ with more than $(1/2+\varepsilon/2)n^2p$ edges satisfies $\mathcal{P}_\delta$. Note that every Hamiltonian graph with property $\mathcal{P}_\delta$ contains a cycle of length t for all even $t \in [8, 2\delta n] \cup [2(1-\delta)n, 2n]$. The proof of Theorem 3.1 is completed. □.

4 Existence of medium cycles of even lengths

In this section we establish the following result, which together with Theorem 3.1 readily gives our main result Theorem 1.2.

Theorem 4.1 *For any $\varepsilon \in (0,1)$ and $\delta \in (0,1)$, there exists a constant $C > 0$ such that if $p \geq Cn^{-2/3}$, then $G(n,n,p)$ a.a.s. satisfies the following. Every Hamiltonian subgraph $G' \subset G(n,n,p)$ with more than $(1+\varepsilon)n^2p/2$ edges contains a cycle of length t for all even $t \in [2\delta n, 2(1-\delta)n]$.*

Fix an allowable labeling of the vertices of $K_{n,n}$ with $[2n]$. We partition the edge set $E(K_{n,n})$ as

$$E(K_{n,n}) = \cup_{i=0}^{n-1} E_i,$$

where $E_i = \{\{x,y\} : x + y \equiv 2i+1 \mod 2n\}$. For each $0 \leq i \leq n-1$, we define an ordering $\leq_i$ for the elements of E_i as follows. If we evenly place the numbers from $[2n]$ on a circle (i.e., C_{2n} as defined above), each set E_i will comprise all parallel edges in some direction. We order them as per their distance from the minor arc connecting i to $i+1$; c.f. Fig. 3. We refer to the elements of E_i as the edges in direction i. Note that $|E_i| = n$ for all i. Two edges $e_1, e_2 \in E(K_{n,n}) \backslash E(C_{2n})$ is said to be crossing if their endpoints are all distinct and lie alternately on the cycle C_{2n}.

In what follows, we will still adopt a similar reasoning as conducted in [10]. We want to highlight a remarkable modification in our methodology.

- The redefinition of total order sets E_i $(i = 0, 1, \cdots, n-1)$ allows a smooth switch to the bipartite structure. This partition appears to be critical in the following development.

We observe that for every $i \in [2n]$ and every even l with $2 \leq l \leq 2n-2$, the graph $C_{2n} \cup \{e_1, e_2\}$, where $e_1 \in E_i$ and $e_2 \in E_{i+l/2}$ are crossing edges, contains cycles of lengths $l+2$ and $2n-l+2$; see Fig. 3.

For $\beta \in (0, 1/6)$, and $k \in \mathbb{N}$ with $1 \leq k \leq 2(1/2-\beta)n$, we define E_i^k be the set of $2\beta n$ consecutive (with respect to $\leq_i$) edges in E_i, beginning from the k-th smallest element of E_i. We may refer to E_i^k as an interval of length $2\beta n$, whose

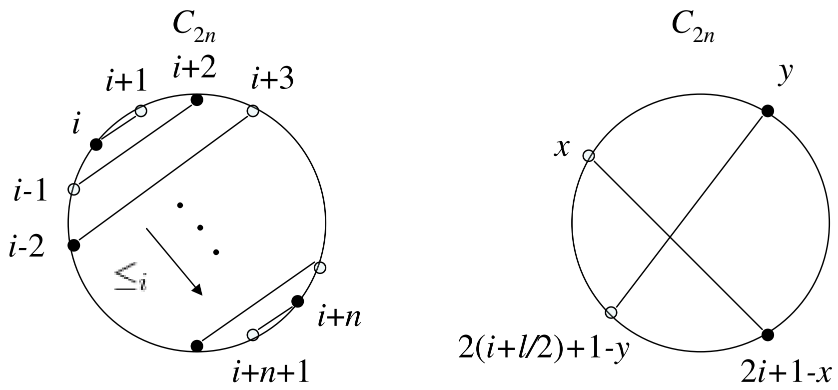

Figure 3: The total order set E_i with arrow pointing from $\leq_i$-smaller to $\leq_i$-larger elements; two crossing edges $e_1 = \{x, 2i+1-x\}$ and $e_2 = \{y, 2(i+l/2)+1-y\}$.

leftmost endpoint is the k-th smallest element of E_i. Denote by $M_i \subset E_i$ the set of $2(1/2-2\beta)n$ middle elements of E_i without the leftmost and rightmost intervals of lengths $2\beta n$. Let G be a $2n$-vertex bipartite balanced graph with an allowable labeling with $[2n]$. For $i \in [2n]$, $\varepsilon' \in (0,1)$ and $p \in [0,1]$, we say that the direction E_i is (β, ε', p)-good in G if for all $k \in \mathbb{N}$ with $1 \leq k \leq 2(1/2-\beta)n$, G satisfies

$$||E(G) \cap E_i^k| - 2\beta np| \leq 2\varepsilon' \beta np, \tag{5}$$

and

$$||E(G) \cap M_i| - 2(1/2-2\beta)np| \leq 2\varepsilon'(1/2-2\beta)np. \tag{6}$$

Lemma 4.2 *Let $\beta, \varepsilon' \in (0, 1/6)$. If $p \geq Cn^{-2/3}$ for some $C > 0$, then a.a.s. for every allowable labeling of vertices of $G(n,n,p)$ with $[2n]$, there are at most $n^{5/6}$ directions that are not (β, ε', p)-good in $G(n,n,p)$.*

Proof. Let G be a graph drawn from $G(n,n,p)$ and fix an allowable labeling of the vertices of G with $[2n]$. It follows from Theorem 2.1 that, for all i and k,

$$P\left(||E(G) \cap E_i^k| - 2\beta np| > 2\varepsilon' \beta np\right) \leq 2e^{-2\varepsilon'^2 \beta np/3} \leq e^{-cnp}$$

and

$$P\left(||E(G) \cap M_i| - 2(1/2-2\beta)np| > 2\varepsilon'(1/2-2\beta)np\right) \leq 2e^{-2\varepsilon'^2(1/2-2\beta)np/3} \leq e^{-cnp},$$

where $c = c(\beta, \varepsilon') > 0$. Therefore, given i, $P(E_i$ is not (β, ε', p)-good$) \leq 4(1/2-\beta)ne^{-cnp} \leq e^{-cnp/2}$ by using (5) and (6). Since the events

$$\{E_i \text{ is not } (\beta, \varepsilon', p)\text{-good}\}_{0 \leq i \leq n-1}$$

are mutually independent, the probability that there are more than $n^{5/6}$ not good directions is at most

$$\binom{n}{n^{5/6}}\left(e^{-cnp/2}\right)^{n^{5/6}} \le 2^n e^{-cn^{11/6}p/2} \le e^{-c'n^{7/6}},$$

where $c' = c'(c, C) > 0$. Since there are $(n!)^2$ different allowable labelings, the probability of there being an allowable labeling with more than $n^{5/6}$ not good directions is at most

$$(n!)^2 e^{-c'n^{7/6}} \le \frac{e^2 n^{2n+1}}{e^{2n+c'n^{7/6}}} = o(1),$$

as $n \to \infty$. The proof is completed. □

For $\beta \in (0, 1/6)$, a crossing between two edges $\{x_1, y_1\}$ and $\{x_2, y_2\}$ is said to be close if

$$\min\{\|x_1 - x_2\|, \|x_1 - y_2\|, \|y_1 - x_2\|, \|y_1 - y_2\|\} \le 2\beta n.$$

The following statements can be proved based on a similar observation in [10, Lemma 3.10]. We leave the proof to the reader.

Lemma 4.3 *For $i \in [2n]$, $\beta \in (0, 1/6)$ and even l with $4\beta n + 1 \le l \le (2 - 4\beta)n - 1$, the following statements are true.*

(i) Every edge in E_i forms close crossings with at most $4\beta n$ edges from $E_{i+l/2}$, and these edges can be covered by a set of the form $E^{k_1}_{i+l/2} \cup E^{k_2}_{i+l/2}$ for some $1 \le k_1, k_2 \le 2(1/2 - \beta)n$.

(ii) At least $(1-4\beta)n$ edges in E_i form close crossings with exactly $4\beta n$ edges from $E_{i+l/2}$, and these $4\beta n$ edges constitute a set of the form $E^{k_1}_{i+l/2} \cup E^{k_2}_{i+l/2}$ for some $1 \le k_1, k_2 \le 2(1/2 - \beta)n$.

(iii) The $(1 - 4\beta)n$ edges in (ii) cover M_i.

Completion of the proof of Theorem 4.1. Let $\varepsilon' = \varepsilon/17$ and $\beta = \min\{\delta/3, \varepsilon'\}$. Let G be a graph drawn from $G(n, n, p)$. By virtue of Lemma 4.2 a.a.s. every allowable labeling of the vertices of G with $[2n]$ yields at most $\varepsilon' n$ directions that are not (β, ε', p)-good in G. It follows from Theorem 2.1 that $e(G) \le (1 + \varepsilon/4)n^2 p$ a.a.s. Fix an even $t \in [2\delta n, 2(1 - \delta)n]$. It suffices to show that, conditioned on the above two events, every Hamiltonian subgraph $G' \subset G$ with more than $(1 + \varepsilon)n^2 p/2$ edges contains a t-cycle.

Fix such a subgraph G' and an allowable labeling of the vertices such that C_{2n} is a Hamilton cycle in G', and set $l = t-2$. Based on our above observation, we only need to show that for some $i \in [2n]$, the graph G' contains two edges $e_1 \in E_i$ and $e_2 \in E_{i+l/2}$ which form a close crossing.

Denote by I the set of directions that are (β, ε', p)-good in G. Hence, $|I| \geq (1-\varepsilon')n$ by our condition. Let X be the number of close crossings between pairs of edges in G which came from E_i and $E_{i+l/2}$ satisfying $i, i+l/2 \in I$. In the following, we assume $l \neq n$ (if n is odd, it clearly holds; if n is even, the proof is similar and we leave it to the reader). Under this assumption, we have $E_{i+l/2} \neq E_{i-l/2}$. So, the number of pairs $\{i, i+l/2\} \subset I$ is at least $(1-2\varepsilon')n$. Fix any such pair of them. By the definition of β and t, we obtain that $l \in (5\beta n, (2-5\beta)n)$. From Lemma 4.3 (ii), (iii) and (6) we know that each of the at least $2(1-\varepsilon')(1/2-2\beta)np$ edges in $M_i \cap E(G)$ forms a close crossing with every edge from some two disjoint sets $E^k_{i+l/2}$ of size $2\beta n$ each. Recall that $i+l/2 \in I$. By (5), the graph G contains at least $2(1-\varepsilon')\beta np$ edges in each such set $E^k_{i+l/2}$. Consequently, we obtain

$$\begin{aligned} X &\geq (1-2\varepsilon')n \cdot 2(1-\varepsilon')\left(\frac{1}{2}-2\beta\right) np \cdot 2 \cdot 2(1-\varepsilon')\beta np \\ &\geq 4(1-4\varepsilon'-4\beta)\beta n^3 p^2. \end{aligned} \tag{7}$$

It follows from Lemma 4.3 (i) that each edge $e_1 \in E_i$ forms close crossings with at most $4\beta n$ edges from $E_{i\pm l/2}$, and these edges are covered by some sets $E^k_{i\pm l/2}$. Therefore, by using (5), every edge in a (β, ε', p)-good direction i forms at most $8(1+\varepsilon')\beta np$ close crossings with edges in a (β, ε', p)-good direction $i \pm i/2$. Let Y be the number of crossings in G that are counted by X but not contained in G'. We have

$$\begin{aligned} Y &\leq (e(G)-e(G')) \cdot 8(1+\varepsilon')\beta np \\ &\leq \left(\left(1+\frac{\varepsilon}{4}\right) n^2 p - (1+\varepsilon)\frac{n^2 p}{2}\right) \cdot 8(1+\varepsilon')\beta np \\ &\leq (4-2\varepsilon+\varepsilon')\beta n^3 p^2. \end{aligned} \tag{8}$$

By our definitions, we have $16\beta + 17\varepsilon' < 2\varepsilon$. Hence, we derive $X > Y$ by (7) and (8). In other words, G' contains two edges from E_i and $E_{i+l/2}$ that form a close crossing. This finally completes the proof of Theorem 4.1. □

5 Open problem

An open problem could be to ask the pancyclicity of random intersection graphs. Since intersection graphs have an underlying bipartite structure, it is hoped that the techniques developed in this work can be applicable in random intersection graph situation (see e. g. [16, 17, 18]).

ACKNOWLEDGEMENTS. I would like to express my indebtedness to the editor and referees for useful comments that improved the presentation of the paper.

References

[1] M. E. K. Abderrezzak and E. Flandrin, *New sufficient conditions for bipancyclic bipartite graphs*, Discrete Math., 286 (2004), 5–13.

[2] D. Amar, *A condition for a Hamiltonian bipartite graph to be bipancyclic*, Discrete Math., 102 (1992), 221–227.

[3] J. Balogh, B. Csaba, and W. Samotij, *Local resilience of almost spanning trees in random graphs*, Rand. Struct. Alg., 38 (2011), 121–139.

[4] S. Ben-Shimon, M. Krivelevich, and B. Sudakov, *Local resilience and Hamiltonicity Maker-Breaker games in random regular graphs*, Comb. Probab. Comput., 20 (2011), 173–211.

[5] J. A. Bondy and M. Simonovits, *Cycles of even length in graphs*, J. Comb. Theory Ser. B, 16 (1974), 97–105.

[6] D. Conlon and W. T. Gowers, *Combinatorial theorems in sparse random sets*, arXiv:1011.4310.

[7] P. E. Haxell, Y. Kohayakawa, and T. Łuczak, *Turán's extremal problem in random graphs: forbidding even cycles*, J. Comb. Theory Ser. B, 64 (1995), 273–287.

[8] S. Janson, T. Łuczak, and A. Ruciński, Random Graphs, Wiley-Interscience, New York, 2000.

[9] M. Krivelevich, C. Lee, and B. Sudakov, *Resilient pancyclicity of random and pseudorandom graphs*, SIAM J. Discrete Math., 24 (2010), 1–16.

[10] C. Lee and W. Samotij, *Pancyclic subgraphs of random graphs*, J. Graph Theory, 71 (2012), 142–158.

[11] C. Lee and B. Sudakov, *Dirac's theorem for random graphs*, Rand. Struct. Alg., 41 (2012), 293–305.

[12] J. Mitchem and E. Schmeichel, *Pancyclic and bipancyclic graphs: a survey*, In: F. Harary and J. S. Maybee (Eds.), Graphs and Applications: Proceedings of the First Colorado Symposium on Graph Theory, Wiley, New York, 1985, 271–278.

[13] J. W. Moon and L. Moser, *On Hamiltonian bipartite graphs*, Israel J. Math., 1 (1963), 163–165.

[14] M. Schacht, *Extremal results for random discrete structures*, Available on: http://www.math.uni-hamburg.de/home/schacht/publikationen.html.en

[15] E. Schmeichel and J. Mitchem, *Bipartite graphs with cycles of all even lengths*, J. Graph Theory, 6 (1982), 429–439.

[16] Y. Shang, *Degree distributions in general random intersection graphs*, Electron. J. Combin., 17 (2010), R23.

[17] Y. Shang, *Joint probability generating function for degrees of active/passive random intersection graphs*, Front. Math. China, 7 (2012), 117–124.

[18] K. B. Singer-Cohen, Random intersection graphs, Ph.D. Thesis, The Johns Hopkins University, Baltimore, MD, 1995.

[19] B. Sudakov and V. Vu, *Local resilience of graphs*, Rand. Struct. Alg., 33 (2008), 409–433.