Arithmetics in the set of beta-polynomials

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Abstract

Let $\beta$ be a formal series with $\text{deg}(\beta) \geq 2$, the aim of this paper is to prove that the maximal length of the finite $\beta$-fractional parts in the $\beta$-expansion of product of two beta-polynomials (a formal series that have not $\beta$-fractional part), denoted $L_\beta(\beta)$ is finite when $\beta$ is Pisot or Salem series. Especially, we give its exact value if $\beta$ have one conjugate with absolute value smaller than $\frac{1}{|\beta|}$ and if $\beta$ is a Pisot series verifying $\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0 = 0$ such that $\text{deg}(\beta) = m \geq 2$ and $\text{deg}(A_0) = s \geq \text{deg}(A_i)$ for $0 \leq i \leq d-2$.

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1 Introduction

The $\beta$-expansion of real numbers was introduced by A. Rényi [12]. Since its introduction in 1975, its properties arithmetic, diophantine and ergodic have been extensively studied by several authors.

Let $\beta > 1$ be a real number. The $\beta$-expansion of a real number $x \in [0,1)$ is defined by the sequence $(x_i)_{i \geq 1}$ with values in $\{0,1,...,\lfloor \beta \rfloor\}$ produced by the $\beta$-transformation $T_\beta : x \rightarrow \beta x \pmod{1}$ as follows:

$$\forall i \geq 1, \ x_i = \lfloor \beta T_{\beta}^{i-1}(x) \rfloor,$$
and thus $x = \sum_{i \geq 1} \frac{x_i}{\beta^i}$.
We write $d_\beta(x) = 0.x_1x_2...$

In [11], Parry showed that for any $x \in [0,1)$, $d_\beta(x)$ is the only transformation of $x$ in base $\beta$ which satisfies the following condition called the Parry condition:

$$\forall n \in \mathbb{N}^*, \ S^n((x_i)_{i \in \mathbb{N}^*}) < \text{lex} \ d_\beta(1)$$

where $S((x_i)_{i \in \mathbb{N}^*}) = (x_{i+1})_{i \in \mathbb{N}^*}$ and $d_\beta^n(1) = \begin{cases} d_\beta(1) & \text{if } d_\beta(1) \text{ is infinite} \\ (t_1...t_{m-1},t_m-1)^\infty & \text{if } d_\beta(1) = 0.t_1...t_m. \end{cases}$

Now let $x > 1$ be a positive real number, so there exist $k \in \mathbb{N}^*$ such that

$$\beta^{k-1} \leq x < \beta^k.$$ So $\frac{x}{\beta^k} \in [0,1)$ and let $d_\beta(\frac{x}{\beta^k}) = (x_i)_{i \geq 1}$ finely we get

$$d_\beta(x) = (x_{i+k+1})_{i \geq -k}.$$ 

Let $d_\beta(x) = (x_i)_{i \geq -n}$, so $x = \sum_{i=0}^{n} x_i \beta^i + \sum_{i>0} x_i \beta^{-i}$. The part with non-negative powers of $\beta$ is called the $\beta$-integer part of $x$, denoted by $[x]_\beta$. The part with negative powers of $\beta$ is called the $\beta$-fractional part of $x$, denoted by $\{x\}_\beta = x - [x]_\beta$, this allows a natural generalization of the definition of development of real number in base 10.

If there exists $n \in \mathbb{N}$ such that $|x| = \sum_{i=0}^{n} x_i \beta^i$, where $x_n \cdots x_0$ is the $\beta$-expansion of $|x|$, then $x$ is called $\beta$-integer and the set of $\beta$-integers is denoted by $\mathbb{Z}_\beta$.

The set $\text{Fin}(\beta)$, introduced in [6], is defined by $\{\beta^{-k}\mathbb{Z}_\beta/k \in \mathbb{N}\}$. It allows to generalize the frame work of numeration to the case of a non-integer base $\beta$.

We know that $\mathbb{Z}_\beta$ and $\text{Fin}(\beta)$ are not stable under usual operations like addition and multiplication.

In order to study arithmetics on $\beta$-integers, we interested on the $\beta$-expansion of the number obtained by addition or multiplication of two $\beta$-integers when the $\beta$-expansion of the sum or the product is finite.

The following notation $L_\oplus(\beta)$ and $L_\odot(\beta)$ are introduced in [8].

**Definition 1.1**

$$L_\oplus(\beta) = \min\{n \in \mathbb{N} \setminus \forall x,y \in \mathbb{Z}_\beta \text{ such that } x+y \in \text{Fin}(\beta) \Rightarrow \beta^n(x+y) \in \mathbb{Z}_\beta\}$$

$$L_\odot(\beta) = \min\{n \in \mathbb{N} \setminus \forall x,y \in \mathbb{Z}_\beta \text{ such that } xy \in \text{Fin}(\beta) \Rightarrow \beta^n(xy) \in \mathbb{Z}_\beta\}$$

Minimum of an empty set is defined to be $+\infty$.

We can see that $L_\oplus(\beta)$ and $L_\odot(\beta)$ represent respectively the maximal possible finite length of the $\beta$-fractional part which may appear when one adds or multiplies two $\beta$-integers, otherwise they designate the maximal finite shift after the comma for the sum or product of two $\beta$-integers.

The computation of these values gives an indication on the difficulty of performing arithmetics on $\mathbb{Z}_\beta$.

Let us explain now why we are interested in the case where $L_\oplus(\beta)$ and $L_\odot(\beta)$ are finite: Indeed, if the sum or the product of two $\beta$-integers belongs to $\text{Fin}(\beta)$, then the length of the $\beta$-fractional part of this sum or product is bounded by a constant which only
depends on \(\beta\).
If the set of the length sums or products of two \(\beta\)-integers is unbounded, then performing arithmetics in \(\mathbb{Z}_\beta\) will be very difficult if not impossible, since one can not compute in a finite time any operation on \(\beta\)-integers.
Let us mention that to our knowledge no example is known of a \(\beta\) such that \(L_\oplus(\beta)\) and \(L_\odot (\beta)\) are infinite, however it has been proven in [6] and [7] that \(L_\oplus (\beta)\) and \(L_\odot (\beta)\) are finite when \(\beta\) is Pisot (a real algebraic integer greater than 1 with all conjugates strictly inside the unit disk). The computation of these values is however not so easy, especially for \(L_\oplus (\beta)\). The case of quadratic Pisot numbers has been studied in [5] when \(\beta\) is a unit. The authors gave exact values for \(L_\oplus (\beta)\) and \(L_\odot (\beta)\), when \(\beta > 1\) is a solution either of equation \(x^2 = mx - 1, m \in \mathbb{N}, m \geq 3\) or of equation \(x = mx + 1, m \in \mathbb{N}\). In the first case \(L_\oplus (\beta) = L_\odot (\beta) = 1\), in the second case \(L_\oplus (\beta) = L_\odot (\beta) = 2\), and in [8] otherwise. However, when \(\beta\) is of higher degree, it is a difficult problem to compute the exact value of \(L_\oplus (\beta)\) or \(L_\odot (\beta)\), and even to compute upper and lower bounds for these two constants.
Several examples are Studied in [2], where a method is described in order to compute upper bounds for \(L_\oplus (\beta)\) and \(L_\odot (\beta)\) for Pisot numbers satisfying additional algebraic properties, for example, in the Tribonacci case, that is, when \(\beta\) is the positive root, of the polynomial \(x^3 - x^2 - x - 1\), we have \(L_\oplus (\beta) = 5\), let’s note that until now, we don’t know the value of \(L_\odot (\beta)\) in the case of the Tribonacci number, it is only proven in [2] that \(4 \leq L_\odot (\beta) \leq 5\).
The condition Pisot is not necessary to have \(L_\oplus (\beta)\) and \(L_\odot (\beta)\) are finite: L. S. Guimond, Z. Masáková and E. Pelantová in [8] prove that if \(\beta\) is an algebraic number such that at least one among its conjugates with modulus smaller than 1, so \(L_\oplus (\beta)\) and \(L_\odot (\beta)\) are finite.

In this paper, we define a similar concepts over the field of formal series. We begin in Section 1 by introducing the field of formal series and the \(\beta\)-expansion over this field. In Section 2, we will show that the condition Pisot is not necessary to have \(L_\oplus (\beta)\) are finite. Especially, we give a sufficient condition for the conjugates of \(\beta\) to obtain \(L_\odot (\beta) = 1\).
In Section 3, we prove the finiteness of \(L_\odot (\beta)\) when \(\beta\) is Pisot or Salem formal power series and we give its exact value in the case of Pisot with some additional conditions.

## 2 Preliminaries

Let \(\mathbb{F}_q\) be a finite field of \(q\) elements, \(\mathbb{F}_q[x]\) is the ring of polynomials with coefficient in \(\mathbb{F}_q\), \(\mathbb{F}_q(x)\) is the field of rational functions and \(\mathbb{F}_q(x, \beta)\) is the field of rational functions in base \(\beta\). Let \(\mathbb{F}_q((x^{-1}))\) be the field of formal power series of the form:

\[
f = \sum_{k=-\infty}^{l} f_k x^k, \quad f_k \in \mathbb{F}_q
\]
Theorem 2.1 [3] Let \( \beta \in \mathbb{F}_q((x^{-1})) \) be an algebraic integer over \( \mathbb{F}_q[x] \) and
\[
P(y) = y^n - A_{n-1}y^{n-1} - \cdots - A_0, \quad A_i \in \mathbb{F}_q[x],
\]
be its minimal polynomial. Then
(i) \( \beta \) is a Pisot element if and only if \( |A_{n-1}| > \max_{0 \leq j \leq n-2} |A_i| \).
(ii) \( \beta \) is a Salem element if and only if \( |A_{n-1}| = \max_{0 \leq j \leq n-2} |A_i| \).

Let \( \beta \in \mathbb{F}_q((x^{-1})) \) with \( |\beta| > 1 \). A representation in base \( \beta \) (or \( \beta \)-representation) of a formal series \( f \in D(0, 1) \) is an infinite sequence \( (x_i)_{i \geq 1}, \quad x_i \in \mathbb{F}_q[x], \) such that
\[
f = \sum_{i \geq 1} x_i \beta^i.
\]
A particular \( \beta \)-representation of \( f \) is called the \( \beta \)-expansion of \( f \) in base \( \beta \), noted \( d_\beta(f) \), which is obtained by using the \( \beta \)-transformation \( T_\beta \) in the unit disk which is given by \( T_\beta(f) = \beta f - |\beta| f \). Then \( d_\beta(f) = (a_i)_{i \geq 1} \) where \( a_i = [\beta T_\beta^{i-1}(f)] \), for better characterization of \( \beta \)-expansion, in [9], M.Hbaib and M.Mkaouar showed the following theorem.

Theorem 2.2 [9] An infinite sequence \( (a_i)_{i \geq 1} \) is the \( \beta \)-expansion of \( f \in D(0, 1) \) if and only if \( |a_i| < |\beta| \) for all \( i \geq 1 \).
Now let \( f \in \mathbb{F}_q((x^{-1})) \) be an element with \( |f| \geq 1 \). Then there is a unique \( k \in \mathbb{N}^* \) such that \( |\beta|^{k-1} \leq |f| < |\beta|^k \), so \( \frac{f}{\beta^k} \) is an integer to the left. Therefore, if \( d_\beta(f) = 0.a_1a_2a_3... \) then \( d_\beta(\beta f) = a_1a_2a_3... \).

Let \( Fin(\beta) \) be the set of \( f \in \mathbb{F}_q((x^{-1})) \) which have a finite \( \beta \)-expansion, so the \( \beta \)-expansion of every \( f \in Fin(\beta) \) has this form:

\[
d_\beta(f) = a_k a_{k-1} \cdots a_1 a_0, a_{-1} a_{-2} \cdots a_m, \text{ where } m \in \mathbb{Z}.
\]

The part \( a_k a_{k-1} \cdots a_1 a_0 \) is called the \( \beta \)-polynomial part of \( f \) and the part \( a_{-1} a_{-2} \cdots a_m \) is called the \( \beta \)-fractional part of \( f \).

We define also \( deg_\beta(f) = k \) and \( ord_\beta(f) = m \).

If \( ord_\beta(f) \geq 0 \) then \( f \) is called \( \beta \)-polynomial and the set of \( \beta \)-polynomials is denoted by \((\mathbb{F}_q[x])_\beta\), who is the analogue of \( \mathbb{Z}_\beta \) in the real case.

We can define by an analogy with the real case the quantity \( L_\oplus(\beta) \) and \( L_\ominus(\beta) \) as follows:

\[
L_\oplus(\beta) = \min\{n \in \mathbb{N} \setminus \forall, g \in (\mathbb{F}_q[x])_\beta \text{ such that } f+g \in Fin(\beta) \Rightarrow \beta^n(f+g) \in (\mathbb{F}_q[x])_\beta\}
\]

\[
L_\ominus(\beta) = \min\{n \in \mathbb{N} \setminus \forall, g \in (\mathbb{F}_q[x])_\beta \text{ such that } f+g \in Fin(\beta) \Rightarrow \beta^n(f+g) \in (\mathbb{F}_q[x])_\beta\}.
\]

Minimum of an empty set is defined to be \(+\infty\).

**Remark 2.3** Let us note that in the case of formal series the quantity \( L_\oplus(\beta) \) is not interesting, because we know that in contrast to the real case, if \( f, g \in \mathbb{F}_q((x^{-1})) \), we have \( d_\beta(f+g) = d_\beta(f)+d_\beta(g) \), so the sum of two \( \beta \)-polynomials is always a \( \beta \)-polynomial.

To calculate \( L_\ominus(\beta) \) for the families of basis \( \beta \), we excluded the case when \( deg(\beta) = 1 \), since in this trivial case, the product of two \( \beta \)-polynomials is a \( \beta \)-polynomial, so we have \( L_\ominus(\beta) = 0 \).

Let \( \beta \in \mathbb{F}_q((x^{-1})) \) with \( |\beta| > 1 \) be an algebraic on \( \mathbb{F}_q[x] \) and let \( \beta_2, \ldots, \beta_n \) be the conjugates of \( \beta \). We associate for every \( f = \sum_{i=0}^{n} a_i \beta^i \), the j-th conjugate which is defined by \( f_j = \sum_{i=0}^{n} a_i \beta_j^i \).

## 3 Main results

### 3.1 Sufficient conditions for finiteness of \( L_\ominus(\beta) \)

In the following theorem, we will prove that \( L_\ominus(\beta) \) is finite when \( \beta \) is an algebraic integer. This result does not have an analogue to the real case.

**Theorem 3.1** Let \( \beta \in \mathbb{F}_q((x^{-1})) \), with \( |\beta| > 1 \) be an algebraic integer on \( \mathbb{F}_q[x] \). Then \( L_\ominus(\beta) \) is finite.
Therefore, we can write

\[ f = a_s \beta^s + a_{s-1} \beta^{s-1} + \ldots + a_0 \quad \text{and} \quad g = b_m \beta^m + b_{m-1} \beta^{m-1} + \ldots + b_0, \]

where \( |a_i| < |\beta| \) for all \( 0 \leq i \leq s \) and \( |b_j| < |\beta| \) for all \( 0 \leq j \leq m \), such that \( fg \in \text{Fin}(\beta) \).

Therefore, we can write

\[ fg = c_n \beta^n + c_{n-1} \beta^{n-1} + \ldots + c_0 + c_{-1} \beta^{-1} + \ldots + c_{-k} \beta^{-k}, \]

with \( |c_j| < |\beta| \) for all \( -k \leq j \leq n \). We denote by \( h_j = c_{-1} \beta_j^{-1} + c_{-2} \beta_j^{-2} + \ldots + c_{-k} \beta_j^{-k} \) for all \( 1 \leq j \leq d \), where \( \beta_1 = \beta \) and \( \beta_j, 2 \leq j \leq d \) are the Galois conjugates of \( \beta \) and \( d \) is the algebraic degree of \( \beta \).

Then

\[ h_j = \sum_{i=0}^{d-1} \alpha_i \beta_j^{-i} \text{ where } \alpha_i \in \mathbb{F}_q[x] \text{ for all } 0 \leq i \leq d - 1. \]

We have |\( h \) | = |\( h_1 \) | < 1, if |\( \beta_j \) | ≥ 1 then |\( h_j \) | = |\( c_{-1} \beta_j^{-1} + c_{-2} \beta_j^{-2} + \ldots + c_{-k} \beta_j^{-k} \) | ≤ |\( \beta \) |, and if |\( \beta_j \) | < 1 then |\( h_j \) | = |\( (a_s \beta_j^s + a_{s-1} \beta_j^{s-1} + \ldots + a_0)(b_m \beta_j^m + b_{m-1} \beta_j^{m-1} + \ldots + b_0) - (c_n \beta_j^n + c_{n-1} \beta_j^{n-1} + \ldots + c_0) \) | ≤ |\( \beta \) |^2.

Since the matrix \( M = (\beta_j^{-k})_{1 \leq j \leq d, 0 \leq k \leq d - 1} \) is non singular, we give that

\[
\begin{pmatrix}
0
\vdots
\alpha_{d-1}
\end{pmatrix} = M^{-1}
\begin{pmatrix}
h_0
\vdots
h_d
\end{pmatrix}.
\]

This implies that |\( \alpha_i \) | < \( C(\beta) \), where \( C(\beta) \) is a constant depend only on \( \beta \), therefore the number of elements \( (\alpha_i)_{0 \leq i \leq d-1} \) is finite. So \( L_\beta(\beta) \) is finite. □

## 3.2 Computation of \( L_\beta(\beta) \)

We propose the following quantitative study over this family of algebraic formal series \( \beta \), that have at least one of its conjugates, say \( \beta_j \), in absolute value smaller than \( \frac{1}{|\beta|} \).

**Theorem 3.2** Let \( \beta \in \mathbb{F}_q((x^{-1})) \) be an algebraic on \( \mathbb{F}_q[x] \) with \( \deg(\beta) \geq 2 \), which have a conjugate, say \( \beta_j \) verifying \( |\beta_j| \leq \frac{1}{|\beta|} \). Then \( L_\beta(\beta) \in \{0, 1\} \).

**Proof.**

Let \( f = \sum_{i=0}^{s} a_i \beta^i \) and \( g = \sum_{i=0}^{r} b_i \beta^i \) where \( |a_i| < |\beta| \) and \( |b_i| < |\beta| \), such that \( fg \in \text{Fin}(\beta) \).

Let \( f_j = \sum_{i=0}^{s} a_i \beta_j^i \) and \( g_j = \sum_{i=0}^{r} b_i \beta_j^i \). Since \( |\beta_j| \leq \frac{1}{|\beta|} \), we have \( |f_j| < |\beta| \) and \( |g_j| < |\beta| \).

So

\[ |f_j g_j| < |\beta|^2 \]

We assume that \( d_\beta(fg) = c_h \ldots c_0 c_{-1} \ldots c_{-m} \) where \( c_{-m} \neq 0 \), so we have

\[ |f_j g_j| = |\sum_{i=0}^{h} c_i \beta_j^i| = |c_{-m} \beta_j^{-m}| \geq |\beta|^m. \]

Therefore \( m < 2 \). □

As application of the above theorem, we have treat some specific cases.
Corollary 3.3 Let \( \beta \) be a quadratic Pisot unit with \( \deg(\beta) \geq 2 \). Then \( L_\odot(\beta) = 1 \).

Proof.
In this case \( \beta \) verify \( \beta^2 + A_1 \beta + A_0 = 0 \) where \( A_0 \in \mathbb{F}_q^* \), so the unique conjugate of \( \beta \) is \( \beta_j \) such that \( |\beta_j| = \frac{1}{|\beta|} \).
By Theorem 3.2, we obtain \( L_\odot(\beta) \in \{0,1\} \). Let now \( A_1 = c_d x^d + \ldots + c_0 \) where \( d = \deg(\beta) \geq 2 \) and \( c_d \neq 0 \), we have \( x, c_d x^{d-1} \in (\mathbb{F}_q[x])_\beta \) and
\[
\beta_d x^d = -\beta - (c_{d-1} x^{d-1} + \ldots + c_0) - \frac{A_0}{\beta}.
\]
So, \( L_\odot(\beta) = 1 \). \( \square \)

Corollary 3.4 Let \( \beta \) be a cubic Salem unit with \( \deg(\beta) \geq 2 \). Then \( L_\odot(\beta) \in \{0,1\} \).

Proof.
In this case \( \beta \) verify \( \beta^3 + A_2 \beta^2 + A_1 \beta + A_0 = 0 \), where \( |A_1| = |A_2| = |\beta| \) and \( A_0 \in \mathbb{F}_q^* \).
So \( \beta \) have two conjugates \( \beta_2 \) and \( \beta_3 \) such that \( |\beta_2| = \frac{1}{|\beta|} \) et \( |\beta_3| = 1 \).
By Theorem 3.2, we obtain \( L_\odot(\beta) \in \{0,1\} \). \( \square \)

Corollary 3.5 Let \( \beta \) be a Salem unit with \( \deg(\beta) \geq 2 \) verifying:
\[
\beta^d + A_{d-1} \beta^{d-1} + \ldots + A_1 \beta + A_0 = 0, \quad (3.1)
\]
where \( A_0 \in \mathbb{F}_q^* \) and \( |A_1| = |A_{d-1}| \). Then \( L_\odot(\beta) \in \{0,1\} \).

Proof.
Let \( \beta = \beta_1 \) be a Salem unit verifying (3.1). Then
\[
|A_0| = \prod_{1 \leq i \leq d} |\beta_i| = 1 \text{ where } \beta_2, \ldots, \beta_d \text{ its conjugates}
\]
\[
|A_{d-1}| = \left| \sum_{i=1}^{d} \beta_i \right| = |A_1| = \left| \sum_{1 \leq i_1 < \ldots < i_{d-1} \leq d} \beta_{i_1} \ldots \beta_{i_{d-1}} \right|
\]
If there exist \( \beta_i \) and \( \beta_j \) (\( i \neq j \)) such that \( |\beta_i| < 1 \) and \( |\beta_j| < 1 \), then we obtain in this case \( |A_1| < |\beta| \) which contradicts the hypothesis that
\[
|\beta| = |A_{d-1}| = |A_1|.
\]
we conclude that \( \beta \) have a unique conjugate \( \beta_j \) such that \( |\beta_j| < 1 \) and the other conjugates of equal absolute value 1. So \( |\beta_j| = \frac{1}{|\beta|} \) and by Theorem 3.2, we obtain \( L_\odot(\beta) \in \{0,1\} \). \( \square \)
The following theorem, allows us to calculate \( L_\odot(\beta) \) for some Pisot series.

Theorem 3.6 let \( \beta \) be a Pisot series satisfying
\[
\beta^d + A_{d-1} \beta^{d-1} + \ldots + A_0 = 0
\]
where $\lambda$ and $\mu$ are two increasing sequences defined by:

$$\mu(n) = -\text{ord}_\lambda(x^n) = (d-1)\left\lfloor \frac{n-s}{m-s} \right\rfloor$$

and $\text{deg}(a^{-\mu(n)}_n) = s + (n - s) = \sup_{0 \leq k \leq d-2} \text{deg}(a^{-\mu(n)+k}_n)$

where $(n - s)$ is the rest of the Euclidean division of $(n - s)$ by $(m - s)$. To prove the above theorem, we will need the following lemma:

**Lemma 3.7** Let $\beta$ be a Pisot series. Then $\text{ord}_\beta(x^{n+1}) \leq \text{ord}_\beta(x^n)$ for all $n \in \mathbb{N}^*$.

**Proof.**

Let $P(y) = y^d + A_{d-1}y^{d-1} + \cdots + A_0$ be the minimal polynomial of $\beta$ and $c$ be the dominant coefficient of $A_{d-1}$. Let $m = \text{deg}(\beta)$, for $n < m - 1$ we have $\text{ord}_\beta(x^{n+1}) = \text{ord}_\beta(x^n) = 0$.

Let now $n \geq m$ we suppose that

$$d_\beta(x^n) = a_n^{\lambda(n)} \cdots a_0^{\mu(n)}$$

with $a^{-\mu(n)}_n \neq 0$.

We will show by induction on $n$ that $\mu(n+1) \geq \mu(n)$ and $\text{deg}(a^{-\mu(n+1)}_{n+1}) \geq s$, where $s = \text{deg}A_0$.

So we have

$$x^m = -c^{-1}\beta - c^{-1}(A_{d-1} - cx^m) - \cdots - c^{-1}A_0\beta^{-(d-1)}.$$  

this implies

$$d_\beta(x^m) = a_n^{\lambda(m)}a_0^{\mu(m)}\cdots a^{-\mu(m)}_m$$

where

$$\begin{cases} 
  a_n^{\lambda(m)} = -c^{-1} \\
  \vdots \\
  a^{-\mu(m)}_m = -c^{-1}A_0 
\end{cases}$$

and $\lambda(m) = 1$, $\mu(m) = d - 1$

Let now

$$\mathcal{A}_n = \{0 \leq i \leq d - 1 \text{ such that } \text{deg}(a^{-\mu(n)+i}_n) = m - 1\}$$

If $\mathcal{A}_n = \emptyset$, then $a^{n+1}_{-\mu(n+1)} = xa^{-\mu(n)}_n$ and $\mu(n+1) = \mu(n)$ and $\text{deg}(a^{-\mu(n+1)}_{n+1}) = 1 + \text{deg}(a^{-\mu(n)}_n)$.

If $\mathcal{A}_n \neq \emptyset$, taking $k_n = \text{min}\mathcal{A}_n$ and $\gamma$ be the dominant coefficient of $a^{-\mu(n)+k_n}_n$.

- If $k_n = d - 1$, then $a^{n+1}_{-\mu(n+1)} = -c^{-1}\gamma A_0 + xa^{-\mu(n)}_n$ and $\mu(n+1) = \mu(n)$.
- If $k_n < d - 1$, then $a^{n+1}_{-\mu(n+1)} = -c^{-1}\gamma A_0$ and $\mu(n+1) = \mu(n) + d - 1 - k_n$.

**Lemma 3.8** Let $\beta$ be a Pisot series, such that $m = \text{deg}(\beta) \geq 2$.

Then $L_\beta(x^m) = -\text{ord}_\beta(x^{2m-2})$. 
Proof.
Let \( f = a_n \beta^n + a_{n-1} \beta^{n-1} + \cdots + a_0 \) and \( g = b_m \beta^m + b_{m-1} \beta^{m-1} + \cdots + b_0 \) in \( \mathbb{F}_q[x]_\beta \) such that \( f.g \in \text{Fin}(\beta) \), since
\[
 f.g = \sum_{k=0}^{n+m} \left( \sum_{p=0}^{k} b_p a_{k-p} \right) \beta^k.
\]
We have \( -\text{ord}_\beta(f.g) \leq \max\{-\text{ord}_\beta(b_p a_{k-p}); 0 \leq p \leq k \leq n + m\} \), using the above lemma we get \( L_\oplus(\beta) \leq -\text{ord}_\beta(x^{2m-2}) \). Or \( f = x^{m-1} \in \mathbb{F}_q[x]_\beta \) and \( -\text{ord}_\beta(f^2) = -\text{ord}_\beta(x^{2m-2}) \), so \( L_\oplus(\beta) = -\text{ord}_\beta(x^{2m-2}) \). □

Proof of Theorem 3.6.
We will show the result by induction on \( n \geq m \).
For \( n = m \), we have
\[
x^n = -c^{-1} \beta - (A_{d-1} - c x^n) - \cdots - c^{-1} A_0 \beta^{-(d-1)}
\]
where \( c \) is the dominant coefficient of \( A_{d-1} \), so
\[
 -\text{ord}_\beta(x^m) = d - 1 = \mu(m) \quad \text{and} \quad \text{deg}(a_{\mu(m)}) = \text{deg}(-c^{-1} A_0) = s + (m-s).
\]
So the result is true for \( n = m \).
Assume that \( \mu(n) = -\text{ord}_\beta(x^n) = (d - 1)[\frac{n-s}{m-s}] \) and \( \text{deg}(a_{\mu(n)}) = s + (n-s) \). We have
\[
x^n = a_{\lambda(n)}^n \beta^{\lambda(n)} + \cdots + a_0^n + a_{-1} \beta^{-1} + \cdots + a_{-\mu(n)} \beta^{\mu(n)}.
\]
Therefore
\[
x^{n+1} = x a_{\lambda(n)}^n \beta^{\lambda(n)} + \cdots + x a_0^n + x a_{-1} \beta^{-1} + \cdots + x a_{-\mu(n)} \beta^{\mu(n)}.
\]
We distinguish two cases:
Case 1: \( \text{deg}(a_{\mu(n)}) = m - 1 \), in this case we have \( (n-s) = m-s-1 \), so \( (n-s) = [\frac{n-s}{m-s}] (m-s) + m-s-1 \), this implies \( (n+1-s) = ([\frac{n-s}{m-s}] + 1)(m-s) \), hence \( [\frac{n+1-s}{m-s}] = [\frac{n-s}{m-s}] + 1 \) and \( (n+1-s) = 0 \). Also we have
\[
\begin{align*}
\mu(n+1) &= -\text{ord}_\beta(x^{n+1}) = -\text{ord}_\beta(x^n) + d - 1 \\
&= (d - 1)[\frac{n-s}{m-s}] + (d - 1) \\
&= (d - 1)((\frac{n-s}{m-s}) + 1) \\
&= (d - 1)[\frac{n+1-s}{m-s}]
\end{align*}
\]
and \( a_{-\mu(n+1)} = -c^{-1}.\gamma A_0 \) where \( \gamma \) is the dominant coefficient of \( a_{-\mu(n)} \), so \( \text{deg}(a_{-\mu(n+1)}) = s + (n+1-s) \).
Case 2: \( \text{deg}(a_{\mu(n)}) < m - 1 \), in this case we have
\[
\begin{align*}
\mu(n+1) &= -\text{ord}_\beta(x^{n+1}) = -\text{ord}_\beta(x^n) = \mu(n)
\end{align*}
\]
and

\[
a_{n+1} - \mu(n+1) = \begin{cases} 
  xa_n - \mu(n) & \text{if } \deg(a_n^{\mu(n)} + d - 1) < m - 1, \\
  -e^{-1} \delta A_0 + xa_n - \mu(n) & \text{if } \deg(a_n^{\mu(n)} + d - 1) = m - 1.
\end{cases}
\]

where \( \delta \) is the dominant coefficient of \( a_{n\mu(n)} + d - 1 \).

Therefore \( \deg(a_{n\mu(n)+1}) = 1 + \deg(a_{n\mu(n)}) = 1 + s + (n - s) = s + (n + 1 - s) \). □

**Corollary 3.9** let \( \beta \) be a Pisot series satisfying

\[
\beta^d + A_{d-1}\beta^{d-1} + \cdots + A_0 = 0
\]

such that \( \deg(\beta) = m \geq 2 \) and \( \deg(A_i) = s \geq \deg(A_i) \quad \forall 0 \leq i \leq d - 2 \). Then \( L_\otimes(\beta) = (d - 1)(\frac{m - 2}{m - s} + 1) \).

**Corollary 3.10** Let \( \beta \) be a Pisot series verifying \( \beta^2 + A_1\beta + A_0 = 0 \) such that \( \deg(\beta) = m \) and \( \deg(A_0) = s \). Then

\[
L_\otimes(\beta) = \left[ \frac{m - 2}{m - s} \right] + 1
\]

4 Open Problem

Study other finiteness conditions and explicitly calculate \( L_\otimes(\beta) \) for other families of algebraic formal power series.

Is it possible to find conditions on \( \beta \) in order that \( L_\otimes(\beta) = 0 \), meaning the set \( \langle I \ F_q[x] \rangle_\beta \) is stable under multiplication?

References


