On a New Weighted Erdős-Mordell Type Inequality

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Abstract
In this short note, a new weighted Erdős-Mordell inequality Involving Interior Point of a triangle is established. By it’s application, some interesting geometric inequalities are derived.

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1 Introduction

Throughout the paper we assume $\triangle ABC$ be a Triangle, and denote by $a$, $b$, $c$ its sides’ lengths, $\Delta$ be the area. Let $P$ be an interior point, Extend $AP, BP, CP$ respectively to meet the opposite sides at $D, E$ and $F$. Let $PD = r'_1, PE = r'_2, PF = r'_3$, $\Delta_1, \Delta_2, \Delta_3$ denote the areas of $\triangle BPC, \triangle CPA, \triangle APB$. $R_a, R_b, R_c$ the circumradii of the triangles $BPC, CPA, APB$, respectively. Let $R_1, R_2, R_3$ be the distances from $P$ to $A, B, C$, and also let $r_1, r_2, r_3$ be the distances from $P$ to the sides $AB, BC, CA$.

Then Erdős-Mordell inequality is true:

**Theorem 1.1.**

$$R_1 + R_2 + R_3 \geq 2 (r_1 + r_2 + r_3)$$

whereat equality holds if and only if the triangle is equilateral and the point $P$ is its center. This inequality was conjectured by Erdős in 1935[1], and was first proved by Mordell in 1937[2].

In the paper[3], D. S. Mitrnović etc noted some generalizations of Erdős-Mordell inequality in 1989. Among their results are the following theorem for three-variable quadratic Erdős-Mordell type inequality:
Theorem 1.2. If \( x, y, z \) are three real numbers, then for any point \( P \) inside the triangle \( ABC \), we have
\[
x^2R_1 + y^2R_2 + z^2R_3 \geq 2(yzx_1 + zxy_2 + xzy_3)
\]
with equality holding if and only if \( x = y = z \) and \( P \) is the center of equilateral \( \triangle ABC \).

Recently, Jiang [6] presented a new weighted Erdős-Mordell type inequality. In this note, we give another new weighted Erdős-Mordell type inequality, as application, some interesting geometric inequalities are also established.

2 Main results

In order to prove Theorem 2.2 below, we need the following lemma.

Lemma 2.1. For any point \( P \) inside \( \triangle ABC \), \( x, y, z \in \mathbb{R} \), then we have
\[
x^2 \sin^2 A + y^2 \sin^2 B + z^2 \sin^2 C \leq \frac{1}{4} \left( \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \right)^2.
\]

Proof. We make use of Kooi’s inequality [4]:
For real numbers \( \lambda_1, \lambda_2, \lambda_3 \) with \( \lambda_1 + \lambda_2 + \lambda_3 \neq 0 \),
\[
(\lambda_1 + \lambda_2 + \lambda_3)^2R^2 \geq \lambda_2\lambda_3a^2 + \lambda_3\lambda_1b^2 + \lambda_1\lambda_2c^2; \tag{4}
\]

Where \( R \) be circumradius of triangle \( ABC \), equality holds if and only if the point with homogeneous barycentric coordinates \( (\lambda_1 : \lambda_2 : \lambda_3) \) with reference to triangle \( ABC \) is the circumcenter of the triangle.

Now, Lemma 2.1 follows from (4) with \( \lambda_1 = \frac{yz}{x}, \lambda_2 = \frac{zx}{y}, \lambda_3 = \frac{xy}{z} \), and the law of sines: \( a = 2R \sin A, b = 2R \sin B, c = 2R \sin C \).

Now we are in a position to state and prove our main result.

Theorem 2.2. For any point \( P \) inside triangle \( ABC \), Extend \( AP, BP, CP \) respectively to meet the opposite sides at \( D, E \) and \( F \). Let \( R_a, R_b, R_c \) the circumradiiuses of triangles \( \triangle BPC, \triangle CPA, \triangle APB \), and let \( PD = r'_1, PE = r'_2, PF = r'_3 \). \( x, y, z \) are positive real numbers, we have
\[
\frac{xr'_1}{\sqrt{R_aR_c}} + \frac{yr'_2}{\sqrt{R_cR_a}} + \frac{zr'_3}{\sqrt{R_aR_b}} \leq \frac{1}{2} \left( \frac{yz}{x} + \frac{zx}{y} + \frac{xy}{z} \right).
\]

with equality holding if and only if \( x = y = z \) and \( P \) is the center of equilateral \( \triangle ABC \).
Proof. Let $\angle BPC = \alpha, \angle CPA = \beta, \angle APB = \gamma$. It is obvious that $0 < \alpha, \beta, \gamma < \pi$ and $\alpha + \beta + \gamma = 2\pi$.

By using spread angle theorem, we have:

$$\frac{\sin \alpha}{r'_1} = \frac{\sin(\pi - \beta)}{R_2} + \frac{\sin(\pi - \gamma)}{R_3}$$

$$= \frac{\sin \beta + \sin \gamma}{R_2} + \frac{\sin \gamma}{R_3}$$

$$\geq 2\sqrt{\frac{\sin \beta \sin \gamma}{R_2R_3}},$$

Thus,

$$2r'_1 \leq \sqrt{R_2R_3 \csc \beta \csc \gamma \sin \alpha}.$$

Make use of $b = 2R_b \sin \beta, c = 2R_c \sin \gamma$, we get

$$\frac{r'_1}{\sqrt{R_bR_c}} \leq \sqrt{\frac{R_2R_3}{bc} \sin \alpha},$$

$$= \sqrt{\frac{\Delta_1}{\Delta}} \sin A \sin \alpha$$

Let

$$A' = \pi - \alpha, B' = \pi - \beta, C' = \pi - \gamma$$

Because

$$\sqrt{\sin A \sin \alpha} \leq \frac{1}{2} (\sin A + \sin \alpha) = \sin \frac{A + A'}{2} \cos \frac{A - A'}{2} \leq \sin \frac{A + A'}{2}$$

we have,

$$\frac{xy'_1}{\sqrt{R_bR_c}} \leq \sqrt{\frac{\Delta_1}{\Delta}} x \sin \frac{A + A'}{2},$$  \hspace{1cm} (6)

By the same way, one can get

$$\frac{y'_{r_2}}{\sqrt{R_cR_a}} \leq \sqrt{\frac{\Delta_2}{\Delta}} y \sin \frac{B + B'}{2},$$  \hspace{1cm} (7)

$$\frac{zr'_3}{\sqrt{R_aR_b}} \leq \sqrt{\frac{\Delta_3}{\Delta}} z \sin \frac{C + C'}{2},$$  \hspace{1cm} (8)
Combining expression (6), (7), (8) and By Cauchy’s inequality, we have
\[
\sum \frac{x_{r_1}}{\sqrt{R_b R_c}} \leq \sum \sqrt{\frac{A}{\Delta}} \sqrt{x \sin \frac{A + A'}{2}}
\]
\[
\leq \sqrt{\sum \frac{A}{\Delta} \sum x^2 \sin^2 \frac{A + A'}{2}},
\]
\[
= \sqrt{\sum x^2 \sin^2 \frac{A + A'}{2}}.
\]
Let
\[
\theta = \frac{A + A'}{2}, \phi = \frac{B + B'}{2}, \varphi = \frac{C + C'}{2}.
\]
Obviously, 0 < 0, 0, 0 < A, A, A and 0 + 0 = 0, so 0, 0, 0 can be angles of a triangle 0, 0, 0. Applying Lemma 2.1 for the triangle 0, 0, 0 we obtain
\[
x^2 \sin^2 \theta + y^2 \sin^2 \phi + z^2 \sin^2 \varphi \leq \frac{1}{4} \left( \frac{y z}{x} + \frac{z x}{y} + \frac{z y}{z} \right)^2.
\]
This conclude that
\[
\frac{x_{r_1}}{\sqrt{R_b R_c}} + \frac{y_{r_2}}{\sqrt{R_c R_a}} + \frac{z_{r_3}}{\sqrt{R_a R_b}} \leq \frac{1}{2} \left( \frac{y z}{x} + \frac{z x}{y} + \frac{z y}{z} \right).
\]
and with equality holding if and only if \( x = y = z \), and P is the center of equilateral \( \triangle ABC \). The proof of Theorem 2.2 is completed.

3 Some application

In this section we give some applications of Theorem 2.2.

Noticed \( r_1 \leq r_1' \) etc, we have
\[
\frac{x_{r_1}}{\sqrt{R_b R_c}} + \frac{y_{r_2}}{\sqrt{R_c R_a}} + \frac{z_{r_3}}{\sqrt{R_a R_b}} \leq \frac{1}{2} \left( \frac{y z}{x} + \frac{z x}{y} + \frac{z y}{z} \right). \tag{9}
\]
By using AM-GM inequality, we have \( \sqrt{R_b R_c} \leq \frac{1}{2} (R_b + R_c) \), then from (5) we have
\[
\frac{x_{r_1}}{R_b + R_c} + \frac{y_{r_2}}{R_c + R_a} + \frac{z_{r_3}}{R_a + R_b} \leq \frac{1}{4} \left( \frac{y z}{x} + \frac{z x}{y} + \frac{z y}{z} \right). \tag{10}
\]
By the same way of (9), the following inequality holds.
\[
\frac{x_{r_1}}{R_b + R_c} + \frac{y_{r_2}}{R_c + R_a} + \frac{z_{r_3}}{R_a + R_b} \leq \frac{1}{4} \left( \frac{y z}{x} + \frac{z x}{y} + \frac{z y}{z} \right). \tag{11}
\]
let $x = y = z = 1$ in (11), we have

$$\frac{r_1}{R_b + R_c} + \frac{r_2}{R_c + R_a} + \frac{r_3}{R_a + R_b} \leq \frac{3}{4}.$$  \tag{12}

In fact, (12) was conjectured by Liu in [5] and here we obtained a proof.

**Corollary 3.1.** If $x,y,z > 0$, then

$$x^2 R_a + y^2 R_b + z^2 R_c \geq 2 (y z r_1' + z x r_2' + x y r_3').$$  \tag{13}

**Proof.** alter $x \to x' \sqrt{R_b R_c}, y \to y' \sqrt{R_c R_a}, z \to z' \sqrt{R_a R_b}(x, y, z > 0)$ in (5), we obtain

$$x' r_1' + y' r_2' + z r_3' \leq \frac{1}{2} \left( \frac{y' x'}{x'} R_a + \frac{z' x'}{y'} R_b + \frac{x' y'}{z'} R_c \right).$$  \tag{14}

and then, let $\frac{x'}{x^2} = x^2, \frac{y'}{y^2} = y^2, \frac{z'}{z^2} = z^2$ in (14), then (13) is obtained. \quad \Box

(13) is similar to (2), that was conjectured by Liu in [5].

Obviously,

$$x^2 R_a + y^2 R_b + z^2 R_c \geq 2 (y z r_1 + z x r_2 + x y r_3).$$  \tag{15}

Let $x = y = z = 1$ in (13) and (15), then we have.

$$R_a + R_b + R_c \geq 2 (r_1' + r_2' + r_3').$$  \tag{16}

and

$$R_a + R_b + R_c \geq 2 (r_1 + r_2 + r_3).$$  \tag{17}

Note that (17) is similar to (1).

let $x = y = z = 1$ in (5) and by AM-GM inequality, we have

$$R_a R_b R_c \geq 8 r_1' r_2' r_3'.$$  \tag{18}

and

$$R_a R_b R_c \geq 8 r_1 r_2 r_3.$$  \tag{19}

## 4 Open problem

At the end, we pose an open problem.

**Open problem:** For an interior point $P$ and positive real numbers $x, y, z,$

Let $AD = w_1', BE = w_2', CF = w_3', R$ and $r$ denote the circumradius and inradius of triangle $ABC$ respectively, then

$$\frac{x w_1'}{\sqrt{R_b R_c}} + \frac{y w_2'}{\sqrt{R_c R_a}} + \frac{z w_3'}{\sqrt{R_a R_b}} \leq \sqrt{2 + \frac{r}{2R} \left( \frac{y z}{x} + \frac{z x}{y} + \frac{x y}{z} \right)}. \quad \tag{20}$$
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References


