Some results on semiprime rings with
generalized derivations

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Abstract

Let $R$ be a semiprime ring, $I$ a nonzero ideal of $R$ and $F$ is a generalized derivation associated with a derivation $d$. If $F$ satisfies any one of the following conditions: (i) $F(xy) - xy \in Z(R)$, (ii) $F(xy) - yx \in Z(R)$, (iii) $F(x)F(y) - xy \in Z(R)$, (iv) $F(x)F(y) - yx \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = (0)$. In particular if $R$ is prime, then either $R$ is commutative or $d = 0$.

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1 Introduction

Throughout this paper, $R$ will represent an associative ring with center $Z(R)$. Recall that $R$ is called prime if $aRb = (0)$ implies $a = 0$ or $b = 0$; it is semiprime if $aRa = (0)$ implies that $a = 0$. Clearly, every prime ring is a semiprime ring.

For $x, y \in R$, $[x, y] = xy - yx$ (resp. $x \circ y = xy + yx$) denote the commutator (resp. the anticommutator) of $x, y$. An additive mapping $d : R \rightarrow R$ is a derivation if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. An additive mapping $F : R \rightarrow R$ is a generalized derivation if there exists a derivation $d$ of $R$ such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$.

Several authors have proved commutativity theorems for prime and semiprime rings admitting derivations or generalized derivations satisfying any one of the properties (i)–(iv) on any appropriate subset. Motivated by this result, our aim in the following paper is to study generalized derivation satisfying properties (i)–(iv) on a a nonzero ideal of a semiprime ring.


2 Main results

In order to prove our theorems we shall need the following facts.

**Fact 2.1** Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. If $x \in I$ is such that $xIx = (0)$, then $x = 0$. In particular, if $xI = (0)$, then $x = 0$.

**Fact 2.2** Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. If $xy \in \mathbb{Z}(R)$ for all $x, y \in I$, then $I \subseteq \mathbb{Z}(R)$.

**Proof.** We have $[x, r]y + x[y, r] = 0$, for all $x, y \in I, r \in R$, replacing $y$ by $yx$ we get $xy[x, r] = 0$, left multiplying last expression by $r$ we obtain $rxy[x, r] = 0$ again replacing $y$ by $ry$ implies that $xry[x, r] = 0$, that is $[x, r]I[x, r] = 0$, thus the semiprimeness together with Fact 2.1 yield $[x, r] = 0$ so that $I \subseteq \mathbb{Z}(R)$.

**Theorem 2.3** Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ satisfying $F(xy) - xy \in \mathbb{Z}(R)$ for all $x, y \in I$, then $[I, R]d(R) = 0$. In particular if $R$ is prime, then either $R$ is commutative or $d = 0$.

**Proof.** We are given that $F(xy) - xy \in \mathbb{Z}(R)$ for all $x, y \in I$.

Replacing $y$ by $yz$ in (1), where $z \in I$, we get $F(xy)z - xyz + yxz = 0$ that is $[xydz, z] = 0$, so that

$$xy[d(z), z] + x[y, z]d(z) + [x, z]yd(z) = 0 \quad \text{for all } x, y, z \in I.$$ (2)

Writing $d(z)x$ instead of $x$ in (2) we obtain

$$[d(z), z]xyz = 0 \quad \text{for all } x, y, z \in I.$$ (3)

Substituting $yz$ for $y$ in (3) we get

$$[d(z), z]xyz = 0 \quad \text{for all } x, y, z \in I.$$ (4)

Right multiplying equation (3) by $z$ we find that

$$[d(z), z]xyz = 0 \quad \text{for all } x, y, z \in I.$$ (5)

Employing equations (4) and (5) yield

$$[d(z), z]xyz = 0 \quad \text{for all } x, y, z \in I.$$ (6)
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Right multiplying equation (6) by $x$ leads to

$$[d(z), z]xI[d(z), z]x = 0 \quad \text{for all } x, z \in I. \quad (7)$$

Hence Fact 2.1 forces that

$$[d(z), z]x = 0 \quad \text{for all } x, z \in I, \quad (8)$$

once again applying Fact 2.1 we get

$$[d(z), z] = 0 \quad \text{for all } z \in I. \quad (9)$$

In view of ([7], main Theorem), equation (9) yields $[I, R]d(R) = 0$.

**Theorem 2.4** Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ satisfying $F(xy) − yx \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = 0$. In particular if $R$ is prime, then either $R$ is commutative or $d = 0$.

**Proof.** We have

$$F(xy) − yx \in Z(R) \quad \text{for all } x, y \in I. \quad (10)$$

Replacing $y$ by $yz$ in (10), where $z \in R$, we get

$$[y, z][x, z] + y[[x, z], z] + [x, z]yd(z) + x[y, z]d(z) + xy[d(z), z] = 0 \quad \text{for all } x, y \in I, \quad z \in R. \quad (11)$$

Substituting $xy$ for $y$ implies that

$$[x, z]y[x, z] + [x, z]xyd(z) = 0 \quad \text{for all } x, y \in I, \quad z \in R. \quad (12)$$

Replacing $z$ by $z + x$ we obtain

$$[x, z]xyd(x) = 0 \quad \text{for all } x, y \in I, \quad z \in R. \quad (13)$$

Writing $d(x)$ instead of $z$ in (13) we get

$$[d(x), x]xyd(x) = 0 \quad \text{for all } x, y \in I. \quad (14)$$

Replacing $y$ by $yx$ in (14) we get

$$[d(x), x]xyxd(x) = 0 \quad \text{for all } x, y \in I. \quad (15)$$

Right multiplying equation (14) by $x$ we obtain

$$[d(x), x]xyd(x)x = 0 \quad \text{for all } x, y \in I. \quad (16)$$
Employing equations (15) and (16) we arrive at $[d(x), x]xy[d(x), x] = 0$, so that
\[ [d(x), x]xI[d(x), x]x = 0 \quad \text{for all } x \in I. \] (17)

In view of equation (17) Fact 2.1 forces that $[d(x), x]x = 0$, so that
\[ [d(x), x]x = 0 \quad \text{for all } x \in I. \] (18)

On the other hand replacing $y$ by $x^2$ in (10) we get
\[ x^2[d(x), x] = 0 \quad \text{for all } x \in I. \] (19)

Comparing equations (18) and (19) we conclude that $[[d(x), x], x^2] = 0$, so
\[ [[d(x^2), x^2], x^2] = 0 \quad \text{for all } x \in I. \] (20)

Accordingly, ([7], main Theorem) assures that $[I, R]d(R) = 0$.

**Theorem 2.5** Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ satisfying $F(x)F(y) - xy \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = 0$. In particular if $R$ is prime, then either $R$ is commutative or $d = 0$.

**Proof.** Suppose that
\[ F(x)F(y) - xy \in Z(R) \quad \text{for all } x, y \in I. \] (21)

Replacing $y$ by $yz$, where $z \in I$, we get
\[ [F(x)y[d(z), z] = 0 \quad \text{for all } x, y, z \in I. \] (22)

Writing $zy$ instead of $y$ in (22) we obtain
\[ [F(x)zy[d(z), z] = 0 \quad \text{for all } x, y, z \in I. \] (23)

Substituting $xz$ for $x$ in (22) and using (23) leads to
\[ [xd(z)y[d(z), z] = 0 \quad \text{for all } x, y, z \in I. \] (24)

Replacing $x$ by $xz$ we arrive at
\[ [xzd(z)y[d(z), z] = 0 \quad \text{for all } x, y, z \in I. \] (25)

Right multiplying (24) by $z$ implies that
\[ [xd(z)y[d(z), z]] = 0 \quad \text{for all } x, y, z \in I. \] (26)
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Subtracting (26) from (25) we find that
\[ [x[d(z)yd(z)], z] = 0 \quad \text{for all } x, y, z \in I. \tag{27} \]
Putting \( d(z)yd(z)x \) instead of \( x \) in (27) we obtain
\[ [d(z)yd(z), z][d(z)yd(z), z] = 0 \quad \text{for all } y, z \in I. \tag{28} \]
Applying Fact 2.1 we deduce
\[ [d(z)yd(z), z] = 0 \quad \text{for all } y, z \in I \tag{29} \]
Now replacing \( y \) by \( yd(z)x \) where \( x \in I \) we obtain
\[ d(z)y[d(z), z]xd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{30} \]
Substituting \( zy \) for \( y \) yield
\[ d(z)zy[d(z), z]xd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{31} \]
Left multiplying equation (30) by \( z \) we get
\[ zd(z)y[d(z), z]xd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{32} \]
Subtracting (32) from (31) we arrive at
\[ [d(z), z][d(z), z]xd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{33} \]
Replacing \( y \) by \( xd(z)y \) implies that
\[ [d(z), z]xd(z)[d(z), z]xd(z) = 0 \quad \text{for all } x, y, z \in I. \tag{34} \]
Hence
\[ [d(z), z]xd(z) = 0 \quad \text{for all } x, z \in I. \tag{35} \]
Replacing \( x \) by \( xz \) in (35) we get
\[ [d(z), z]xzd(z) = 0 \quad \text{for all } x, z \in I. \tag{36} \]
Right multiplying (35) by \( z \) we result
\[ [d(z), z]xd(z)z = 0 \quad \text{for all } x, z \in I. \tag{37} \]
Thus
\[ [d(z), z][d(z), z] = 0 \quad \text{for all } z \in I, \tag{38} \]
so that
\[ [d(z), z] = 0 \quad \text{for all } z \in I. \tag{39} \]
Applying ([7], main Theorem) we conclude that \([I, R]d(R) = 0\). \hfill \blacksquare
Theorem 2.6 Let $R$ be a semiprime ring and $I$ a nonzero ideal of $R$. If $R$ admits a generalized derivation $F$ associated with a derivation $d$ satisfying $F(x)F(y) - yx \in Z(R)$ for all $x, y \in I$, then $[I, R]d(R) = 0$. In particular if $R$ is prime, then either $R$ is commutative or $d = 0$.

Proof. We have

$$F(x)F(y) - yx \in Z(R) \quad \text{for all } x, y \in I. \quad (40)$$

Replacing $y$ by $yz$, where $z \in I$, we get

$$(F(x)F(y) - yx)z + y[x, z] + F(x)yd(z) \in Z(R) \quad \text{for all } x, y, z \in I, \quad (41)$$

so that

$$[y[x, z], z] + [F(x)yd(z), z] = 0 \quad \text{for all } x, y, z \in I. \quad (42)$$

Writing $xz$ instead of $x$, we obtain

$$[y[x, z], z]z + [F(x)zyd(z) + xd(z)yd(z), z] = 0 \quad \text{for all } x, y, z \in I. \quad (43)$$

Again replacing $y$ by $zy$ in (42) we arrive at

$$z[y[x, z], z] + [F(x)zyd(z), z] = 0 \quad \text{for all } x, y, z \in I. \quad (44)$$

Subtracting (44) from (43) we get

$$[[y[x, z], z], z]z + [xd(z)yd(z), z] = 0 \quad \text{for all } x, y, z \in I. \quad (45)$$

Substituting $xz$ for $x$ in (45) we obtain

$$[[y[x, z], z], z]z + [xz]d(z)yd(z), z] = 0 \quad \text{for all } x, y, z \in I, \quad (46)$$

Right multiplying equation (45) by $z$ lead to

$$[[y[x, z], z], z]z + [zd(z)yd(z), z] = 0 \quad \text{for all } x, y, z \in I, \quad (47)$$

Employing (46) and (47) yield

$$[x[d(z)yd(z), z], z] = 0 \quad \text{for all } x, y, z \in I, \quad (48)$$

Since (48) is the same as (27) reasoning as in Theorem 2.5, we get the required result. \hfill \blacksquare
3 Open Problems

To end this paper we introduce the following open questions:

(i) Does the condition $F([x, y]) - [x, y] \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

(ii) Does the condition $F(x \circ y) - x \circ y \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

(iii) Does the condition $[F(x), F(y)] - [x, y] \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

(iv) Does the condition $F(x) \circ F(y) - x \circ y \in Z(R)$ for all $x, y \in I$ implies that $[I, R]d(R) = 0$?

References


