Int. J. Open Problems Compt. Math., Vol. 6, No. 2, June 2013
ISSN 1998-6262; Copyright ©ICSRS Publication, 2013
www.i-csrs.org

Annulets of Stone lattices generated by pseudo-complements

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Abstract

The notion of pseudo-annulets is introduced in Stone lattices and characterized in terms of prime filters. Two operator α and β are introduced and obtained that their composition $\beta \circ \alpha$ is a closure operator on the class of all filters of a Stone lattice. A congruence θ is introduced on a Stone lattice L and proved that the quotient lattice L/θ is a Boolean algebra.

Keywords: *Stone lattice, pseudo-annulet, dense element, prime filter.*
2010 Mathematics Subject Classification: 06D30.

1 Introduction

The theory of pseudo-complements was introduced and extensively studied in semi-lattices and particularly in distributive lattices by Orrin Frink [4] and Garret Birkhoff [2]. Later the problem of characterizing the class of Stone lattices has been studied by several authors like Raymond Balbes [1], Orrin Frink [4], George Gratzer [5] etc.

In this paper, the concept of pseudo-annulets is introduced in Stone lattices and proved that the class $\mathcal{A}^+(L)$ of all pseudo-annulets of a Stone lattice L forms a complete Boolean algebra. An operations α is defined on the Stone lattice L and proved that, for any filter F of L, $\alpha(F)$ is an ideal in the lattice $\mathcal{A}^+(L)$. For any prime filter P of a Stone lattice, we define a set $\ell(P) = \{x \in L \mid x^* \notin P\}$ and proved that a pseudo-annulet $(a)^+$ is equal to the intersection of all $\ell(P)$ where $a \in P$. A Glivenko type congruence relation θ is introduced on a Stone lattice in terms of pseudo-annulets. Finally, it is proved that the quotient lattice $L_{/\theta}$ is a Boolean algebra.

2 Preliminaries

The reader is refereed to [2] for the notions and notations. However, some of the preliminary definitions and results are presented for the ready reference of the reader. Throughout the rest of this note L stands for a Stone lattice $(L, \vee, \wedge, ^*, 0, 1)$, unless otherwise mentioned.

Definition 2.1 *[2] For any element a of a distributive lattice L, the pseudo-complement a^* of a is an element satisfying the following property for all $x \in L$:*

$$a \wedge x = 0 \Leftrightarrow a^* \wedge x = x \Leftrightarrow x \leq a^*$$

A distributive lattice L in which every element has a pseudo-complement is called a pseudo-complemented distributive lattice.

Theorem 2.2 *[2] For any two elements a, b of a pseudo-complemented distributive lattice, we have the following:*

(1) $0^{**} = 0$
(2) $a \wedge a^* = 0$
(3) $a \leq b$ *implies* $b^* \leq a^*$
(4) $a \leq a^{**}$
(5) $a^{***} = a^*$
(6) $(a \vee b)^* = a^* \wedge b^*$
(7) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$

An element x of a pseudo-complemented lattice L is called dense if $x^* = 0$ and the set $D(L)$ of all dense element of L forms a filter of L.

Theorem 2.3 *[4] Let F be a filter and I an ideal of a distributive lattice L such that $F \cap I = \emptyset$. Then there exists a prime filter P of L such that $F \subseteq P$ and $P \cap I = \emptyset$.*

Definition 2.4 *[2] A pseudo-complemented distributive lattice L is called a Stone lattice if, for all $x \in L$, it satisfies the property: $x^* \vee x^{**} = 1$.*

Theorem 2.5 *[2] The following conditions are equivalent in a pseudo-complemented distributive lattice L.*

(1) *L is a Stone lattice*
(2) $S_L = \{x^* \mid x \in L\}$ *is a sublattice of L*
(2) $(x \wedge y)^* = x^* \vee y^*$
(3) $(x \vee y)^{**} = x^{**} \vee y^{**}$

A binary relation θ on a Stone lattice L is a Glivenko type congruence if it satisfies the following properties.

(i). $(x, y) \in \theta$ implies $(x \wedge c, y \wedge c) \in \theta$, $(x \vee c, y \vee c) \in \theta$ for any $c \in L$.
(ii). $(x, y) \in \theta$ if and only if $x^* = y^*$ for all $x, y \in L$.

3 Main results

In this section, the concept of pseudo-annulets is introduced in a Stone lattice. Some operations are introduced on Stone lattices and the lattice of pseudo-annulets. A Glivenko type congruence is introduced on a Stone lattice and proved that the quotient lattice is a Boolean algebra.

Definition 3.1 *For any subset A of a Stone lattice L, define the set A^+ as follows: $A^+ = \{x \in L \mid x^* \wedge a = 0$ for all $a \in A\}$.*

If $A = \{a\}$, then for brevity we denote $(\{a\})^+$ by $(a)^+$. Then it can be easily observed that $(0)^+ = L$ and $L^+ = (1)^+ = D$.

Lemma 3.2 *For any subset A of a Stone lattice L, A^+ is a filter of L.*

Proof. Clearly $1 \in A^+$. Let $x, y \in A^+$. Then $x^* \wedge s = 0$ and $y^* \wedge t = 0$ for all $s, t \in A$. Now for any $a \in A, (x \wedge y)^* \wedge a = (x^* \vee y^*) \wedge a = (x^* \wedge a) \vee (y^* \wedge a) = 0 \vee 0 = 0$. Hence $x \wedge y \in A^+$. Let $x \in A^+$ and $x \leq y$. Then for any $c \in L, y^* \wedge c \leq x^* \wedge c = 0$. Hence $y \in A^+$. Therefore A^+ is a filter of L.

Proposition 3.3 *Let A and B be two subsets of a Stone lattice L. Then the following conditions hold.*

(1) $A^+ = \bigcap\limits_{a \in A} (a)^+$
(2) $A \subseteq B \Rightarrow B^+ \subseteq A^+$
(3) $A \subseteq A^+$
(4) $A^{++} = A^+$
(5) For any two filters F, G of $L, (F \vee G)^+ = F^+ \cap G^+$

Proof. (1). It is clear.
(2). Suppose $A \subseteq B$. Let $x \in B^+$. Then $x^* \wedge b = 0$ for all $b \in B$. Hence $x^* \wedge a = 0$ for all $a \in A$. Hence $x \in A^+$. Therefore it concludes $B^+ \subseteq A^+$.
(3). Let $x \in A$. Then $x \wedge x^* = 0$. Hence it yields $x \in A^+$. Therefore $A \subseteq A^+$.
(4). By (3), we get $A^{++} \subseteq A^+$. Also $A^+ \subseteq (A^+)^+$. Therefore $A^{++} = A^+$.
(5). Clearly $(F \vee G)^+ \subseteq F^+ \cap G^+$. Conversely, let $x \in F^+ \cap G^+$. Let $c \in F \vee G$. Then $c = f \wedge g$ for some $f \in F, g \in G$. Now $x^* \wedge c = x^* \wedge (f \wedge g) = 0 \wedge g = 0$. Hence $x \in (F \vee G)^+$. Therefore $(F \vee G)^+ = F^+ \cap G^+$. Some properties of pseudo-annulets can be observed in the following Lemma.

Lemma 3.4 *For any $a, b \in L$, we have the following:*

(1) $a \in (a)^+$

(2) $[a) \subseteq (a)^+$

(3) $(a)^{++} = (a)^+$

(4) $a \leq b$ *implies* $(b)^+ \subseteq (a)^+$

(5) $a \in (b)^+$ *implies* $(a)^+ \subseteq (b)^+$

(6) $(a)^+ = L$ *if and only if* $a = 0$

(7) $a \wedge b = 0$ *implies* $(a)^+ \vee (b)^+ = L$

Proof. (1). It is clear.
(2). From (1), it is clear.
(3). It is clear by the above Proposition 3.3.
(4). Let $a \leq b$. Let $x \in (b)^+$. Then $x^* \wedge a \leq x^* \wedge b = 0$. Therefore $x \in (a)^+$.
(5). Suppose $a \in (b)^+$. Then $a^* \wedge b = 0$. Let $t \in (a)^+$. Then $t^* \wedge a = 0$. Hence $t^* \leq a^*$. Thus $t^* \wedge b \leq a^* \wedge b = 0$. Hence $t \in (b)^+$. Therefore $(a)^+ \subseteq (b)^+$.
(6). Suppose $(a)^+ = L$. Then we get $0 \in (a)^+$. Hence it yields that $a = 1 \wedge a = 0^* \wedge a = 0$. Converse is clear.
(7). Let $a, b \in L$ be such that $a \wedge b = 0$. Suppose $(a)^+ \vee (b)^+ \neq L$. Then there exists a maximal filter M such that $(a)^+ \vee (b)^+ \subseteq M$. Hence $(a)^+ \subseteq M$ and $(b)^+ \subseteq M$. Now $(a)^+ \subseteq M$ implies $a \in (a)^+ \subseteq M$. Hence $a^* \notin M$. Similarly, we can get $b^* \notin M$. Since M is a prime, we get $1 = (a \wedge b)^* = a^* \vee b^* \notin M$, which is a contradiction. Therefore $(a)^+ \vee (b)^+ = L$.

Let us denote the class of all pseudo-annulets of L by $\mathcal{A}^+(L)$. Then in the following, we prove that $\mathcal{A}^+(L)$ is a complete Boolean algebra.

Theorem 3.5 *For any Stone lattice $L, \mathcal{A}^+(L)$ is a Boolean algebra.*

Proof. Let $(a)^+, (b)^+ \in \mathcal{A}^+(L)$. We first prove the existence of infimum and suprimum for $(a)^+$ and $(b)^+$ in $\mathcal{A}^+(L)$. Clearly $(a \vee b)^+ \subseteq (a)^+ \cap (b)^+$. Conversely let $x \in (a)^+ \cap (b)^+$. Then $x^* \wedge (a \vee b) = (x^* \wedge a) \vee (x^* \wedge b) = 0 \vee 0 = 1$. Hence $x \in (a \vee b)^+$. Therefore $(a)^+ \cap (b)^+ = (a \vee b)^+$. Again, clearly $(a)^+ \vee (b)^+ \subseteq (a \wedge b)^+$. Let $x \in (a \wedge b)^+$. Then $(x^* \wedge a) \wedge (x^* \wedge b) = x^* \wedge a \wedge b = 0$. Hence by Lemma 3.4(7), we get $(x^* \wedge a)^+ \vee (x^* \wedge b)^+ = L$. Thus $x \in L = (x^* \wedge a)^+ \vee (x^* \wedge b)^+$. Hence $x = r \wedge s$ for some $r \in (x^* \wedge a)^+$ and $s \in (x^* \wedge b)^+$. Now

$$\begin{aligned} r \in (x^* \wedge a)^+ &\Rightarrow r^* \wedge x^* \wedge a = 0 \\ &\Rightarrow (r \vee x)^* \wedge a = 0 \\ &\Rightarrow r \vee x \in (a)^+ \end{aligned}$$

Similarly, we can get $s \vee x \in (b)^+$. Hence

$$\begin{aligned} x &= x \vee x \\ &= x \vee (r \wedge s) \\ &= (x \vee r) \wedge (x \vee s) \in (a)^+ \vee (b)^+ \end{aligned}$$

Hence $(a \wedge b)^+ \subseteq (a)^+ \vee (b)^+$. Therefore $(a)^+ \vee (b)^+ = (a \wedge b)^+$. Therefore $\langle \mathcal{A}^+(L), \wedge, \vee \rangle$ forms a distributive lattice with greatest element $(0)^+$ and the least element D. Moreover $\mathcal{A}^+(L)$ is a Boolean algebra where the complement of each $(x)^+, x \in L$ is precisely $(x^*)^+$.

We now introduce two operation α and β in the following.

Definition 3.6 *For any filter F of L, define*

$$\alpha(F) = \{(x)^+ \mid x \in F\}$$

Definition 3.7 *For any ideal I of $\mathcal{A}^+(L)$, define*

$$\beta(I) = \{x \in L \mid (x)^+ \in I\}$$

We first prove some basic properties of the above operations α and β.

Lemma 3.8 *For any Stone lattice L, we have the following:*

(1). For any filter F of $L, \alpha(F)$ is an ideal in $\mathcal{A}^+(L)$
(2). for any ideal I of $\mathcal{A}^+(L), \beta(I)$ is a filter in L
(3). α and β are isotones

Proof. (1). Let F be filter of L. Since $1 \in F$, we get $(1)^+ \in \alpha(F)$. Let $(x)^+, (y)^+ \in \alpha(F)$. Then $(x)^+ = (a)^+$ and $(y)^+ = (b)^+$ for some $a, b \in F$. Hence $(x)^+ \vee (y)^+ = (a)^+ \vee (b)^+ = (a \wedge b)^+ \in \alpha(F)$, because of $a \wedge b \in F$. Again, let $(x)^+ \in \alpha(F)$ and $(r)^+ \in (A)^+(L)$. Then it yields $(x)^+ = (a)^+$ for some $a \in F$. Now we get that $(x)^+ \cap (r)^+ = (a)^+ \cap (r)^+ = (a \vee r)^+ \in \alpha(F)$, because of $a \vee r \in F$. Therefore $\alpha(F)$ is an ideal in $\mathcal{A}^+(L)$.

(2). Let I be an ideal of $\mathcal{A}^+(L)$. Since $(1)^+$ is the smallest element of $\mathcal{A}^+(L)$ and I is an ideal of $\mathcal{A}^+(L)$, we get $(1)^+ \in I$. Hence $1 \in \beta(I)$. Let $x, y \in \beta(I)$. Then we get $(x)^+, (y)^+ \in I$. Since I is an ideal, we get $(x \wedge y)^+ = (x)^+ \vee (y)^+ \in I$. Hence $x \wedge y \in \beta(I)$. Again, let $x \in \beta(I)$ and $r \in L$. Then we get $(x)^+ \in I$ and $(r)^+ \in \mathcal{A}^+(L)$. Since I is an ideal, we get $(x \vee r)^+ = (x)^+ \cap (r)^+ \in I$. Hence $x \vee r \in \beta(I)$. Therefore it concludes that $\beta(I)$ is a filter of L.

(3). Suppose F, G are two filters of L such that $F \subseteq G$. Let $(x)^+ \in \alpha(F)$. Then we get $(x)^+ = (a)^+$ for some $a \in F \subseteq G$. Hence it yields $(x)^+ \in \alpha(G)$. Therefore $\alpha(F) \subseteq \alpha(G)$. Similarly, we can get $\beta(F) \subseteq \beta(G)$.

In the following, we prove that the operation $\beta \circ \alpha$ is a closure operator.

Theorem 3.9 *Let L be a Stone lattice and F a filter of L. Then the map $F \longrightarrow \beta \circ \alpha(F)$ is a closure operator. That is:*

(i). $F \subseteq \beta \circ \alpha(F)$

(ii). $\beta \circ \alpha[\ \beta \circ \alpha(F)\ ] = \beta \circ \alpha(F)$

(iii). $F \subseteq G \ \Rightarrow\ \beta \circ \alpha(F) \subseteq \beta \circ \alpha(G)$ *for any two filters F, G of L*

Proof. (i). Let $x \in F$. Then we get that $(x)^+ \in \alpha(F)$. Since $\alpha(F)$ is an ideal in $\mathcal{A}^+(L)$, it yields that $x \in \beta \circ \alpha(F)$. Therefore $F \subseteq \beta \circ \alpha(F)$.
(ii). Since $\beta \circ \alpha(F)$ is a filter in L, from the condition (i) of this Theorem, we get that $\beta \circ \alpha(F) \subseteq \beta \circ \alpha[\ \beta \circ \alpha(F)\ ]$. Conversely, let $x \in \beta \circ \alpha[\ \beta \circ \alpha(F)\ ]$. Then $(x)^+ \in \alpha[\ \beta \circ \alpha(F)\ ]$. Then $(x)^+ = (y)^+$ for some $y \in \beta \circ \alpha(F)$. Hence $\beta \circ \alpha[\ \beta \circ \alpha(F)\ ] \subseteq \beta \circ \alpha(F)$. Therefore $\beta \circ \alpha[\ \beta \circ \alpha(F)\ ] = \beta \circ \alpha(F)$.
(iii). Suppose $F \subseteq G$. Let $(x)^+ \in \alpha(F)$. Then $(x)^+ = (y)^+$ for some $y \in F$. So $(x)^+ = (y)^+$ for some $y \in G$. Now $y \in G$ implies $(y)^+ \in \alpha(G)$. Hence $(x)^* \in \alpha(G)$. Therefore $x \in \beta \circ \alpha(G)$. Hence $\beta \circ \alpha(F) \subseteq \beta \circ \alpha(G)$.

Definition 3.10 *For any prime filter M of L, define*

$$\ell(P) = \{x \in L \mid x^* \notin P\}$$

Proposition 3.11 *For any prime filter P of L, $\ell(P)$ is a filter of L such that $P \subseteq \ell(P)$.*

Proof. Assume that P is a prime filter of L. Clearly $1 \in \ell(P)$. Let $x, y \in \ell(P)$. Then $x^* \notin P$ and $y^* \notin P$. Since P is a prime filter, we get $(x \wedge y)^* = x^* \vee y^* \notin P$. Hence it yield that $x \wedge y \in \ell(P)$. Let $x \in \ell(P)$ and $r \in L$. Then we get $x^* \notin P$. Hence $(x \vee r)^* \notin P$, otherwise $x^* \in P$. Thus $x \vee r \in \ell(P)$. Therefore $\ell(P)$ is a filter of L. Let $x \in P$. Then $x^* \notin P$, otherwise we get $0 = x \wedge x^* \in P$. Hence $x \in \ell(P)$. Therefore $P \subseteq \ell(P)$.

Let us denote the class of all prime filters of L by $\wp$ and $\wp_a = \{P \in \wp \mid a \in P\}$. Then we have the following:

Theorem 3.12 *For any $a \in L$, $(a)^+ = \bigcap\limits_{P \in \wp_a} \ell(P)$*

Proof. Let $F_0 = \bigcap\limits_{P \in \wp_a} \ell(P)$. Let $x \in (a)^+$ and $P \in \wp_a$. Then $x^* \wedge a = 0$. If $x^* \in P$, then $0 = x^* \wedge a \in P$, which is a contradiction. Hence we get $x^* \notin P$. Thus $x \in \ell(P)$. This is true for all $P \in \wp_a$. Hence it yields that $(a)^+ \subseteq F_0$. Conversely, let $x \in F_0$. Then $x \in \ell(P)$ for all $P \in \wp_a$. Suppose $x^* \wedge a \neq 0$. Then there exists a maximal filter M_0 of L such that $x^* \wedge a \in M_0$. Hence $x^* \in M_0$ and $a \in M_0$. Since M_0 is a prime filter and $a \in M_0$, by our assumption $x \in \ell(M_0)$, which implies that $x^* \notin M_0$, which is a contradiction. Hence $x^* \wedge a = 0$. Thus $x \in (a)^+$. Hence $F_0 \subseteq (a)^+$. Therefore $(a)^+ = F_0$.

Corollary 3.13 *Let $P \in \wp$. If $a \in P$, then $(a)^+ \subseteq \ell(P)$.*

We now introduce a congruence on L in terms of pseudo-annulets.

Proposition 3.14 *For any $x, y \in L$, define a relation θ on L as follows:*

$$(x, y) \in \theta \text{ if and only if } (x)^+ = (y)^+$$

Then θ is a congruence on L.

Proof. Clearly θ is an equivalence relation on L. Let $(a, b) \in \theta$. Then we get $(a)^+ = (b)^+$. Now for any $c \in L$, $(a \wedge c)^+ = (a)^+ \sqcup (c)^+ = (b)^+ \sqcup (c)^+ = (b \wedge c)^+$. Also $(a \vee b)^+ = (a)^+ \cap (c)^+ = (b)^+ \cap (c)^+ = (b \vee c)^+$. Hence $(a \wedge c, b \wedge c), (a \vee c, b \vee c) \in \theta$. Therefore θ is a congruence on L.

It is a well known fact that the quotient algebra $L_{/\theta} = \{\theta(x) \mid x \in L\}$, where $\theta(x)$ is a congruence class of x with respect to θ, is a distributive lattice with respect to the operations given by

$$\theta(x) \cap \theta(y) = \theta(x \wedge y) \text{ and } \theta(x) \vee \theta(y) = \theta(x \vee y)$$

The bounds of the above lattice $L_{/\theta}$ are given in the following lemma.

Theorem 3.15 *Let θ be the congruence defined above on L. Then $L_{/\theta}$ is a Boolean algebra.*

Proof. Clearly $\{0\}$ is the smallest congruence class of $L_{/\theta}$. We now show that D is the unit congruence class of $L_{/\theta}$. Let $x, y \in D$. Then $x^* = y^*$. Let $t \in (x)^+$. Then $t^* \wedge x = 0$. Hence $t^* \leq x^* = y^*$. Thus $t^* \wedge y \leq y^* \wedge y = 0$. Therefore $t \in (y)^+$. Thus $(x)^+ \subseteq (y)^+$. Similarly, we can get $(y)^+ \subseteq (x)^+$. Therefore $(x, y) \in \theta$. Thus D is a congruence class of $L_{/\theta}$. Now, let $a \in D$ and $x \in L$. Since D is a filter, we get $a \vee x \in D$. Hence $\theta(x) \vee \theta(a) = \theta(x \vee a) = D$. Therefore D is the unit congruence class of $L_{/\theta}$. Let $x \in L$. Then clearly $\theta(x) \cap \theta(x^*) = \theta(x \wedge x^*) = \theta(0) = \{0\}$. Also $\theta(x) \vee \theta(x^*) = \theta(x \vee x^*) = D$, because of $x \vee x^* \in D$. Therefore $L_{/\theta}$ is a Boolean algebra.

4 Open Problem

(1). For any two filters F, G of a Stone lattice, it can be easily observed that $F^+ \vee G^+ \subseteq (F \cap G)^+$. It is under investigation that whether the equality exists or not for any two filter of a Stone lattice.

(2). The operation $\beta \circ \alpha$ is a closure operation on the class of all filters of a Stone lattice. Still some investigation has to be carried to derive some set of equivalent conditions for the existence of equality $F = \beta \circ \alpha(F)$.

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