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Fourier spectral methods for solving some nonlinear partial differential equations

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Abstract

The spectral collocation or pseudospectral (PS) methods (Fourier transform methods) combined with temporal discretization techniques to numerically compute solutions of some partial differential equations (PDEs). In this paper, we solve the Korteweg-de Vries (KdV) equation using a Fourier spectral collocation method to discretize the space variable, leap frog and classical fourth-order Runge-Kutta scheme (RK4) for time dependence. Also, Boussinesq equation is solving by a Fourier spectral collocation method to discretize the space variable, finite difference and classical fourth-order Runge-Kutta scheme (RK4) for time dependence. Our implementation employs the Fast Fourier Transform (FFT) algorithm.

Keywords: *Fourier spectral method, Fast Fourier transform, Boussinesq equation, Korteweg-de Vries equation; leap frog, finite difference.*

1 Introduction

Let us consider the equation

$$u_t + \alpha\, u\, u_x + \beta\, u_{xxx} = 0\,, \tag{1}$$

The equation (1) is known as the Korteweg-de Vries (KdV) equation (with non–linearity uu_x , and dispersion u_{xxx}) with real constants α , β . The KdV equation is one of the popular soliton equations. More than a century later the KdV equation has been found in many other applications such as magnetohydro dynamic waves in a cold plasma , longitudinal vibrations of an enharmonic discrete-mass string , ion-acoustic waves in a cold plasma, pressure waves in liquids-gas bubble mixture , rotating flow down a tube, and longitudinal dispersive waves in elastic rods . The exact solutions of (1) is

$$u(x,t)=\frac{3c}{\alpha}\operatorname{sec}h^2\left(\frac{1}{2}\sqrt{\frac{c}{\beta}}(x-ct-x_0)\right), \tag{2}$$

where x_0 is an arbitrary integration constant.

We investigate the numerical solution of the KdV equation using the Fourier-Leap Frog methods, due to Fornberg and Whitham [2], and the Fourier based fourth-order Runge-Kutta (RK4) method for better stability of the solution.

Consider the equation

$$u_{tt}-u_{xx}+3(u^2)_{xx}+u_{xxxx}=0, \tag{3}$$

The equation (3) is known as the Boussinesq equation, where subscripts *x* and *t* denote differentiation, was introduced to describe the propagation of long waves in shallow water. The Boussinesq equation also arises in several other physical applications including one-dimensional nonlinear lattice waves, vibrations in a nonlinear string, and ion sound waves in a plasma. It is well known that The Boussinesq equation (3) has a bidirectional solitary wave solution

$$u(x,t)=2b^2\operatorname{sec}h^2(b(x-x_0-ct)). \tag{4}$$

representing a solitary wave, where $c=\pm\sqrt{1-4b^2}$ is the propagation speed and *b*, x_0 arbitrary constants determining the height and the position of the maximum height of the wave, respectively. From the form of *c* it is apparent that the solution can propagate in either direction (left or right). In the present work, we have applied to (3) two numerical methods to study soliton solutions and investigate their interactions upon collision: a combination of finite differences and a Fourier pseudospectral method

2 Analysis the method for the Korteweg-de Vries (KdV) equation

Now, consider the standard KdV equation in the form

$$u_t+\alpha\,u\,u_x+\beta\,u_{xxx}=0, \qquad x\in[-L/2,L/2], \tag{5}$$

where $u = u(x, t)$, subscripts x and *t* denote differentiation, where *L* is a given number , usually large. We have changed the solution interval from $[-L/2,L/2]$ to $[0,2\pi]$, with the change of variable $x\to\ \pi+\frac{2\pi x}{L}$.

Thus, the equation (5) becomes

$$u_t + \frac{2\pi}{L}\alpha u u_x + (\frac{2\pi}{L})^3 \beta u_{xxx} = 0, \quad x \in [0, 2\pi], \tag{6}$$

$u(x, t)$ is transformed into Fourier space with respect to x, and derivatives (or other operators) with respect to x. Applying the inverse Fourier transform

$$\frac{\partial^n u}{\partial x^n} = F^{-1}\{(ik)^n F(u)\}, \quad n = 1, 2, \cdots. \tag{7}$$

using this with $n = 1$ and $n = 3$,

$u_x = F^{-1}\{ikF(u)\}, \quad u_{xxx} = F^{-1}\{-ik^3 F(u)\}$.

The equation (6) becomes

$$u_t = -\frac{2\pi}{L}\alpha u F^{-1}\{ikF(u)\} - (\frac{2\pi}{L})^3 \beta F^{-1}\{-ik^3 F(u)\}, \tag{8}$$

In practice, we need to discretize the equation (8). For any integer $N > 0$, we consider

$$x_j = j\Delta x = \frac{2\pi j}{N}, \; j = 0, 1, \ldots, N-1.$$

Let $u\,(x, t)$ be the solution of the KdV equation (5). Then, we transform it into the discrete Fourier space as

$$\hat{u}(k,t) = F(u) = \frac{1}{N}\sum_{J=0}^{N-1} u(x_j,t)\, e^{-ikx_j}, -\frac{N}{2} \le k \le \frac{N}{2} - 1. \tag{9}$$

From this, using the inversion formula, we get

$$u(x_j,t) = F^{-1}(\hat{u}) = \sum_{k=-N/2}^{N/2-1} \hat{u}(k,t)\, e^{ikx_j}, \quad 0 \le j \le N-1. \tag{10}$$

where we denote the discrete Fourier transform and the inverse Fourier transform by F and F^{-1}, respectively. Now, replacing F and F^{-1} in (8) by their discrete counterparts, and discretizing (8) gives

$$\frac{du(x_j,t)}{dt} = -\frac{2\pi}{L}\alpha\, u(x_j,t) F^{-1}\{ikF(u)\} - (\frac{2\pi}{L})^3 \beta F^{-1}\{-ik^3 F(u)\}, 0 \le j \le N-1. \tag{11}$$

$[u(x_0,t), u(x_1,t), \cdots, u(x_{N-1},t)]^T$. Letting $\mathbf{U} = [u(x_0,t), u(x_1,t), \cdots, u(x_{N-1},t)]^T$.

The equation (11) can be written in the vector form

$$\mathbf{U}_t = \mathbf{F}\,(\mathbf{U}) \tag{12}$$

where **F** defines the right hand side of (11).

2.1 Fourier Leap-Frog Method for KdV Equation

A time integration known as a Leap-Frog method (a two step scheme) is given by

$$\mathbf{U}^{n+1} = \mathbf{U}^{n-1} + 2\Delta t\, \mathbf{F}^n$$

where the superscripts denote the time level at which the term is evaluated. In this subsection, we use a Fourier collocation method and this scheme to solve the system given in (12) numerically. Here, we follow the analysis of Fornberg and Whitham [10]. Using the Leap-Frog scheme to advance in time, we obtain

$$\mathbf{U(t+\Delta t) = U(t - \Delta t) + 2\Delta t F(U(t))}$$

This is called the Fourier Leap-Frog (FLF) scheme for the KdV equation (5). FLF needs two levels of initial data. Usually, the first one is given by the initial condition $u(x, 0)$ and the second level $u(x, \Delta t)$ can be obtained by using a higher-order one-step method, for example, a fourth –order Runge- Kutta method. Thus, the time discretization for (11) is given by

$$u(x,t+\Delta t) = u(x,t-\Delta t) - 2\Delta t\frac{2\pi}{L}\alpha u(x_j,t)F^{-1}\{ikF(u)\} - 2\Delta t(\frac{2\pi}{L})^3\beta F^{-1}\{-ik^3F(u)\}. \quad (13)$$

where we denote $\quad v^3 = \beta(\frac{2\pi}{L})^3 k^3.$

In general, the Fourier-Leap Frog scheme defined in (13) is accurate for low enough wave numbers, but it loses accuracy rapidly for increasing wave numbers.

2.2 Fourier Based RK4 Method for KdV Equation

Notice that the Fourier-Leap-Frog method is a second-order scheme and has some disadvantage in the way that the solution of the model problem is subject to a temporal oscillation with period $2\Delta t$ a commonly suggested alternative method is the Runge-Kutta methods. The classical fourth-order Runge-Kutta methods for the system (11) is given by

$$\begin{aligned}
\mathbf{K_1} &= \mathbf{F(U^n, t^n)} \\
\mathbf{K_2} &= \mathbf{F(U^n + \tfrac{1}{2}\Delta t K_1, t^n + \tfrac{1}{2}\Delta t)} \\
\mathbf{K_3} &= \mathbf{F(U^n + \tfrac{1}{2}\Delta t K_2, t^n + \tfrac{1}{2}\Delta t)} \\
\mathbf{K_4} &= \mathbf{F(U^n + \tfrac{1}{2}\Delta t K_3, t^n + \Delta t)} \\
\mathbf{U^{n+1}} &= \mathbf{U^n + \frac{\Delta t}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]}
\end{aligned} \qquad (14)$$

where the superscripts denote the time level at which the term is evaluated.

2.3 Linear stability analysis

For an analysis of stability we use the standard Fourier analysis to find the condition imposed on the time step Δt. For simplicity we let $\beta = 1$ and consider the KdV equation in the form

$$u_t + \alpha u_x + u_{xxx} = 0, \qquad (15)$$

2.3.1 By using FLF scheme

We approximate this equation by

$$u(x,t+\Delta t)-u(x,t-\Delta t)+2\Delta t\alpha F^{-1}\{i\nu F(u)\}+2\Delta tF^{-1}\{-i\nu^3F(u)\}=0. \tag{16}$$

We look a solution to (16) of the form

$$u(x,t)=\kappa^{t/\Delta t}e^{i\nu x}$$

Substitution in (16) gives

$$\kappa^{(t+\Delta t)/\Delta t}e^{i\nu x}-\kappa^{(t-\Delta t)/\Delta t}e^{i\nu x}+2i\,\nu\,\Delta t\,\kappa^{t/\Delta t}e^{i\nu x}-2i\,\nu^3\Delta t\,\kappa^{t/\Delta t}e^{i\nu x}=0,$$

i.e.

$$\kappa-\kappa^{-1}+2i\,\nu\alpha\Delta t-2i\,\nu^3\Delta t=0$$

$$\kappa^2-2if\,(\Delta t,\nu,\alpha)\kappa-1=0,$$

where

$$f\,(\Delta t,\nu,\alpha)=\nu^3\Delta t-\nu\alpha\Delta t$$

The scheme is conditionally stable if and only if $f\,(\Delta t,\nu,\alpha)$ is real and less than one in magnitude. Let us again assume for simplicity that the period is [0,2π] and that this interval is discretized with N equidistant mesh points, that is

$$\Delta x=2\pi/N$$

The wave number ν takes the values

$$\nu=0,\pm1,\pm2,\ldots,\pm N/2.$$

We want to find the largest value of Δt such that

$$\left|f\,(\Delta t,\nu,\alpha)\right|<1$$

is true for all ν .The most severe restriction on Δt is imposed for the ν ,which are largest in magnitude, i.e. for $\nu=\pm\nu_{\max}$, $\nu_{\max}=N/2=\pi/\Delta x$. Thus, we obtain

$$f\,(\Delta t,\nu_{\max},\alpha)=\Delta t(\frac{\pi}{\Delta x})^3-\Delta t(\frac{\pi}{\Delta x})\alpha$$

If we assume that α >0, then $\left|f\,(\Delta t,\nu,\alpha)\right|$ <1 when

$$<1\quad \Delta t\,(\frac{\pi}{\Delta x})^3$$

So, the stability condition becomes

$$\frac{\Delta t}{\Delta x^3}<\frac{1}{\pi^3}\approx0.0322515$$

2.3.2 By using RK4 method

To do stability analysis of the RK4 scheme for the KdV equation, we could use the approach used in analyzing the stability of the Fourier Leap-Frog schemes as in the previous subsection. To do this, we substitute $u(x,t)=\kappa^{t/\Delta t}e^{i\nu x}$ into (16).

After a very tedious and long derivation, we are led to the stability condition or after we numerically experiment with the scheme, we see that the appropriate time step is the one satisfying the condition

$$\frac{\Delta t}{\Delta x^3} \le 0.062$$

The right-hand side of the system of ODES in time given in (12) F (U) is

$$\mathbf{F(U_j)} = -\frac{2\pi}{L}\alpha u(x_j,t)F^{-1}\{ikF(u)\} - \beta F^{-1}\{-i\left(\frac{2\pi}{L}\right)^3 k^3 F(u)\}, \quad 0 \le j \le N-1.$$

2.4 Numerical Results and Examples

In order to show how good the numerical solutions are in comparison with the exact ones, we will use the L_2 and L_∞ error norms defined by

$$L_2 = \left\| u^{exact} - u^{num} \right\|_2 = \left[\Delta x \sum_{i=1}^{N} \left| u_i^{exact} - u_i^{num} \right|^2 \right]^{1/2},$$

$$L_\infty = \left\| u^{exact} - u^{num} \right\|_\infty = \max_i \left| u_i^{exact} - u_i^{num} \right|. \tag{17}$$

To implement the performance of the method, three test problems will be considered: the motion of a single solitary wave, the interaction of two positive solitary waves, the interaction of three positive solitary waves and other solutions.

2.4.1 The motion of a single solitary wave

Consider the KdV equation (5) with $\alpha = 6$, $\beta = 1$ and $L = 40$.

$$u_t + 6uu_x + u_{xxx} = 0, \qquad x \in [-L/2, L/2].$$

The simplest exact solution is

$$u(x,t) = 2\sec h^2(x-4t).$$

and initial condition

$$u(x,0) = 2\sec h^2(x).$$

Example 1

In this example we compute the numerical solutions u(x, 1) using the Fourier Leap-Frog scheme. The numerical solution at t = 0, 0.5, 1, 1.5 and 2 in Figure 3.1 with N = 128. It is clear from Figure 1 that the proposed method performs the motion of propagation of a solitary wave satisfactorily, which moved to the right with the preserved amplitude. In Figure 2, we plot the exact solution and Numerical solution at t = 1 with N = 128, Figure 3 show the error at each collection point at t = 1 with N =128, Table 1 displays the values of error norms obtained at N = 64, N = 128 and N = 256.

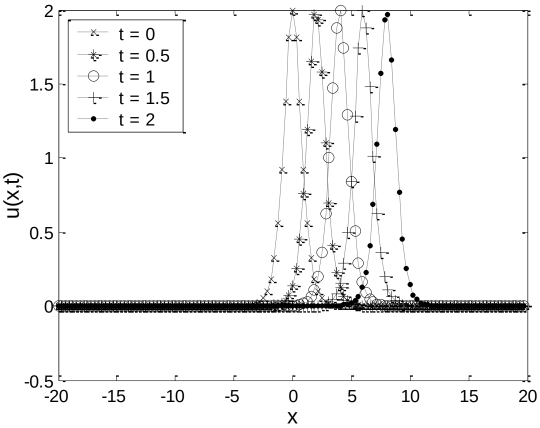


Fig. 1: Fourier spectral solution of the KdV equation using FLF scheme with N = 128 at t = 0, 0.5, 1, 1.5 and 2.

N	Δt	$L_2 \times 10^3$	$L_\infty \times 10^3$
64	0.0078739	213.5352	87.9812
128	0.000984	6.0557	4.5391
256	0.000123	0.9342	0.6951

Table 1: Error norms of FLF scheme of the KdV equation at t = 1 with N = 64,128 and 256.

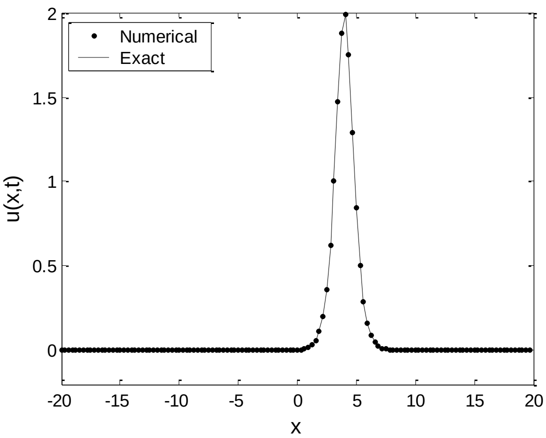


Fig. 2: Fourier spectral solution of the KdV equation using FLF scheme with N = 128 at t = 1

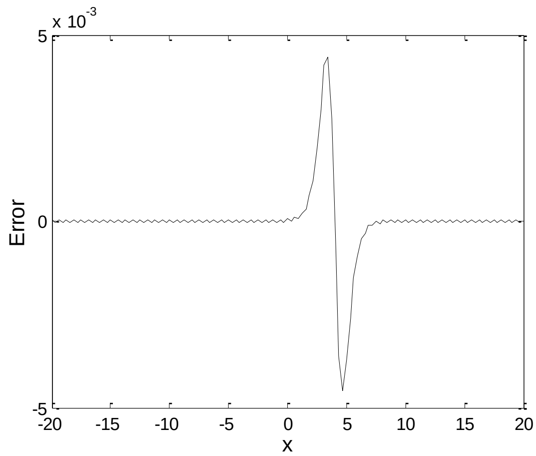


Fig. 3: The error (error = exact – numerical) distributions in FLF scheme for the KdV equation with N = 128 at t = 1.

Example 2

Now, we solve the same problem using the RK4 scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. Figure 4 shows simulation of the solution computed using N = 128. In Figure 5, we plot the exact solution and Numerical solution at t = 1 with N = 128, Figure 6 show the error at each collection point at t = 1 with N =128. Table 2 displays error norms obtained at N = 64, N = 128 and N = 256. From these results we can see that by carefully choosing the time steps, RK4 is more accurate than the Fourier Leap-frog, but Fourier Leapfrog is easier than RK4.

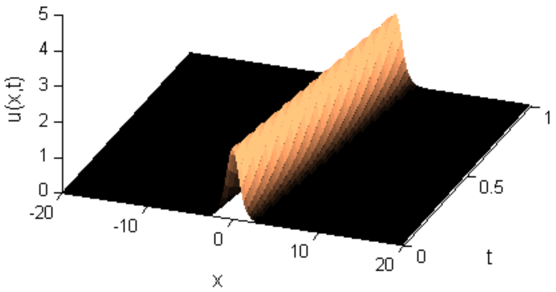


Fig. 4: Numerical simulation of Fourier spectral solution of the KdV equation using RK4 scheme with N = 128.

N	Δt	$L_2 \times 10^3$	$L_\infty \times 10^3$
64	0.0152	159.58	69.0868
128	0.0019	2.7825	3.67
256	0.000248	0.4993	0.6354

Table 2: Error norms of Fourier spectral solution of the KdV equation using RK4 scheme at t = 1 with N = 64,128 and 256.

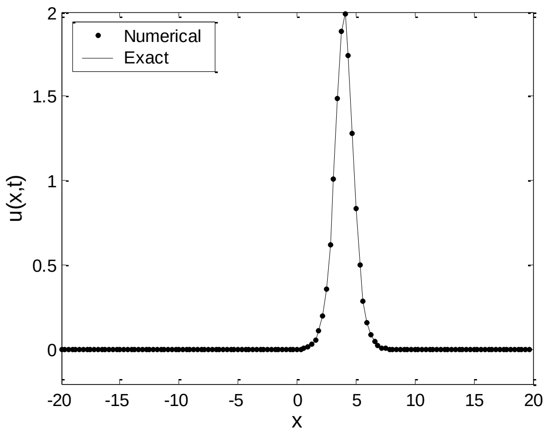


Fig. 5: Fourier spectral solution of the KdV equation using RK4 scheme at t = 1 with N = 128.

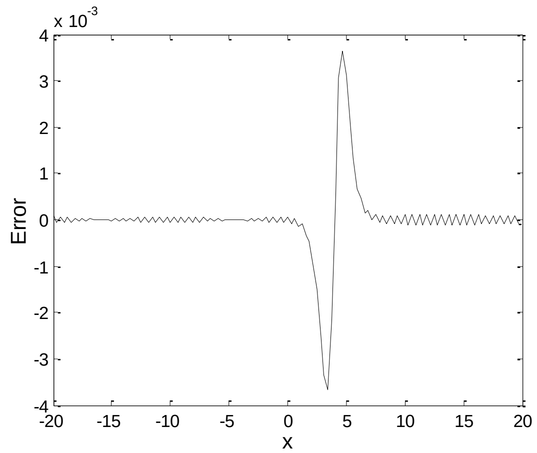


Fig. 6: The error distributions in Fourier spectral solution of the KdV equation using RK4 scheme at t = 1 with N = 128.

2.4.2 The interaction of two positive solitary waves

Consider the KdV equation (5) with $\alpha = 6$, $\beta = 1$, and L = 40. The initial data is

$$u(x,0) = n(n+1)\operatorname{sec}h^2(x) \tag{18}$$

results in *n* solitons solution that propagate with different velocities. The initial for n = 2 is

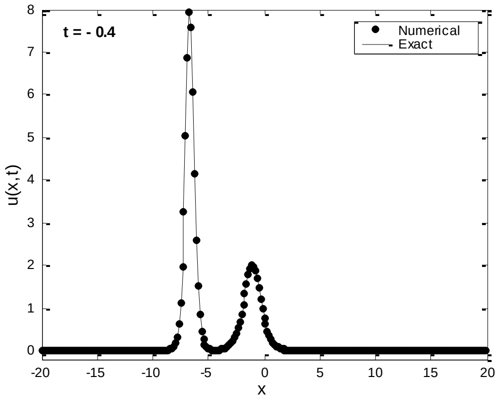
$$u(x,0) = 6\, sech^2(x) \tag{19}$$

The exact solution can be expressed as [11]

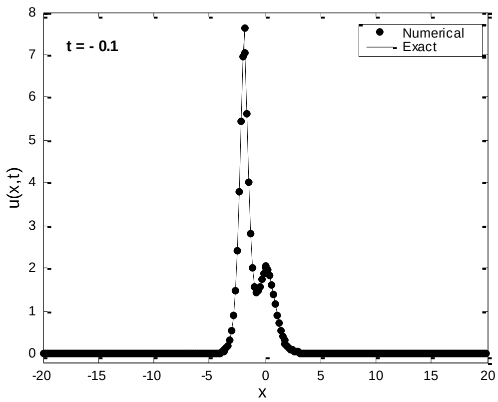
$$u(x,t) = 12\frac{3+4\cosh(2x-8t)+\cosh(4x-64t)}{[3\cosh(x-28t)+\cosh(3x-36t)]^2}$$

Example 3

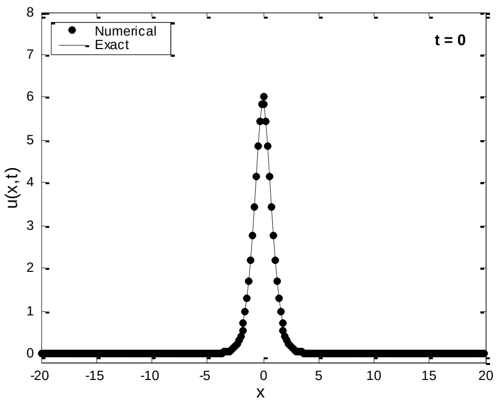
We solve equation (5) with initial solution (19) using the FLF scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. We plot the exact solution and the numerical solutions at t = -0.4,-0.1, 0, 0.1, and 0.4 at N = 256 and $\Delta t = 0.000123$ in Figure 7.



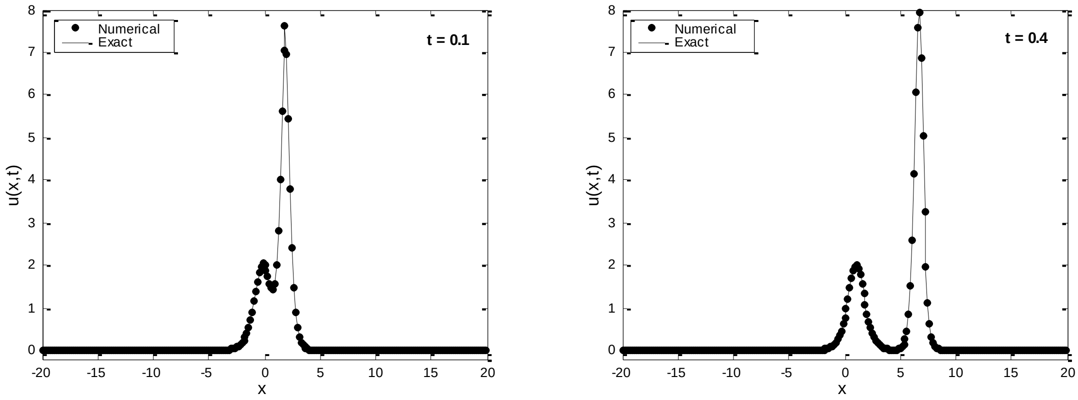
(a)



(b)



(c)



(d) (e)

Fig. 7: Fourier spectral solution for interaction of two waves of the KdV equation using FLF scheme with initial condition (19) and N =256.

Example 4

We solve example 3 using the RK4 scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. Figure 8 show simulation of the solution computed. We plot the exact solution and the numerical solutions at t = -0.4,-0.1, 0, 0.1, and 0.4 with N = 256 and $\Delta t = 0.000237$ in Figure 9.

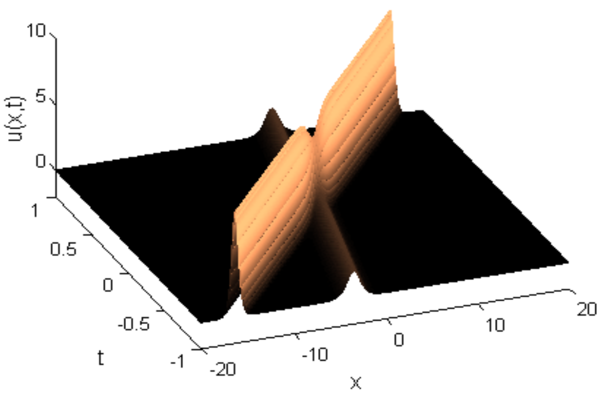
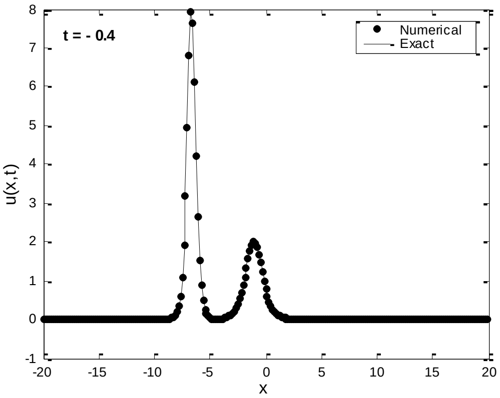
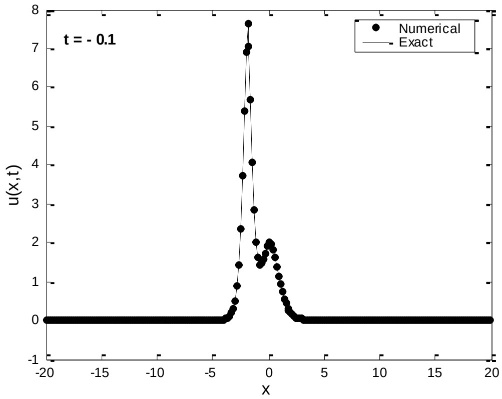


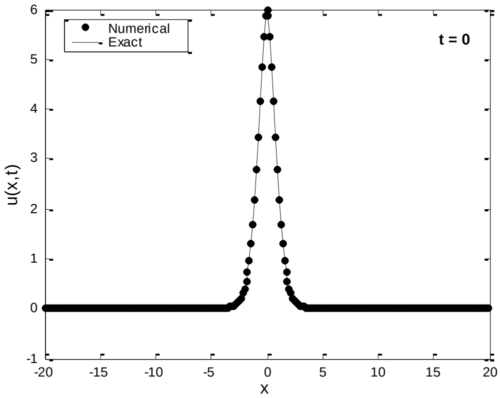
Fig. 8: Numerical simulation of Fourier spectral of interaction of two waves of the KdV equation using RK4 scheme with initial condition (19) and N =256.



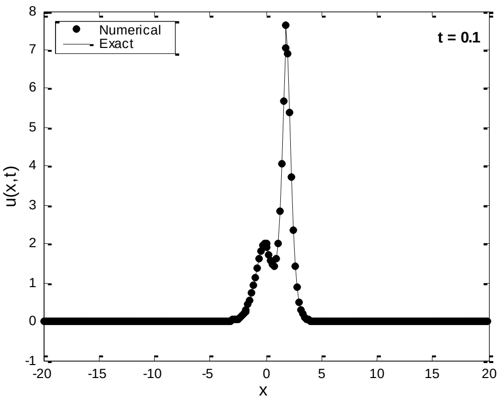
(a)



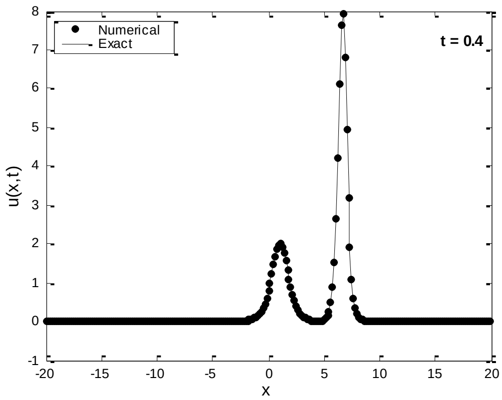
(b)



(c)



(d)



(e)

Fig. 9: Fourier spectral solution for interaction of two waves of the KdV equation using RK4 scheme with initial condition (19) and N =256**.**

In next method, we can study the interaction of n solitary waves by using the initial condition given by the linear sum of n separate solitary waves of various amplitudes

$$u(x,0)=\sum_{i=1}^{n}0.5c_i \sec h^2\left(0.5c_i^{1/2}\left(x-x_i\right)\right). \tag{20}$$

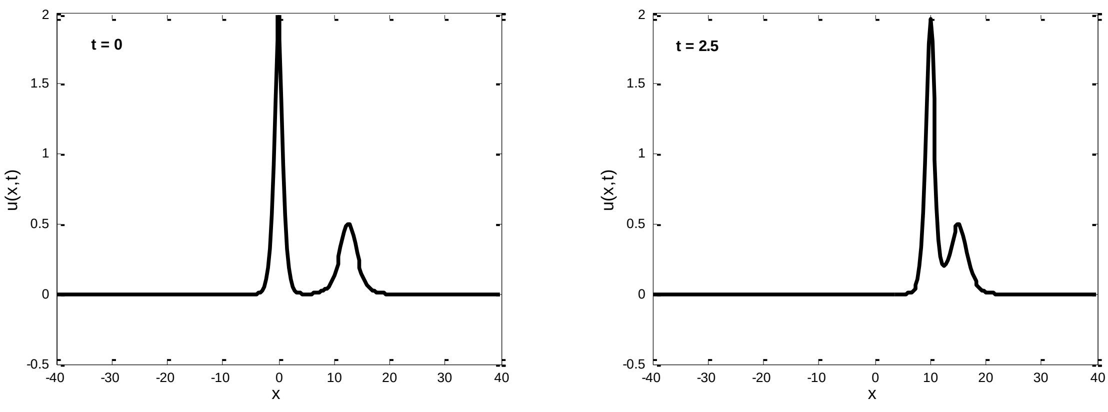
Interaction of two positive solitary waves can be studied by using the initial condition given by the linear sum of two separate solitary waves of various amplitudes

$$u(x,0)=u_1(x,0)+u_2(x,0)$$

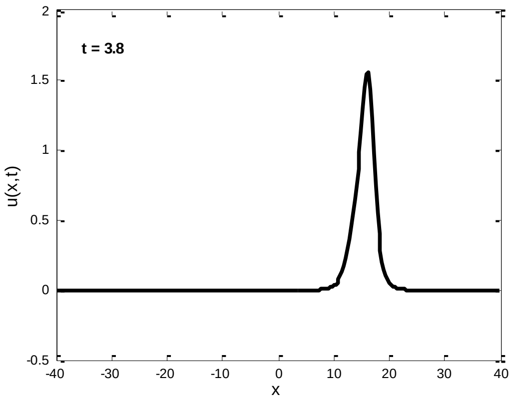
$$u(x,0)=2\sec h^2(x)+0.5\sec h^2(0.5(x-4\pi)) \tag{21}$$

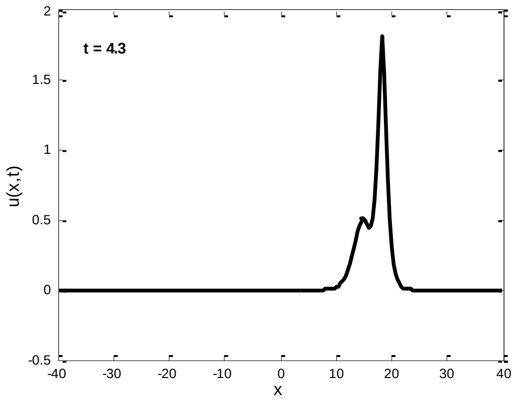
Example 5

The calculation is carried out with the time step Δt = 0.000984 and N = 256 over the region -40 $\leq x \leq$ 40, we solve using the FLF scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. We plot the numerical solutions at t = 0, 2.5, 3.8, 4.3 and 6.5 with N = 256, in Figure 10, respectively. The initial function was placed on the left side of the region with the larger wave to the left of the smaller one as seen in the Figure 10, both waves move to the right with velocities dependent upon their magnitudes. The shapes of the two solitary waves is graphed during the interaction at t = 3.8 and after the interaction at time t = 6.5, which is seen to have separated the larger wave from the smaller one. According to the figure, the larger wave catches up the smaller wave at about t = 2.5, the overlapping process continues until the time t = 5, then two solitary waves emerge from the interaction and resume their former shapes and amplitudes.

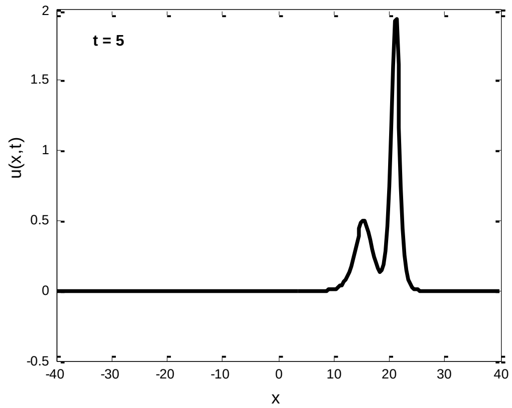


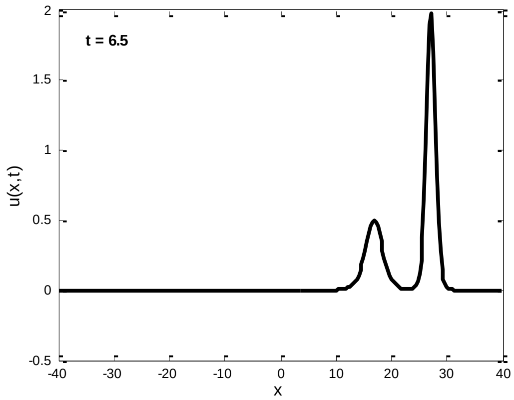
(a) (b)





(c) (d)





(e) (f)

Fig. 10: Fourier spectral solution for interaction of two waves of the KdV equation using FLF scheme with initial condition (21) and N = 256**.**

Example 6

We solve example 5 using the RK4 scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. The calculation is carried out with the time step Δt = 0.0019 and N = 256 over the region $-40 \le x \le 40$, Figure 11 shows simulation of the solution computed. We plot the numerical solutions at t = 0, 2.5, 3.8, 4.3 and 6.5 with N = 256, in Figure 12, respectively.

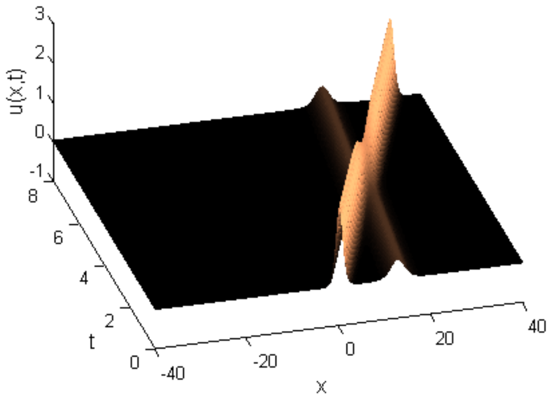
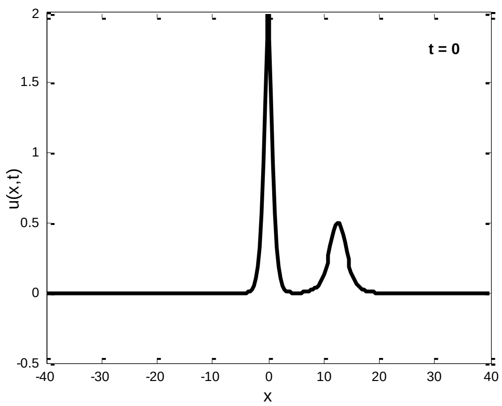
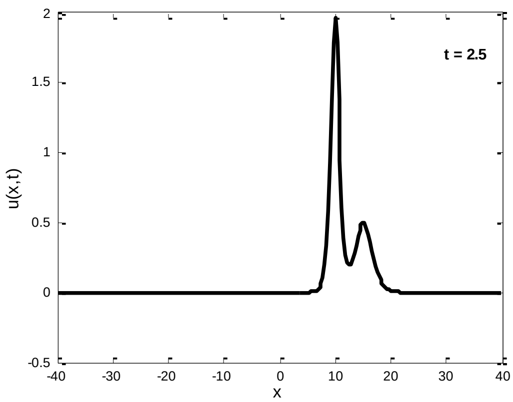


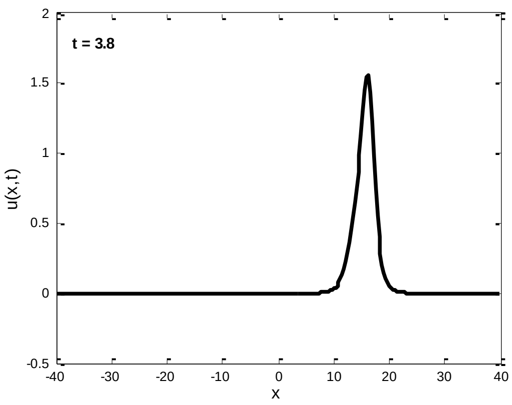
Figure 11: Numerical simulation of Fourier spectral of interaction of two waves of the KdV equation using RK4 scheme with initial condition (21) and N =256**.**



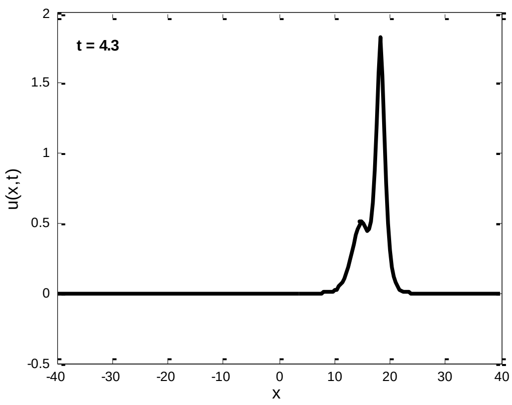
(a)



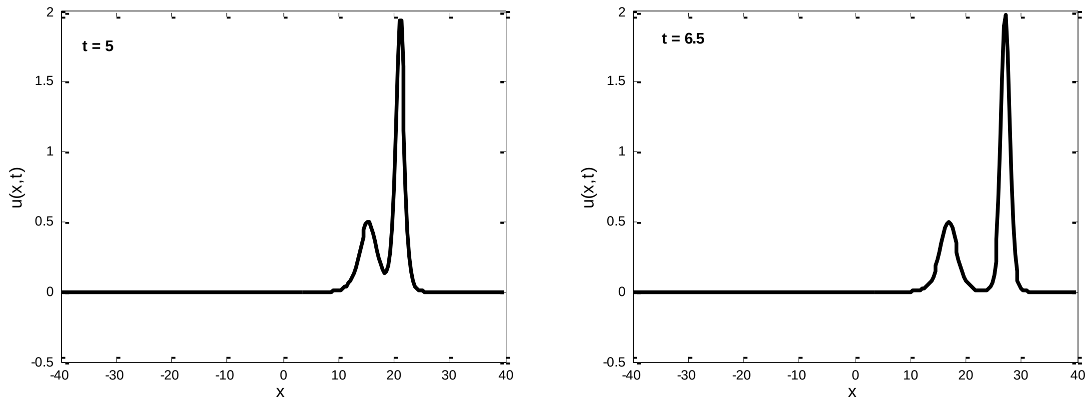
(b)



(c)



(d)



(e) (f)

Fig. 12: Fourier spectral solution for interaction of two waves of the KdV equation using RK4 scheme with initial condition (21) and N =256.

2.4.3 The interaction of three positive solitary waves

Consider the KdV equation (5) with $\alpha = 6$, β =1, and L = 40

$$u_t + 6uu_x + u_{xxx} = 0, \qquad x \in [-20, 20],$$

The initial for n =3 in (18) is

$$u(x,0) = 12\sec h^2(x) \tag{22}$$

Example 7

We solve (5) with initial condition (22).The calculation is carried out with the time step Δt = 0.000123 and N = 256 over the region -20 ≤ x ≤ 20, we solve using the FLF scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. We plot the numerical solutions at t = - 0.3, - 0.1, - 0.05,0, 0.05 and 0.3 with N =256, in Figure 13, respectively.

Example 8

We solve example 7 using the RK4 scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. The calculation is carried out with the time step Δt = 0.000237 and N = 256 over the region -20 ≤ x ≤ 20, we plot the numerical solutions at t = - 0.3, - 0.1, - 0.05,0, 0.05, and 0.3 with N =256, in Figure 14, respectively.

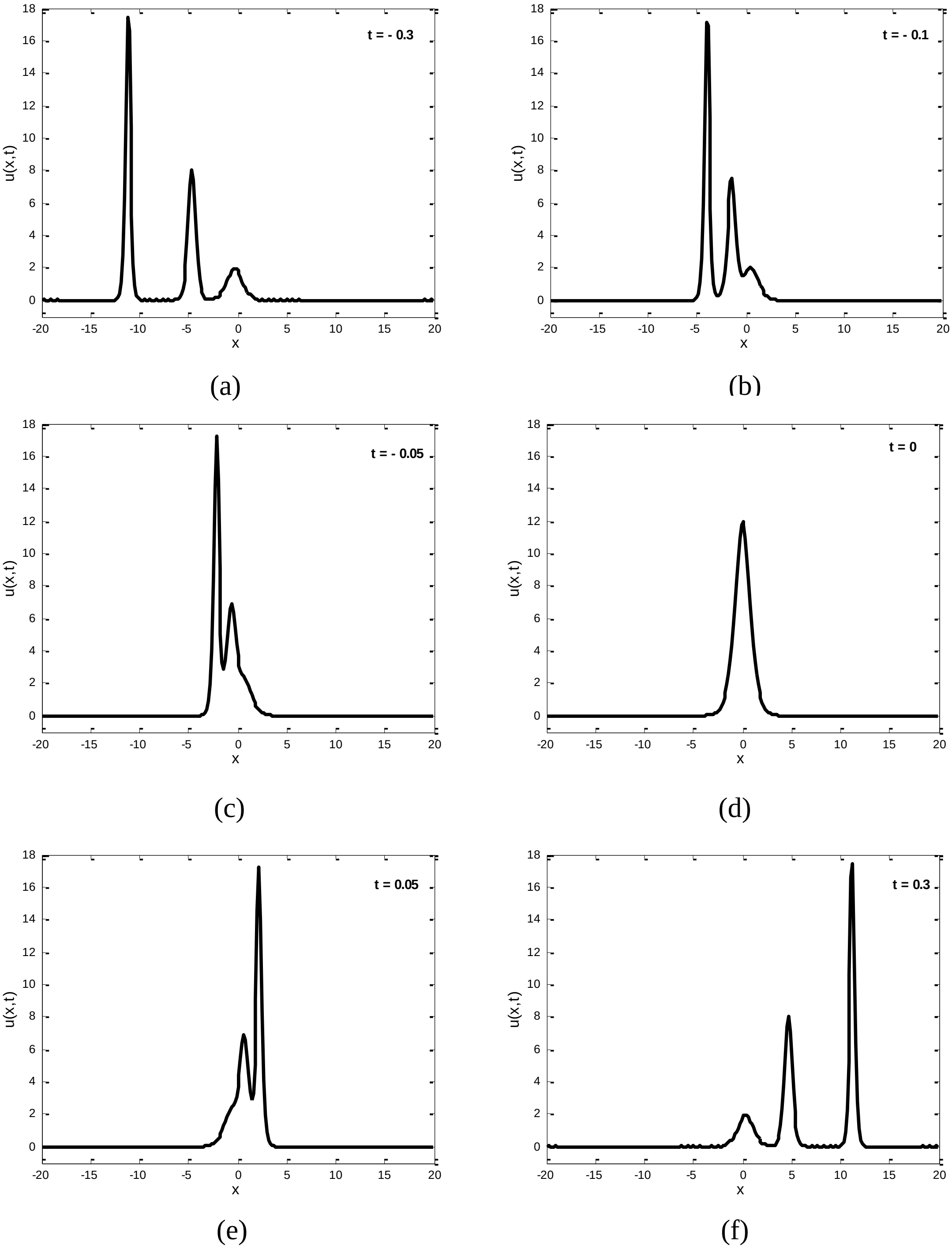
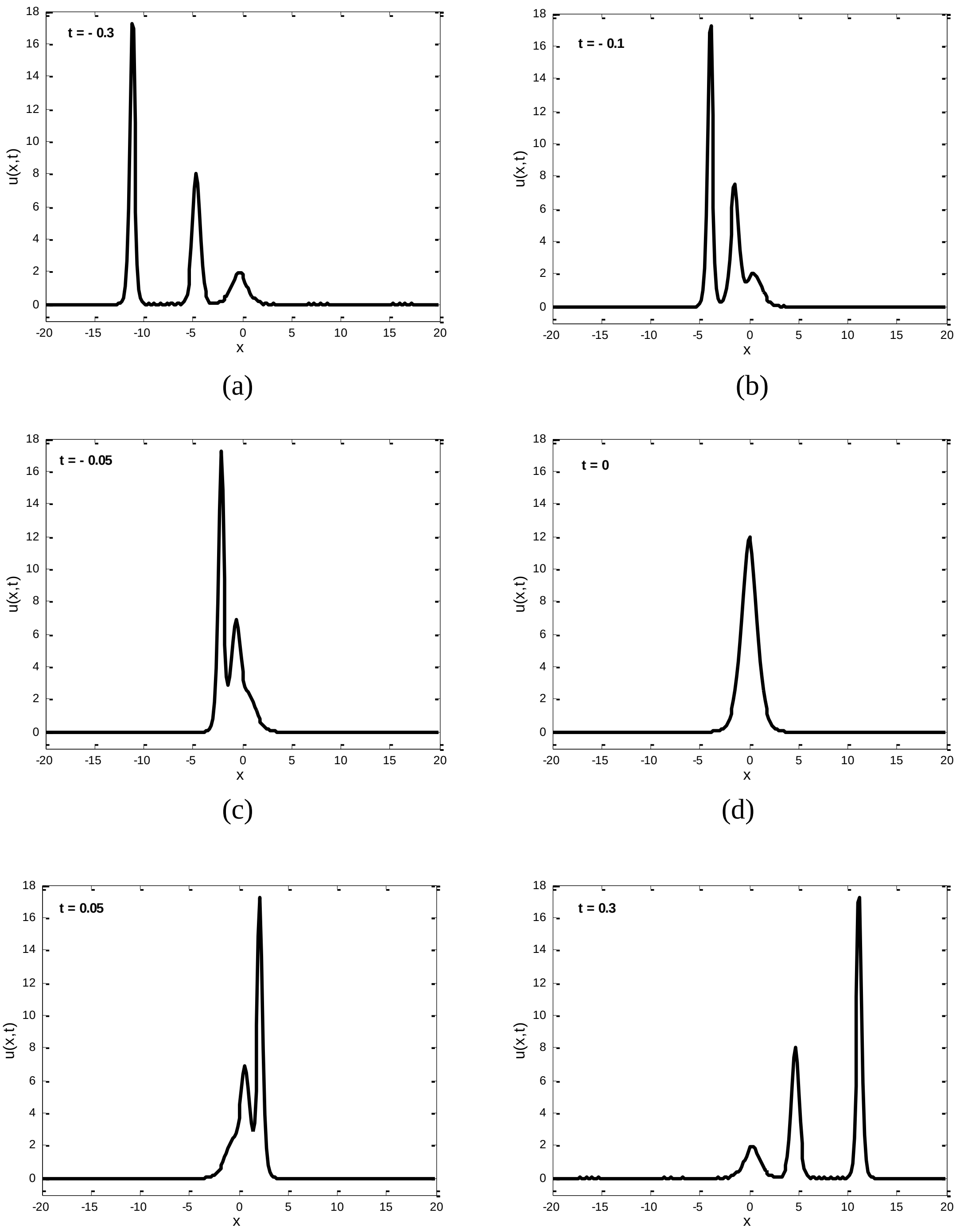


Fig. 13: Fourier spectral solution for interaction of three waves of the KdV equation using FLF scheme with initial condition (22) and N =256**.**



(a) (b) (c) (d) (e) (f)

Fig. 14: Fourier spectral solution for interaction of three waves of the KdV equation using RK4 scheme with initial condition (22) and N = 256**.**

In this interaction of three positive solitary waves is studied by using the initial condition given by the linear sum of three separate solitary waves of various amplitudes from (20)

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0)$$

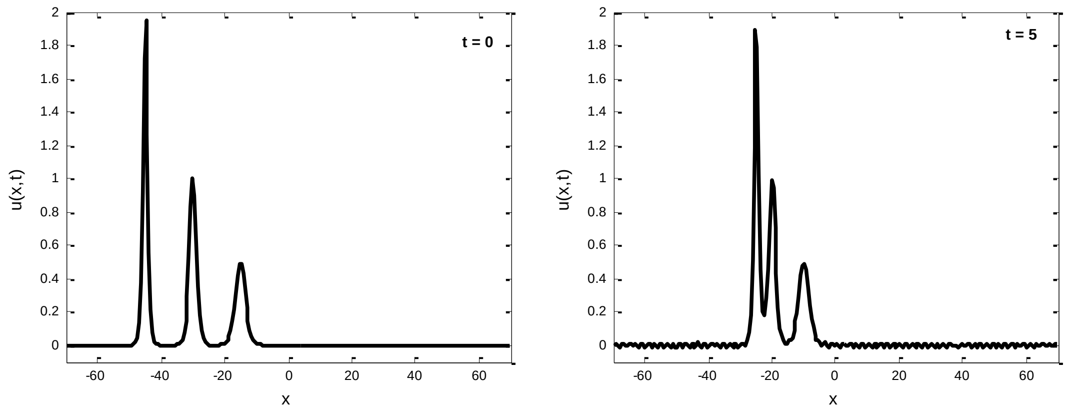
$$u(x,0) = 2\sec h^2(x+45) + \sec h^2(0.5(2)^{1/2}(x+30)) + 0.5\sec h^2(0.5(x+15)) \qquad (23)$$

Example 9

We solve (5) using initial condition (23).The calculation is carried out with the time step Δt = 0.00527 and N = 256 over the region -70 ≤ x ≤ 70, we solve using the FLF scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. Figure 15 shows the numerical solution at t = 0, 5, 8, 10, 12, 18.As can be seen the three pulses travel with time to the right. But the taller soliton moves faster, hence the three occasionally merge and then split apart again.

Example 10

We solve example 9 using the RK4 scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. The calculation is carried out with the time step Δt = 0.01 and N = 256 over the region $-70 \le x \le 70$, we solve using the RK4 scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. Figure 16 and 17 shows simulation and plane view of the solution computed.



(a) (b)

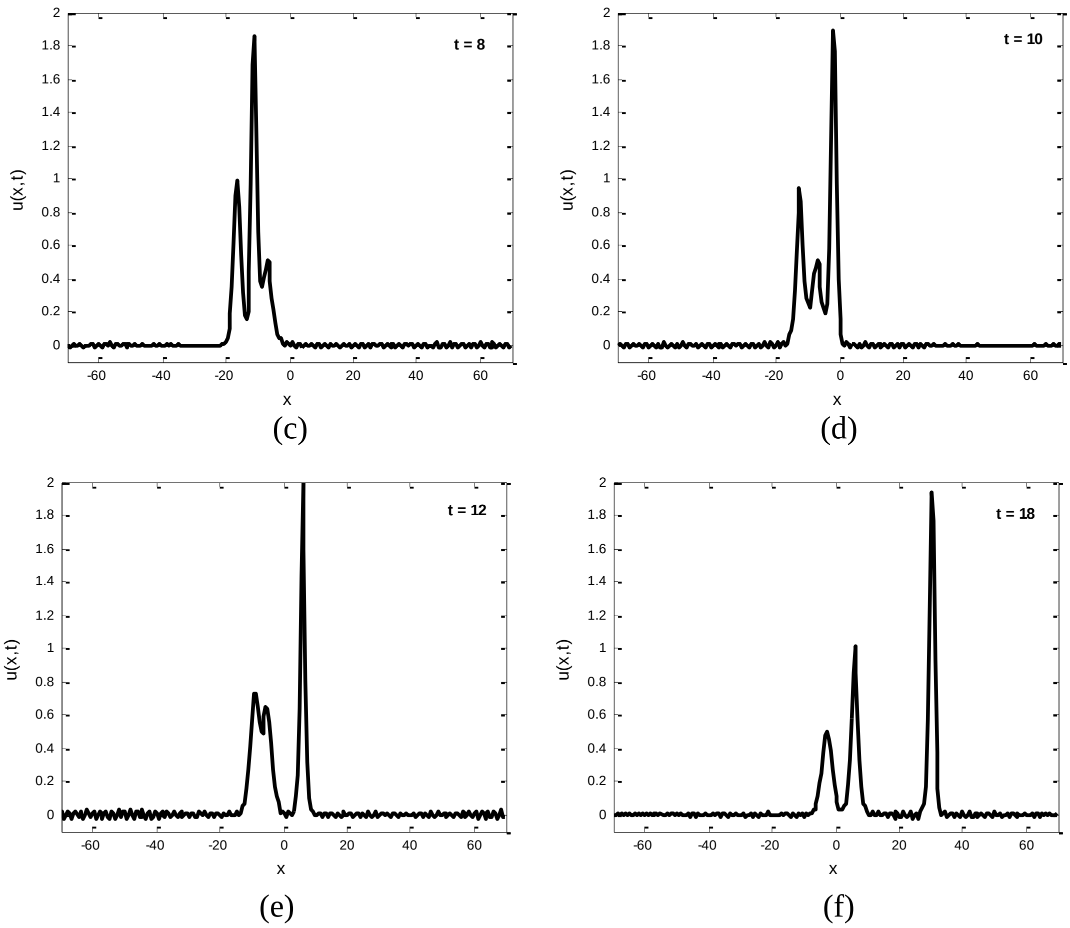


Fig. 15: Fourier spectral solution for interaction of three waves of the KdV equation using FLF scheme with initial condition (22) and N = 256.

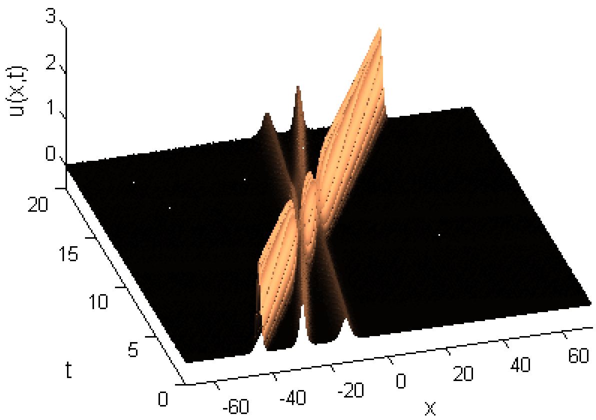


Fig. 16: Numerical simulation of Fourier spectral of interaction of three waves of the KdV equation using RK4 scheme with initial condition (22) and N =256.

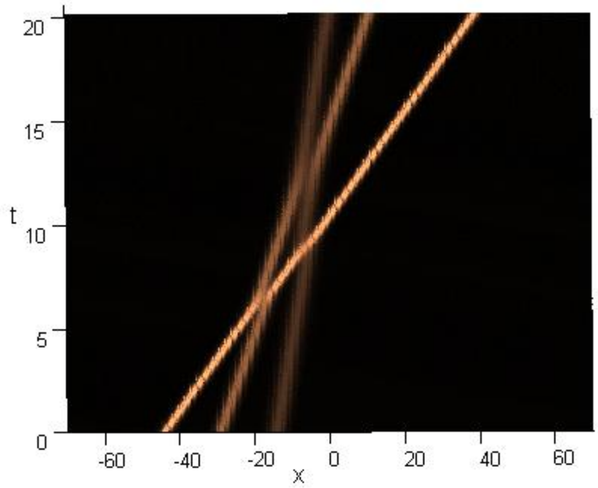


Fig. 17: Plan view of Fourier spectral of interaction of three waves of the KdV equation using RK4 scheme with initial condition (22) and N = 256.

2.4.4 Other solutions

Now suppose that the initial condition is such that is does not just produce one or more solitons. Let us choose as initial condition [11]

$$u(x,0) = 4\sec h^2(x) \tag{24}$$

with $\alpha = 6$, $\beta = 1$, $L = 40$,$\Delta t = 0.0019$ and $N = 128$ using the RK4 scheme to march the solution in time and the Fourier spectral method to take care of the spatial domain. Four is not of the form $n\ (n + 1)$.The peak moving to the right. In addition, there are waves moving to the left. These will disperse and lose their form with time.

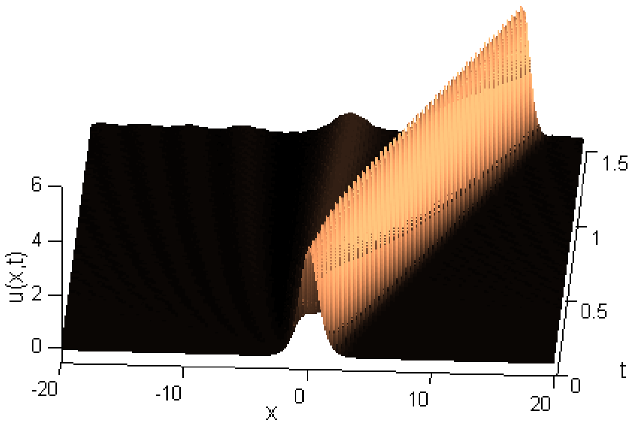


Fig. 18: Numerical simulation of Fourier spectral solution of the KdV equation using RK4 scheme with initial condition (24) and N = 128.

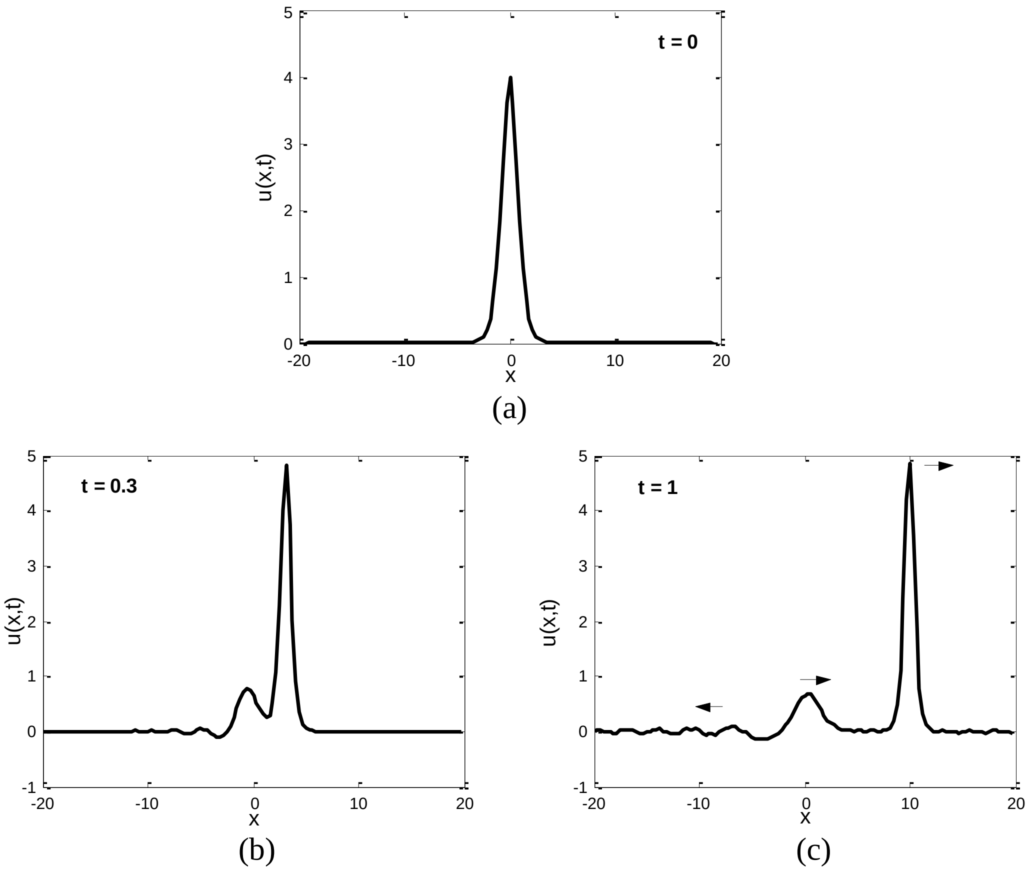


Fig. 19: Fourier spectral solution of the KdV equation using RK4 scheme with initial condition (24) and N =128.

3 Analysis the method for the Boussinesq equation

3.1 Fourier based RK4 method for Boussinesq equation

Consider the Boussinesq equation (3) in the following form:

$$u_{tt} = u_{xx} - 6uu_{xx} - 6u_x^2 - u_{xxxx}, \qquad x \in [a, z] \tag{25}$$

We solve this equation by combination of RK4 method with respect to t and a Fourier pseudospectral method with respect to *x*. To prepare the equation for numerical solution we introduce the auxiliary variable $v = u_t$. This reduces the second-order equation in time to the first order system

$$\begin{aligned} u_t &= v \\ v_t &= u_{xx} - 6uu_{xx} - 6u_x^2 - u_{xxxx} \end{aligned} \tag{26}$$

We need two initial conditions. The initial conditions we use to numerically solve equation (25) can thus be extracted from the above relation (6) for $t = 0$ at $u(x, t)$ and $u_t(x,t)$

$$u(x,0) = 2b^2 \sec h^2(b(x - x_1))$$

$$v(x,0) = u_t(x,0) = -4b^3 c \sec h^2(b(x - x_1)) \tanh(b(x - x_1)) \tag{27}$$

We have changed the solution interval from [a, z] to [0,2π], with the change of variable.

$$x \to \frac{2\pi}{L}(x - a)$$

where L = z – a, thus the equation (4) become

$$\begin{aligned} u_t &= v \\ v_t &= \left(\frac{2\pi}{L}\right)^2 u_{xx} - 6\left(\frac{2\pi}{L}\right)^2 uu_{xx} - 6\left(\frac{2\pi}{L}\right)^2 u_x^2 - \left(\frac{2\pi}{L}\right)^4 u_{xxxx} \end{aligned} \tag{28}$$

$u(x, t)$ is transformed into Fourier space with respect to *x*, and derivatives (or other operators) with respect to *x*. Applying the inverse Fourier transform using

$$u_x = F^{-1}\{ikF(u)\}$$

$$u_{xx} = F^{-1}\{-k^2F(u)\}$$

$$u_{xxxx} = F^{-1}\{k^4F(u)\}$$

The equation (28) becomes

$$
\begin{aligned}
u_t &= v \\
v_t &= \left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 uF^{-1}\{-k^2F(u)\} \\
&\quad -6\left(\frac{2\pi}{L}\right)^2 \left[F^{-1}\{ikF(u)\}\right]^2 - \left(\frac{2\pi}{L}\right)^4 \left[F^{-1}\{k^4F(u)\}\right],
\end{aligned}
\tag{29}
$$

In practice, we need to discretize the equation (26). For any integer N > 0, we consider

$$x_j = j\Delta x = \frac{2\pi}{N},\ j = 0,1,\ldots,N-1.$$

Let u (x, t), $v(x, t)$ be the solution equation (29). Then, we transform it into the Discrete Fourier space as

$$
\begin{aligned}
\hat{u}(k,t) = F(u) &= \frac{1}{N}\sum_{J=0}^{N-1} u(x_j,t)\, e^{-ikx_j},\ -\frac{N}{2} \le k \le \frac{N}{2} - 1 \\
\hat{v}(k,t) = F(v) &= \frac{1}{N}\sum_{J=0}^{N-1} v(x_j,t)\, e^{-ikx_j},\ -\frac{N}{2} \le k \le \frac{N}{2} - 1.
\end{aligned}
$$

from this, using the inversion formula, we get

$$
\begin{aligned}
u(x_j,t) = F^{-1}(\hat{u}) &= \sum_{k=-N/2}^{N/2-1} \hat{u}(k,t)\, e^{ikx_j}, \quad 0 \le j \le N-1 \\
v(x_j,t) = F^{-1}(\hat{v}) &= \sum_{k=-N/2}^{N/2-1} \hat{v}(k,t)\, e^{ikx_j}, \quad 0 \le j \le N-1
\end{aligned}
$$

replacing F and F^{-1} in (29) by their discrete counterparts, and discretizing (29) gives

$$
\begin{aligned}
\frac{du(x_j,t)}{dt} &= v(x_j,t) \\
\frac{dv(x_j,t)}{dt} &= \left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 u(x_j,t)F^{-1}\{-k^2F(u)\} \\
&\quad -6\left(\frac{2\pi}{L}\right)^2 \left[F^{-1}\{ikF(u)\}\right]^2 - \left(\frac{2\pi}{L}\right)^4 \left[F^{-1}\{k^4F(u)\}\right]
\end{aligned}
\tag{30}
$$

Letting $\mathrm{U} = [u(x_0,t), u(x_1,t), \cdots, u(x_{N-1},t)]^T$,

$\mathrm{V} = [v(x_0,t), v(x_1,t), \cdots, v(x_{N-1},t)]^T$,

$$\mathbf{w} = \begin{bmatrix} \mathbf{U} \\ \mathbf{V} \end{bmatrix}, \quad \mathbf{w_t} = \begin{bmatrix} \mathbf{U_t} \\ \mathbf{V_t} \end{bmatrix}.$$

The system of equations (30) can be written in the vector form

$$\mathbf{w_t} = \mathbf{F}\,(\mathbf{U}, \mathbf{V}) \tag{31}$$

where F defines the right hand side of (30).

3.2 Combination of finite differences and a Fourier pseudospectral method

The numerical scheme used is based on a combination of finite differences and a Fourier pseudospectral method.
After we have changed the solution interval from [a, z] to [0,2π] and $u(x, t)$ is transformed into Fourier space with respect to x, and derivatives (or other operators) with respect to x, equation (25) become

$$u_{tt} = \left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 uF^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 \left[F^{-1}\{ikF(u)\}\right]^2 - \left(\frac{2\pi}{L}\right)^4 \left[F^{-1}\{k^4F(u)\}\right]. \tag{32}$$

In practice, we need to discretize the equation (32). For any integer N > 0, Let u (x, t) be the solution equation (32). Then, we transform it into the Discrete Fourier space using the inversion formula, we get

$$\frac{d^2u(x_j,t)}{dt^2} = \left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 u(x_j,t)F^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 \left[F^{-1}\{ikF(u)\}\right]^2 - \left(\frac{2\pi}{L}\right)^4 \left[F^{-1}\{k^4F(u)\}\right]. \tag{33}$$

$[u(x_0,t),u(x_1,t),\cdots,u(x_{N-1},t)]^T$. Letting U =

The equation (33) can be written in the vector form

$$\mathbf{U_{tt} = F\,(U)} \tag{34}$$

where F defines the right hand side of (33).
The time derivative in equation (34) is discretized using a finite difference approximation, in terms of central differences

$$\frac{u^{n+1} - 2u^n + u^{n-1}}{(\Delta t)^2} = \mathbf{F\,(U)}$$

or

$$u(x,t+\Delta t) = 2u(x,t) - u(x,t-\Delta t) + (\Delta t)^2\left[\left(\frac{2\pi}{L}\right)^2 F^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 u(x_j,t)F^{-1}\{-k^2F(u)\} - 6\left(\frac{2\pi}{L}\right)^2 \left[F^{-1}\{ikF(u)\}\right]^2 - \left(\frac{2\pi}{L}\right)^4 \left[F^{-1}\{k^4F(u)\}\right]\right]. \tag{35}$$

we need two level initial conditions $u\,(x,-\Delta t)$,$u\,(x,0)$. The initial condition $u\,(x, 0)$ given in (27), from the central difference of $u_t(x,t)$

$$u_t(x,t)=\frac{u^{n+1}-u^{n-1}}{2\Delta t}$$

where Δt is time step, we have the approximation

$$u^{n-1}=u^{n+1}-2\Delta t u_t(x,t)$$

we can get $u\,(x,-\Delta t)$

$$u(x,-\Delta t)=u(x,\Delta t)-2\Delta t u_t(x,0)$$
$$=u(x,\Delta t)-2\Delta t\left[\,4b^3c\operatorname{sec}h^2(b(x-x_1))\tanh(b(x-x_1))\right].$$

then we substitute $u\,(x,-\Delta t)$ and $u\,(x,0)$ in (34) to get $u\,(x,\Delta t)$

$$u(x,\Delta t)=u(x,0)+\Delta t\,u_t(x,0)+\frac{(\Delta t)^2}{2}\left[\left(\frac{2\pi}{L}\right)^2F^{-1}\{-k^2F(u)\}-6\left(\frac{2\pi}{L}\right)^2\right.$$
$$\left.u(x_j,0)F^{-1}\{-k^2F(u)\}-6\left(\frac{2\pi}{L}\right)^2\left[F^{-1}\{ikF(u)\}\right]^2-\left(\frac{2\pi}{L}\right)^4\left[F^{-1}\{k^4F(u)\}\right]\right]. \quad (36)$$

So we substitute $u\,(x,0)$ and $u\,(x,\Delta t)$ in (34) to get $u\,(x,2\Delta t)$ and so on until we get $u(x,t)$ at time t. Various values of N (128 to 1024) and time step Δt = 0.0001 to 0.02.

3.3 Numerical results and examples

In order to show how good the numerical solutions are in comparison with the exact ones, we will use L_2 and L_∞ error norms.

$$L_2=\left\|u^{exact}-u^{num}\right\|_2=\left[\Delta x\sum_{i=1}^{N}\left|u_i^{exact}-u_i^{num}\right|^2\right]^{1/2}$$

$$L_\infty=\left\|u^{exact}-u^{num}\right\|_\infty=\max_i\left|u_i^{exact}-u_i^{num}\right|.$$

To implement the performance of the method, test problems will be considered: the motion of a single solitary wave in right direction, the motion of a single solitary wave in left direction. Several tests have been made for the wave solution of the Boussinesq verifying that for various values of N (128 to 1024) and time step Δt = 0.0001 to 0.02 using Combination of finite differences and a Fourier pseudospectral method. As b increases, the stability of the wave propagating in time breaks down and for values of $|b|$ close to 0.5 the wave blows up. For stability, we have found that the maximum value of b is 0.4.

We take the Boussinesq equation of form (25) with periodic boundary condition

$$u(a,t)=u(z,t)=0$$

Example 11

We solve the equation (25) with the initial conditions (27) ,with b = 0.2, $x_1 = 0$, $c = +\sqrt{1-4b^2}$ and Δx = 1, with Δt = 0.001 .It is clear from Figure 1 that the proposed method performs the motion of propagation of a solitary wave satisfactorily, which moved to the right with the preserved amplitude.

t	$L_2 \times 10^3$	$L_\infty \times 10^3$
10	3.3286	1.2964
20	6.4411	2.3372
30	9.5471	3.3493
40	12.6338	4.3381

Table 3: Error norms of Fourier spectral solution of the Boussinesq equation using RK4 scheme at t =10, 20, 30 and 40 with N = 128.

N	$L_2 \times 10^3$	$L_\infty \times 10^3$	Amplitude
128	12.6338	4.3381	0.0792055
256	6.3224	2.1867	0.0795634
512	3.1624	1.0945	0.0797424

Table 4: Error norms of Fourier spectral solution of the Boussinesq equation using RK4 scheme with N = 128, 256 and 512.

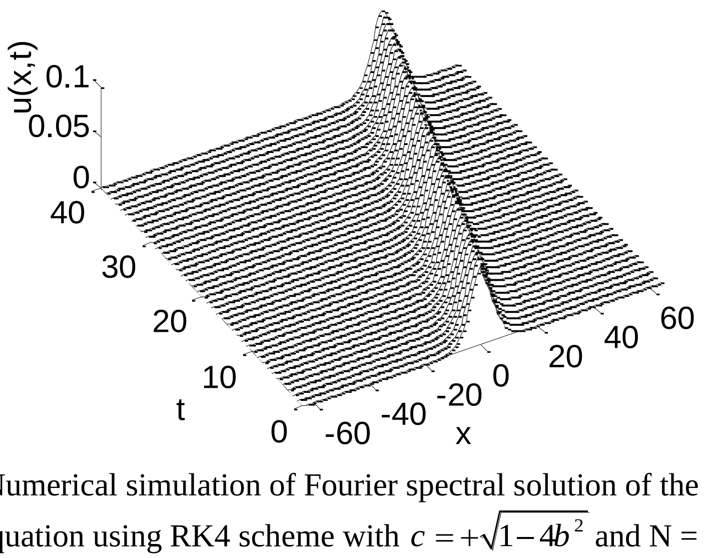


Fig. 20 Numerical simulation of Fourier spectral solution of the Boussinesq equation using RK4 scheme with $c = +\sqrt{1-4b^2}$ and N = 128.

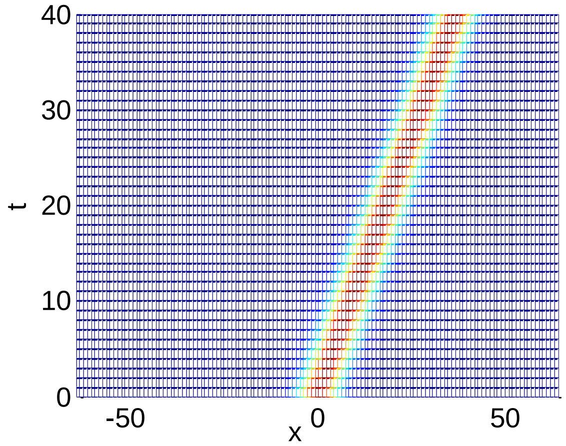


Fig. 21. Plan view of Fourier spectral solution of the Boussinesq equation using RK4 scheme with $c = +\sqrt{1-4b^2}$ and N = 128.

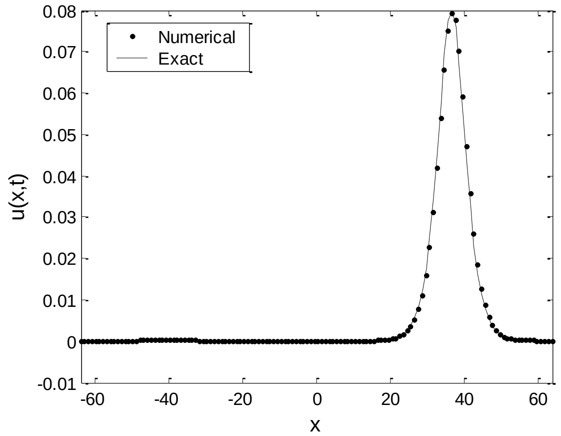


Fig. 22. Fourier spectral solution of the Boussinesq equation using RK4 scheme at t = 40 with $c = +\sqrt{1-4b^2}$ and N = 128.

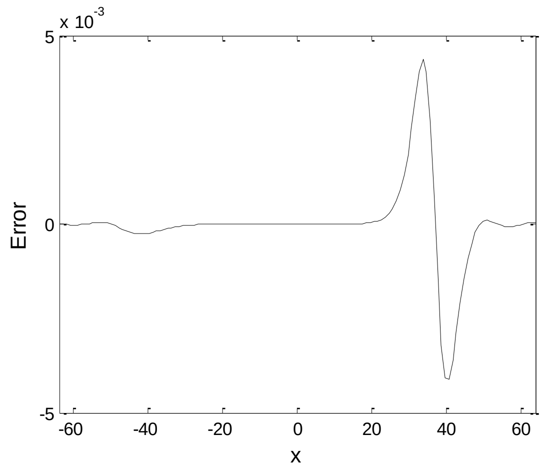


Fig. 23. The error distributions in Fourier spectral solution of the Boussinesq equation using RK4 scheme at t = 40 with $c = +\sqrt{1-4b^2}$ and N = 128.

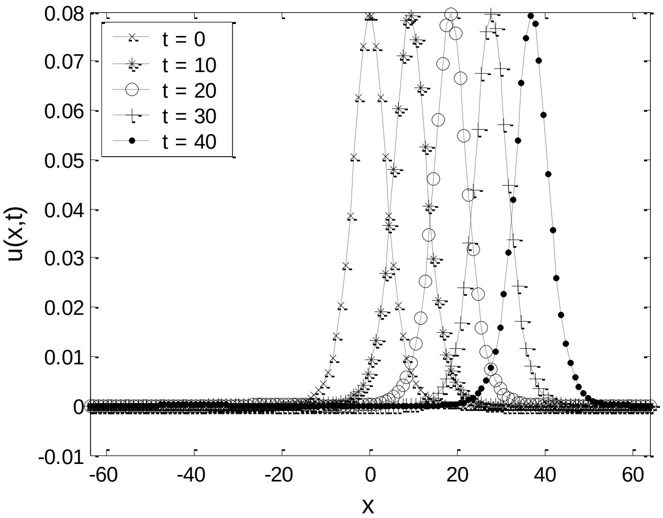


Fig. 24. Fourier spectral solution of the Boussinesq equation using a finite difference scheme at t = 0, 10, 20, 30 and 40 with $c = +\sqrt{1-4b^2}$ and N = 128.

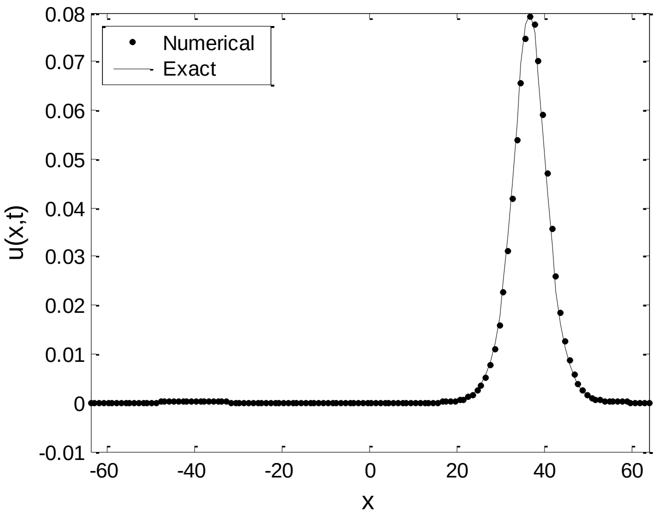


Fig.25. Fourier spectral solution of the Boussinesq equation using a finite difference scheme at t = 40 with $c = +\sqrt{1-4b^2}$ and N = 128.

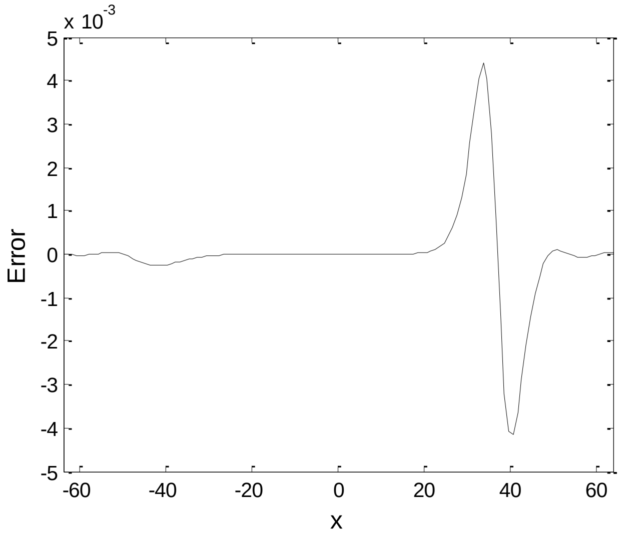


Fig. 26.The error distributions of Fourier spectral solution of the Boussinesq equation using a finite difference scheme at t = 40 with $c = +\sqrt{1-4b^2}$ and N = 128.

t	$L_2 \times 10^3$	$L_\infty \times 10^3$
10	3.3500	1.3185
20	6.4407	2.3477
30	9.5468	3.3693
40	12.6671	4.3895

Table 5. Error norms of Fourier spectral solution of the Boussinesq equation using a finite difference scheme at t = 0, 10, 20, 30 and 40 with N = 128.

N	$L_2 \times 10^3$	$L_\infty \times 10^3$	Amplitude
128	12.6671	4.3895	0.0792055
256	6.3558	2.2006	0.0795644
512	3.1957	1.1056	0.0797412
1024	1.6148	0.5583	0.0798291

Table 6. Error norms for the single soliton of Fourier spectral solution of the Boussinesq equation using a finite difference scheme with N = 128,256, 512 and 1024.

Now, we solve the same problem with b = 0.2, $x_1 = 0$, $c = -\sqrt{1-4b^2}$ and $\Delta x = 1$ with $\Delta t = 0.001$. It is clear from Figure 27 that the proposed method performs the motion of propagation of a solitary wave satisfactorily, which moved to the left with the preserved amplitude.

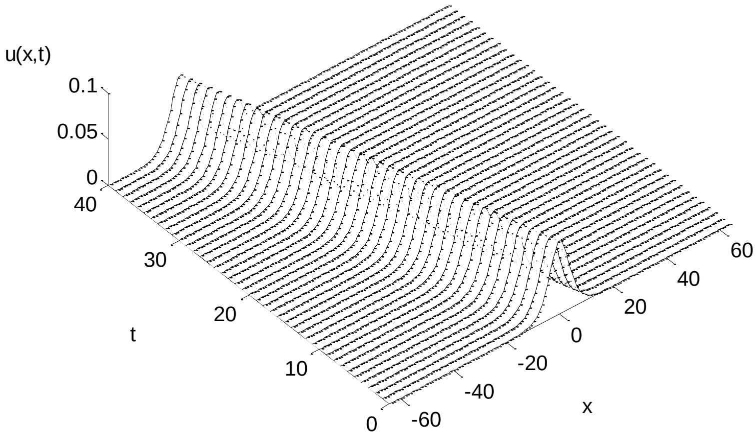


Fig. 27. Numerical simulation of Fourier spectral solution of the Boussinesq equation using RK4 scheme with $c = -\sqrt{1-4b^2}$ and N = 128.

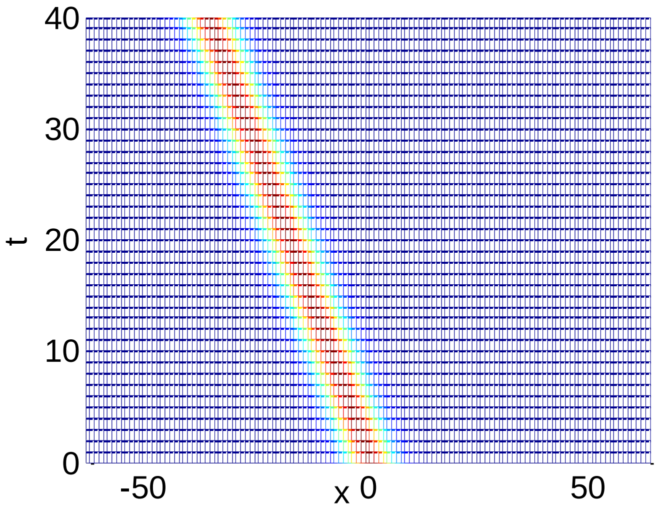


Fig. 28. Plan view of Fourier spectral solution of the Boussinesq equation using RK4 scheme with $c = -\sqrt{1-4b^2}$ and N = 128.

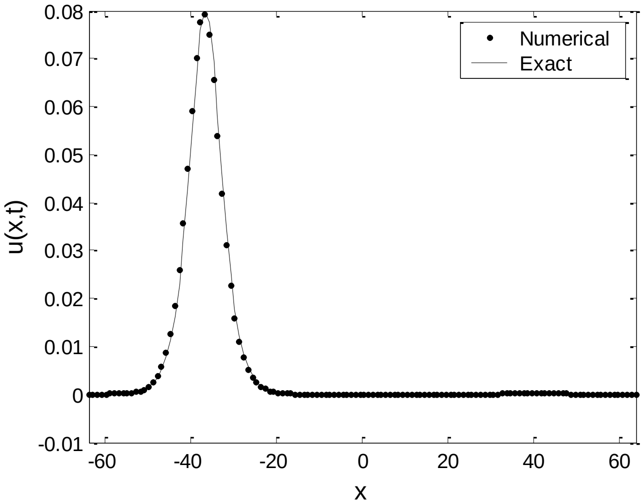


Fig.29. Fourier spectral solution of the Boussinesq equation using RK4 scheme at t = 40 with $c = -\sqrt{1-4b^2}$ and N = 128.

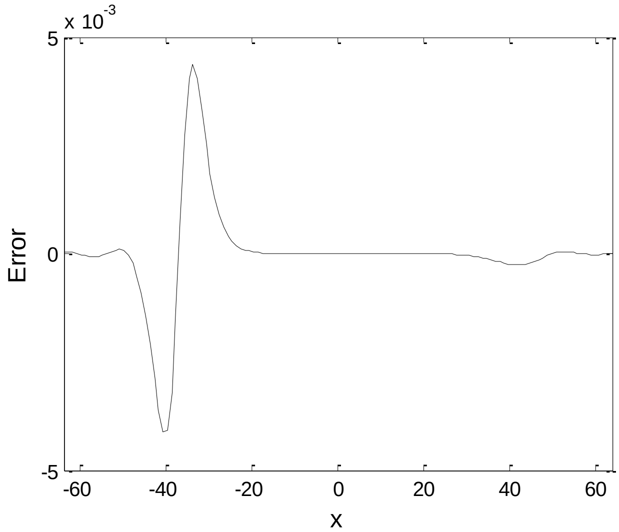


Fig.30. The error distributions of Fourier spectral solution of the Boussinesq equation using RK4 scheme at t = 40 with $c = -\sqrt{1-4b^2}$ and N = 128.

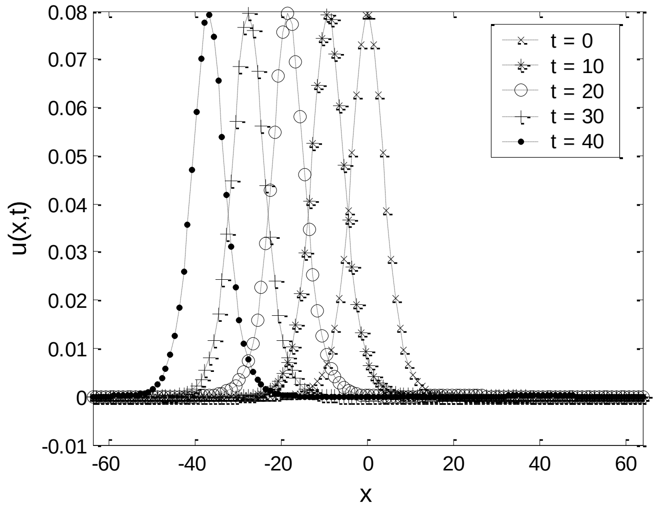


Fig. 31. Fourier spectral solution of the Boussinesq equation using a finite difference scheme at $c = -\sqrt{1-4b^2}$, t = 0, 10, 20, 30 and 40 with N = 128.

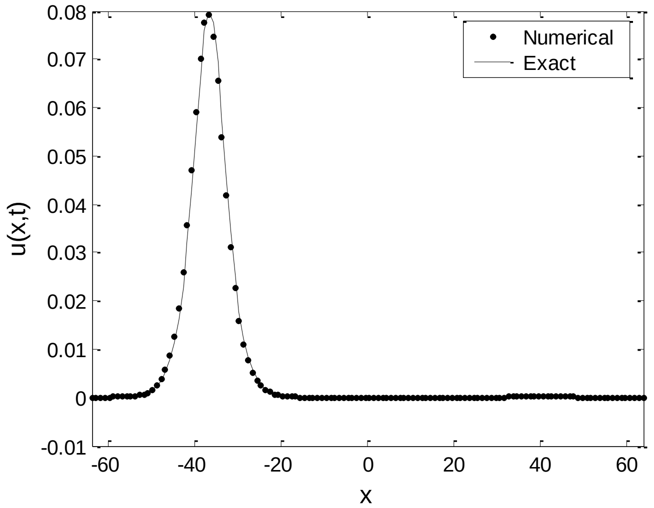


Fig.32. Fourier spectral solution of the Boussinesq equation using a finite difference scheme at $c = -\sqrt{1-4b^2}$, t = 40 with N = 128.

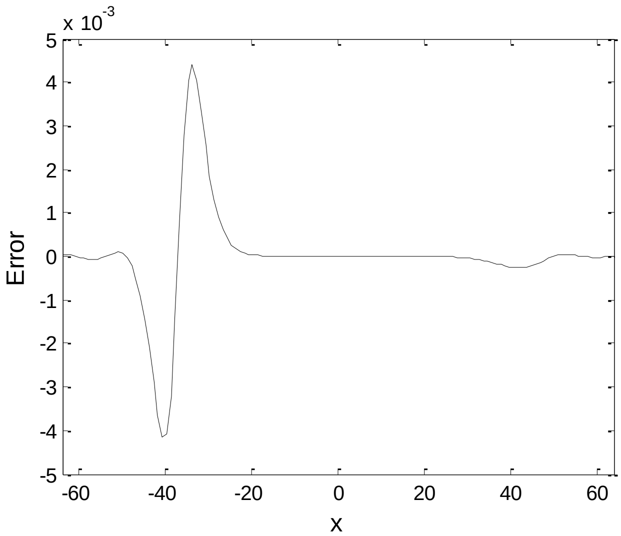


Fig. 33. The error distributions in Fourier spectral solution of the Boussinesq equation using a finite difference scheme with $c = -\sqrt{1-4b^2}$ at t = 40 with N = 128.

4 Conclusions

In Our study, we applied Fourier spectral collocation methods to solve partial differential equations of the form $u_t = F(u)$ particularly the Korteweg-de Vries equation (KdV) , and the form $u_{tt} = F(u)$ particularly the Boussinesq equation. A finite differences or a fourth-order Runge-Kutta method is used to march our solution in time, with the help of the fast Fourier transform algorithm our methods cost only (number of operations) $O\left(N \log_2 N\right)$, provided that *N* is a power of 2. We applied the leap-frog scheme combined with the Fourier spectral collocation, called the Fourier Leap-Frog method, to find numerical solution of the KdV equation, with α = 6, and β = 1. Also we applied the fourth-order Runge-Kutta (RK4) method combined with the Fourier spectral collocation to the same problem. We presented stability conditions for these schemes, and we found out that the Fourier Leap-Frog method is less stable and far less accurate compared to the classical RK4 scheme. Even though it is quite costly to use, RK4 is easy to implement and needs only one level of initial data, whereas two levels of the initial data are needed in the Fourier Leap-Frog methods.

We then applied the finite difference scheme combined with the Fourier spectral collocation to find numerical solution of the Bossinesq equation. Also we applied the fourth-order Runge-Kutta (RK4) method combined with the Fourier spectral collocation to the same problem. In order to show how good the numerical solutions are in comparison with the exact ones, we will use L_2 and L_∞ error norms. It is apparently seen that Fourier spectral collocation method is powerful and efficient technique in finding numerical solutions for wide classes of nonlinear partial differential equations.

5 Open Problem

The work presented in this paper transform some types of partial differential equations like the Korteweg-de Vries equation (KdV) , Boussinesq equation to simple ordinary differential equations using Fourier spectral method, can be solved by simple techniques (Leap frog, finite difference, Runge Kutta, …), the question here what the results of using other spectral methods to solve these equations instead of Fourier spectral method. The other question, in this paper we applied the used method to solve solitons, what happened when we solve equations by non solitons spectral methods.

References

[1] Fornberg B. A Practical Guide to Pseudospectral Methods. Cambridge University Press, New York. 1996.

[2] Fornberg B and Whitham G B. A numerical and Theoretical Study of Certain Nonlinear Wave Phenomena. Philos. Trans. Soc. London, 289, May 1978, pp. 373-404.

[3] Hany N. Hassan and Hassan K. Saleh, The Solution of the Regularized Long Wave Equation Using the Fourier Leap-Frog Method, Z. Naturforsch. 65a, 268 – 276 (2010)

[4] Boyd J P. Chebyshev and Fourier Spectral Methods. Second Edition Dover Publications Inc., New York, 2000.

[5] Whitham G B. Linear and Nonlinear Waves. John Wiley &Sons, 1974.

[6] Weideman, J.A.C., and Reddy, S.C., "A MATLAB Differentiation Matrix Suite", ACM Transactions on Mathematical Software, 26, No. 4, December 2000, pp. 465–519.

[7] Efstratios, E., Tzirtzilakis, Charalampos D., Skokos, and Tassos C. Bountis, "Numerical Solution of the Bossinesq equation using Spectral Methods and Stability of Solitary Wave Propagation", first International Conference "From Scientific Computing to Computational Engineering", Athens, 8-10 September, 2004.

[8] Eric Gourgoulhon, "Introduction to spectral methods", 4th EU Network Meeting, Palma de Mallorca, Sept. 2002. http://www.luth.obspm.fr

[9] Tzirtzilakis, E. , Marinakis, V. , Apokis, C. , and Bountis, T. , "Soliton-like solutions of higher order wave equations of the Korteweg de Vries type", Journal of Mathematical Physics, 43 No.12, December 2002, pp. 6151–6165.

[10] Bengt Fornberg and Tobin A. Driscoll, "A Fast Spectral Algorithm for Nonlinear Wave Equations with Linear Dispersion", Journal of Computational Physics155, 1999, pp.456–467.
[11] Randall, J. LeVeque, "Finite Difference Methods for Differential Equations", A Math 585, Winter Quarter 2006, University of Washington, Version of January, 2006.
[12] Drazin, P.G. and Johnson, R.S., "Solitons: an introduction", Cambridge University Press, Cambridge, 1989.
[13] Idris Dag , M. Naci Ozer, "Approximation of the RLW equation by the least square cubic B-spline finite element method" Applied Mathematical Modelling 25 (2001) 221-231
[14] H. Jafari, H. Hosseinzadeh and S. Mohamadzadeh "Numerical Solution of System of Linear Integral Equations by using Legendre Wavelets" *Int. J. Open Problems Compt. Math., Vol. 3, No. 5, December 2010*
[15] Hany N. Hassan and M. A. El-Tawil, "A new technique of using homotopy analysis method for solving high-order nonlinear differential equations", Mathematical Methods in the Applied Sciences, 34, 728–742 (2011).