Int. J. Open Problems Compt. Math., Vol. 6, No. 2, June 2013
ISSN 1998-6262; Copyright ©ICSRS Publication, 2013
www.i-csrs.org

Completion of Normed Hyperquotient Spaces

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(Communicated by Iqbal H. Jebril)

Abstract

In consonant with the prime object of this paper as to initiate the concept of hyperquotient structure, necessarily and essentially the notion of hyperquotient sets is the basic to begin with subsequently what comes as the formations of norm on this spaces with the assistance offered by the norm on hypervector spaces.

In conclusion, an adequate condition for a normed hyperquotient space to be Banach space has been constituted by us.

Keywords: *Norm, hyperquotient space, normed hyperquotient space, completeness.*

2010 Mathematics Subject Classification: 54B15, 54B20.

1 Introduction

After having instituted the notion of hyperstructure first by F. Marty [2] in 1934, the exposition of hypergroup was made by him in 1935 in the paper

[3]. Since then, from various angles, such as [1, 4] etc, several researchers have explored this domain. It is to be noted that exclusively the multiplication structure as a hyperstructure has been accepted by a lot of researchers [5] in defining of hypervector space. In the paper [6] this concept has been reshaped in keep with a panoramic-view through rethinking on all structures of hypervector space as a hyperstructure followed by some strongly grounded significant consequences and theorems. In practice, this papers [7, 8] enfold more wide spread form of the concept aforementioned.

Primarily this paper is endowed with an exposition of the quotient of a hypervector space. Next comes the foundation of hyperquotient space by weighing on all the structures as a hyperstructure. In the following feature a few elucidation and proposition have been presented. Then, norms on the hyperquotient space have been placed and finally, an effective condition for a normed hyperquotient space to a Banach space has been endeavored to be engrafted.

2 Preliminaries

We quote some definitions and statement of a proposition which will be needed in the sequel.

Definition 2.1 *[4] A hyperoperation over a non-empty set X is a mapping from $X\times X$ into the set of all non-empty subsets of X. A non-empty set X with exactly one hyperoperation '$\#$' is called a hypergroupoid. Let $(X, \#)$ be a hypergroupoid. For every point $x \in X$ and every non-empty subset A of X, we define $x \#A = \bigcup_{a\in A}\{x \# a\}$.*

Definition 2.2 *[4] A hypergroupoid $(X, \#)$ is called a hypergroup if*
(i) $x\# (y \# z) = (x \# y) \#z$ for all $x, y, z \in X$.
(ii) There exists $0 \in X$ such that for every $a \in X$, there is unique element $b \in X$ for which $0 \in a\#b$ and $0 \in b\#a$. Here b is denoted by $-a$.
(iii) For all $a, b, c \in X$ if $a \in b\#c$, then $b \in a\#(-c)$.

Proposition 2.3 *[4] (i) In a hypergroup $(X, \#)$, $-(-a) = a$ for all $a \in X$.*
(ii) $0 \# a = \{a\}$, for all $a \in X$, if $(X, \#)$ is a commutative hypergroup.
(iii) In a commutative hypergroup $(X, \#)$, 0 is unique.

Definition 2.4 *[6] A hyperring is a non-empty set equipped with a hyperaddition '$\#$' and a multiplication '$.$' such that $(X, \#)$ is a commutative hypergroup and $(X, .)$ is a semigroup and the multiplication is distributive across the hyperaddition both from the left and from the right and $a.0 = 0.a = 0$ for all $a \in X$, where 0 is the zero element of the hyperring.*

Definition 2.5 *[6] A hyperfield is a non-empty set X equipped with a hyperaddition '#' and a multiplication '.' such that*
(i) $(X, \#, .)$ *is a hyperring.*
(ii) *There exists an element* $1 \in X$, *called the identity element such that* $a.1 = a$ *for all* $a \in X$.
(iii) *For each non zero element* $a \in X$, *there exists an element* a^{-1} *such that* $a.a^{-1}=1$.
(iv) $a.b = b.a$ *for all* $a, b \in X$.

Definition 2.6 *[7] Let* $(F, \oplus, .)$ *be a hyperfield and* $(V, \#)$ *be an additive commutative hypergroup. Then* V *is said to be a hypervector space over the hyperfield* F *if there exists a hyperoperation* $* : F \times V \to P^*(V)$ *such that*
(i) $a * (\alpha \# \beta) \subseteq a * \alpha \# a * \beta$ *for all* $a \in F$ *and for all* $\alpha, \beta \in V$.
(ii) $(a \oplus b) * \alpha \subseteq a * \alpha \# b * \alpha$ *for all* $a, b \in F$ *and for all* $\alpha \in V$.
(iii) $(a . b) * \alpha = a * (b * \alpha)$ *for all* $a, b \in F$ *and for all* $\alpha \in V$.
(iv) $(-a) * \alpha = a * (-\alpha)$ *for all* $a \in F$ *and for all* $\alpha \in V$.
(v) $\alpha \in 1_F * \alpha$, $\theta \in 0 * \alpha$ *and* $0 * \theta = \theta$ *for all* $\alpha \in V$.
where 1_F *is the identity element of* F, 0 *is the zero element of* F *and* θ *is zero vector of* V *and* $P^*(V)$ *is the set of all non-empty subset of* V.

Definition 2.7 *[6] A subset W of a hypervector space V over a hyperfield F is called a hypersubspace of V if W is a hypervector space over F with the hyperoperations of addition and the scalar multiplication defined on V. Therefore a subset W of a hypervector space V is a hypersubspace of V if and only if the following four properties hold.*
(i) $\alpha \# \beta \subseteq W$ *for all* $\alpha, \beta \in W$.
(ii) $a * \alpha \subseteq W$ *for all* $\alpha \in W$ *and for all* $a \in F$.
(iii) *W has a zero vector.*
(iv) *each vector of W has an additive inverse.*

Definition 2.8 *[7] Let* $(V, \#, *, F)$ *be a hypervector space. A subset* $A = \{\alpha_\lambda\}_{\lambda \in \Lambda}$ *of* V *is set to a linearly dependent set if there exists a finite subset* $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ *of* A *s.t* $\theta \in \lambda_1 * \alpha_1 \# \lambda_2 * \alpha_2 \# \cdots \# \lambda_n * \alpha_n$ *for some* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *(not all zeros)* $\in F$. *Otherwise* A *is said to a linearly independent set.*

Definition 2.9 *[6] Let* V *be a hypervector space over a hyperfield* F. *Then the vector* $\alpha \in V$ *is said to be a linear combination of the vectors* $\alpha_1, \alpha_2, \cdots, \alpha_n \in V$ *if there exist* $a_1, a_2, \cdots, a_n \in F$ *such that* $\alpha \in a_1 * \alpha_1 \# a_2 * \alpha_2 \# \cdots \# a_n * \alpha_n$.

Definition 2.10 *[7] An independent subset* A *of* V *is called a basis of* V *if for every* $\alpha \in V$, *there are* n $(\in \mathbf{N})$ *elements* $\lambda_1, \lambda_2, \cdots, \lambda_n \in F$ *and* $\alpha_1, \alpha_2, \cdots, \alpha_n \in A$ *such that* $\alpha \in \lambda_1 * \alpha_1 \# \lambda_2 * \alpha_2 \# \cdots \# \lambda_n * \alpha_n$. *The hypervector space is said to be finite dimensional if it has a finite basis.*

Definition 2.11 *[7] Let* $\mathbf{R}$ *be the set of all real number. The hyperfield defined on* $\mathbf{R}$ *is called the real hyperfield.*

Definition 2.12 *[9] Let* $(V, \#, *)$ *be a hypervector space over the real hyperfield* $\mathbf{R}$*. A norm on* V *is a mapping* $\|\cdot\| : V \to \mathrm{R}$*, where* R *is a usual real space, such that for all* $a \in \mathbf{R}$ *and* $\alpha, \beta \in V$ *has the following properties*
(i) $\|\alpha\| \geq 0$.
(ii) $\|\alpha\| = 0$ *if and only if* $\alpha = \theta$.
(iii) $\sup \|\alpha\#\beta\| \leq \|\alpha\| + \|\beta\|$, *where* $\|\alpha \# \beta\| = \{\, \|x\|, \;\; x \in \alpha \# \beta \,\}$.
(iv) $\sup \|a * \alpha\| \;\leq\; |a| \cdot \|\alpha\|$, *where* $\|a * \alpha\| = \{\, \|x\|, \;\; x \in a * \alpha \,\}$.

Definition 2.13 *[9] A sequence* $\{\alpha_n\}_n$ *in a normed hypervector space* V *is said to converge to a point* $\alpha \in V$ *if for any* $\epsilon\,(> 0)$*, there exists a positive integer* n_0 *such that* $\inf \|\alpha_n\#(-\alpha)\| < \epsilon$ *for all* $n \geq n_0$.

Definition 2.14 *[9] A sequence* $\{\alpha_n\}_n$ *in a normed hypervector space* V *is said to be a Cauchy sequence if for any* $\epsilon\,(> 0)$*, there exists a positive integer* n_0 *such that* $\inf \|\alpha_n\#(-\alpha_m)\| < \epsilon$ *for all* $m, n \geq n_0$*. The normed hypervector space* V *is said to be complete if every cauchy sequence in* V *converges to some point in* V.

3 Hyperquotient spaces

In this section, we define quotient sets, hyperquotient spaces and also establish a few theorems related to hyperquotient spaces.

Definition 3.1 *Let* M *be hypersubspace of a hypervector space* V *over the hyperfield* F*. Then the set* $\{\alpha\#M : \;\alpha \in V\}$ *is called the quotient set of* V *with respect to* M *and it is denoted by* V/M.

Theorem 3.2 *Let* M *be a hypersubspace of a hypervector space* V *over the hyperfield* F *and* $\alpha\#M \in V/M$*. If* $\beta \in \alpha\#M$, $\alpha\#M = \beta\#M$.

Proof: Since $\beta \in \alpha\#M$, $\beta\#M \subseteq \alpha\#M\#M \subseteq \alpha\#M$. So $\beta\#M \subseteq \alpha\#M$.
Again $\beta \in \alpha\#M$, so $\beta \in \alpha\#m$ for some $m \in M$.
Therefore $\alpha \in \beta\#(-m)$. So $\alpha\#M \subseteq \beta\#(-m)\#M$, that is, $\alpha\#M \subseteq \beta\#M$.
Hence $\alpha\#M = \beta\#M$.

Corollary 3.3 *Let* M *be a hypersubspace of a hypervector space* V *over the hyperfield* F *and* V/M *be a quotient set. Then any two members of* V/M *are either equal or disjoint.*

Proof: Follows from theorem 3.2.

Proposition 3.4 *Let $(V, \#, *)$ be a hypervector space over the hyperfield F. Then $a*(A\#B) \subseteq a*A\#a*B$ for all $a \in F$ and for all $A, B \subseteq V$.*

Proof: Let $\alpha \in A$ and $\beta \in B$. Then $a*(\alpha\#\beta) \subseteq a*\alpha\#a*\beta \subseteq a*A\#a*B$.
Therefore $a*(\alpha\#\beta) \subseteq a*A\#a*B$ for all $\alpha \in A$ and for all $\beta \in B$.
So $\bigcup_{\alpha\in A,\,\beta\in B} a*(\alpha\#\beta) \subseteq a*A\#a*B$.
Hence $a*(A\#B) \subseteq a*A\#a*B$.

Result 3.5 *Let '$+$' and '$\odot$' be two mapping define as $+ : V/M \times V/M \to P^*(V/M)$ by $(\alpha\#M) + (\beta\#M) = \{x\#M : x \in \alpha\#\beta\}$ and $\odot : F \times V/M \to P^*(V/M)$ by $a \odot (\alpha\#M) = \{x\#M : x \in a*\alpha\}$, where $\alpha\#M, \beta\#M \in V/M$ and $a \in F$. Then these two mappings are well defined.*

Proof: Let $\alpha_1\#M = \alpha_2\#M$ and $\beta_1\#M = \beta_2\#M$. Then $\alpha_1\#m \subseteq \alpha_2\#M$ for all $m \in M$. Therefore $\alpha_1\#m\#(-m) \subseteq \alpha_2\#M\#(-m)$.
That is, $\alpha_1 \in \alpha_1\#m\#(-m) \subseteq \alpha_2\#M$. Thus $\alpha_1 \in \alpha_2\#M$.
Similarly, $\beta_1 \in \beta_2\#M$.
Now let $x\#M \in (\alpha_1\#M) + (\beta_1\#M)$, then without loss of generality we may assume that $x \in \alpha_1\#\beta_1$. Then $x \in (\alpha_2\#M)\#(\beta_2\#M) \subseteq (\alpha_2\#\beta_2)\#M$.
Therefore $x \in y\#m$ for some $y \in \alpha_2\#\beta_2$ and $m \in M$ $\cdots$ (1)
So $x\#M \subseteq (y\#m)\#M \subseteq y\#M$.
Again from (1) we have $y \in x\#(-m)$. Then $y\#M \subseteq (x\#(-m))\#M \subseteq x\#M$.
Therefore $x\#M = y\#M \in (\alpha_2\#M) + (\beta_2\#M)$, which implies $(\alpha_1\#M) + (\beta_1\#M) \subseteq (\alpha_2\#M) + (\beta_2\#M)$.
Similarly, $(\alpha_2\#M) + (\beta_2\#M) \subseteq (\alpha_1\#M) + (\beta_1\#M)$.
Hence $(\alpha_1\#M) + (\beta_1\#M) = (\alpha_2\#M) + (\beta_2\#M)$.
Therefore $'+'$ is well defined.
Next let $\alpha\#M = \beta\#M \in V/M$ and $a \in F$.
Now $a \odot (\alpha\#M) = \{x\#M : x \in a*\alpha\}$ and $a \odot (\beta\#M) = \{x\#M : x \in a*\beta\}$.
Let $x\#M \in a \odot (\alpha\#M)$. Then without loss of generality we may assume that $x \in a*\alpha$. So $x \in a*(\beta\#M)$ [as $\alpha\#M = \beta\#M$, $\alpha \in \beta\#M$].
Then $x \in a*\beta\#a*M$, by proposition 3.4.
Therefore $x \in y\#m$, where $y \in a*\beta$ and $m \in a*M = M$. $\cdots$ (2)
So $x\#M \subseteq y\#M$.
Again from (2) we have $y \in x\#(-m)$. So $y\#M \subseteq x\#M$.
Therefore $x\#M = y\#M \in a \odot (\beta\#M)$, which implies $a \odot (\alpha\#M) \subseteq a \odot (\beta\#M)$. Similarly, $a \odot (\beta\#M) \subseteq a \odot (\alpha\#M)$.
Hence $a \odot (\alpha\#M) = a \odot (\beta\#M)$. Therefore '$\odot$' is well defined.

Theorem 3.6 *Let M be a hypersubspace of a hypervector space $(V, \#, *)$ over the hyperfield $(F, \oplus, .)$. Then the quotient set V/M forms a hypervector space over the hyperfield $(F, \oplus, .)$ with respect to the binary compositions $'+'$ and '$\odot$'.*

Proof: We first show that $(V/M\,,+)$ is a commutative hypergroup.
Since '#' is commutative on V, '+' is commutative on V/M.
Let $\alpha\#M$, $\beta\#M$, $\gamma\#M \in V/M$.
Now $((\alpha\#M)+(\beta\#M))+(\gamma\#M)$
$=\{x\#M : x\in\alpha\#\beta\}+(\gamma\#M)$
$=\bigcup_{x\in\alpha\#\beta}\{y\#M : y\in x\#\gamma\}$
$=\{y\#M : y\in\alpha\#\beta\#\gamma\}$
$=\bigcup_{z\in\beta\#\gamma}\{y\#M : y\in\alpha\#z\}$
$=(\alpha\#M)+\{z\#M : z\in\beta\#\gamma\}$
$=(\alpha\#M)+((\beta\#M)+(\gamma\#M))$.
This implies that '+' is associative on V/M, that is, the first condition of hypergroup is satisfied.
Since $\theta\in V$, $\theta\#M\in V/M$. So $M\in V/M$.
Let $\alpha\#M\in V/M$. Then $\alpha\in V$. Therefore there exists a unique element $-\alpha\in V$ such that $\theta\in\alpha\#(-\alpha)$. Therefore $M=\theta\#M\in(\alpha\#M)+(-\alpha\#M)$.
If possible, let there be another element $\beta\#M\in V/M$ such that
$M\in(\alpha\#M)+(\beta\#M)$, that is, $M=x\#M$ for some $x\in\alpha\#\beta$.
Again $M=x\#M$. So $x\in M$. Therefore there exists $m\in M$ such that
$m\in\alpha\#\beta$, that is, $\beta\in m\#-\alpha$. So $\beta\in-\alpha\#M$. Therefore $\beta\#M=-\alpha\#M$ (by theorem 3.2), which is a contradiction. Thus for every $\alpha\#M\in V/M$, there is a unique element $-\alpha\#M\in V/M$ such that $M\in(\alpha\#M)+(-\alpha\#M)$.
So the second condition of hypergroup is satisfied.
Next let $\alpha\#M$, $\beta\#M$, $\gamma\#M\in V/M$ such that $\alpha\#M\in(\beta\#M)+(\gamma\#M)=\{x\#M : x\in\beta\#\gamma\}$. Then there exists $x\in\beta\#\gamma$ such that $\alpha\#M=x\#M$. Since $x\in\beta\#\gamma$, we have $\beta\in x\#(-\gamma)$. So $\beta\#M\in(x\#M)+(-\gamma\#M)=(\alpha\#M)+(-\gamma\#M)$. Therefore the last condition of the hypergroup is satisfied.
Hence $(V/M\,,+)$ is a commutative hypergroup.
We now verify the remaining five conditions of hypervector space.
Let $a\in F$ and $\alpha\#M$, $\beta\#M\in V/M$.
Now $a\odot((\alpha\#M)+(\beta\#M))$
$=a\odot\{x\#M :\ x\in\alpha\#\beta\}$
$=\bigcup_{x\in\alpha\#\beta}\{y\#M :\ y\in a*x\}$
$=\{y\#M :\ y\in a*(\alpha\#\beta)\}$
$\subseteq\{y\#M : y\in a*\alpha\#a*\beta\}$, as $a*(\alpha\#\beta)\subseteq a*\alpha\#b*\beta$
$=\{y\#M :\ y\in\alpha_1\#\beta_1,\alpha_1\in a*\alpha,\beta_1\in a*\beta\}$
$=\bigcup_{\alpha_1\in a*\alpha,\beta_1\in a*\beta}\{y\#M : y\in\alpha_1\#\beta_1\}$
$=\bigcup_{\alpha_1\in a*\alpha,\beta_1\in a*\beta}\{(\alpha_1\#M)+(\beta_1\#M)\}$
$=\{\alpha_1\#M :\ \alpha_1\in a*\alpha\}+\{\beta_1\#M :\ \beta_1\in a*\beta\}$
$=a\odot(\alpha\#M)+a\odot(\beta\#M)$.
Therefore $a\odot((\alpha\#M)+(\beta\#M))\subseteq a\odot(\alpha\#M)+a\odot(\beta\#M)$.
Thus the first condition is satisfied.
Next let $a,b\in F$ and $\alpha\#M\in V/M$.

Now $(a \oplus b) \odot (\alpha \# M)$
$= \bigcup_{c \in a \oplus b} c \odot (\alpha \# M)$
$= \bigcup_{c \in a \oplus b} \{x \# M : \ x \in c * \alpha\}$
$= \{x \# M : \ x \in (a \oplus b) * \alpha\}$
$\subseteq \{x \# M : \ x \in a * \alpha \# b * \alpha\}$
$= \{x \# M : \ x \in \alpha_1 \# \beta_1, \alpha_1 \in a * \alpha, \beta_1 \in b * \alpha\}$
$= \bigcup_{\alpha_1 \in a * \alpha, \beta_1 \in b * \alpha} \{x \# M : \ x \in \alpha_1 \# \beta_1\}$
$= \bigcup_{\alpha_1 \in a * \alpha, \beta_1 \in b * \alpha} \{(\alpha_1 \# M) + (\beta_1 \# M)\}$
$= \{\alpha_1 \# M : \ \alpha_1 \in a * \alpha\} + \{\beta_1 \# M : \ \beta_1 \in b * \alpha\}$
$= a \odot (\alpha \# M) + b \odot (\alpha \# M)$.
Therefore $(a \oplus b) \odot (\alpha \# M) \subseteq a \odot (\alpha \# M) + b \odot (\alpha \# M)$.
Thus the second condition is satisfied.
Next let $a, b \in F$ and $\alpha \# M \in V/M$.
Now $(a.b) \odot (\alpha \# M)$
$= \{x \# M : \ x \in (a.b) * \alpha\}$
$= \{x \# M : \ x \in a * (b * \alpha)\}$
$= \{x \# M : \ x \in a * \alpha_1, \alpha_1 \in b * \alpha\}$
$= \bigcup_{\alpha_1 \in b * \alpha} \{x \# M : \ x \in a * \alpha_1\}$
$= \bigcup_{\alpha_1 \in b * \alpha} a \odot (\alpha_1 \# M)$
$= a \odot \{\alpha_1 \# M : \ \alpha_1 \in b * \alpha\}$
$= a \odot (b \odot (\alpha_1 \# M))$.
Therefore $(a.b) \odot (\alpha \# M) = a \odot (b \odot (\alpha \# M))$.
Thus the third condition is satisfied.
Next let $a \in F$ and $\alpha \# M \in V/M$.
Now $(-a) \odot (\alpha \# M)$
$= \{x \# M : \ x \in (-a) * \alpha\}$
$= \{x \# M : \ x \in a * (-\alpha)\}$, as $(-a) * \alpha = a * (-\alpha)$
$= a \odot (-\alpha \# M)$
$= a \odot (-(\alpha \# M))$.
Therefore the fourth condition is satisfied.
Next let $\alpha \# M \in V/M$.
Now $1_F \odot (\alpha \# M) = \{x \# M : \ x \in 1_F * \alpha\}$.
Since $\alpha \in 1_F * \alpha$, $\alpha \# M \in 1_F \odot (\alpha \# M)$.
Next $0 \odot (\alpha \# M) = \{x \# M : \ x \in 0 * \alpha\}$.
Since $\theta \in 0 * \alpha$, $M = \theta \# M \in 0 \odot (\alpha \# M)$.
Next $0 \odot M = 0 \odot (\theta \# M) = \{x \# M : \ x \in 0 * \theta\}$.
Since $\theta = 0 * \theta$, $0 \odot M = \theta \# M = M$.
Thus the last conditions are satisfied.
Hence $(V/M, +, \odot)$ is a hypervector space over the hyperfield $(F, \oplus, .)$.

Definition 3.7 *Let M be a hypersubspace of a hypervector space V over the hyperfield F, then the hypervector space $(V/M, +, \odot)$ is called hyperquotient space over the hyperfield F.*

Theorem 3.8 *Let M be a proper hypersubspace of a finite dimensional hypervector space V over the hyperfield F and $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a basis of V. Then the set $\{\alpha \# M : \alpha \in S\}$ is linearly dependent subset of the hyperquotient space V/M.*

Proof: Let $m(\neq \theta) \in M$. Since S is a basis of V and M is a proper hypersubspace of V, then there exist $c_1, c_2, \cdots, c_n \in F$, not all zeros such that $m \in c_1 * \alpha_1 \# c_2 * \alpha_2 \# \cdots \# c_n * \alpha_n$.
Then $m \# M \in \{x \# M : \ x \in c_1 * \alpha_1 \# c_2 * \alpha_2 \# \cdots \# c_n * \alpha_n\}$.
Therefore $M \in \{x \# M : \ x \in x_1 \# x_2 \# \cdots \# x_n,\ x_1 \in c_1 * \alpha_1,\ x_2 \in c_2 * \alpha_2,\ \cdots,\ x_n \in c_n * \alpha_n\}$. This implies
$M \in \bigcup_{x_1 \in c_1 * \alpha_1, x_2 \in c_2 * \alpha_2, \cdots, x_n \in c_n * \alpha_n} \{x \# M : \ x \in x_1 \# x_2 \# \cdots \# x_n\}$.
So $M \in \bigcup_{x_1 \in c_1 * \alpha_1, x_2 \in c_2 * \alpha_2, \cdots, x_n \in c_n * \alpha_n} \{(x_1 \# M) + (x_2 \# M) + \cdots + (x_n \# M)\}$.
So $M \in \{x_1 \# M : \ x_1 \in c_1 * \alpha_1\} + \{x_2 \# M : \ x_2 \in c_2 * \alpha_2\} + \cdots + \{x_n \# M : x_n \in c_n * \alpha_n\}$.
Thus $M \in c_1 \odot (\alpha_1 \# M) + c_2 \odot (\alpha_2 \# M) + \cdots + c_n \odot (\alpha_n \# M)$ for some non-zeros c_i, $i = 1, 2, \cdots, n$. Hence the set $\{\alpha \# M : \alpha \in S\}$ is linearly dependent subset of the hyperquotient space V/M.

Theorem 3.9 *Let M be a proper hypersubspace of a finite dimensional hypervector space V over the hyperfield F and $S = \{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ be a basis of V. Then the set $\{\alpha \# M : \alpha \in S\}$ generates the hyperquotient space V/M.*

Proof: Obvious.

4 Normed hyperquotient spaces

In this section, we establish a few results related to normed hypervector space and define norm on a hyperquotient space.

Definition 4.1 *Let $(V, \#, *, \| \cdot \|)$ be a normed hypervector space over a real hyperfield $\mathbf{R}$. Then we define*
$\sup \|A\| = \sup_{\alpha \in A} \|\alpha\|$ *for all* $A \subseteq V$,
$\sup \|a * A\| = \sup_{\alpha \in A} \sup \|a * \alpha\|$ *for all* $a \in \mathbf{R}$ *and for all* $A \subseteq V$,
$\sup \|A \# B\| = \sup_{\alpha \in A, \beta \in B} \sup \|\alpha \# \beta\|$ *for all* $A \subseteq V$ *and for all* $B \subseteq V$,
$\inf \|A\| = \inf_{\alpha \in A} \|\alpha\|$ *for all* $A \subseteq V$,
$\inf \|a * A\| = \inf_{\alpha \in A} \inf \|a * \alpha\|$ *for all* $a \in \mathbf{R}$ *and for all* $A \subseteq V$,
and $\inf \|A \# B\| = \inf_{\alpha \in A, \beta \in B} \inf \|\alpha \# \beta\|$ *for all* $A \subseteq V$ *and for all* $B \subseteq V$.

Proposition 4.2 *Let $(V, \#, *, \| \cdot \|)$ be a normed hypervector space over a real hyperfield $\mathbf{R}$, $A \subseteq V$ and $a \in \mathbf{R}$. Then* $\sup \|a * A\| \leq |a| \sup \|A\|$.

Proof: Let $\alpha \in A$. Then $\sup \|a * \alpha\| \leq |a| \cdot \|\alpha\| \leq |a| \sup \|A\|$.
Therefore $\sup \|a * \alpha\| \leq |a| \sup \|A\|$ for all $\alpha \in A$
Thus $\sup_{\alpha \in A} \sup \|a * \alpha\| \leq |a| \sup \|A\|$.
Hence $\sup \|a * A\| \leq |a| \sup \|A\|$.

Proposition 4.3 *Let* $(V, \#, *, \|\cdot\|)$ *be a normed hypervector space over a real hyperfield* $\mathbf{R}$ *and* $A, B \subseteq V$. *Then* $\sup \|A\#B\| \leq \sup \|A\| + \sup \|B\|$.

Proof: Let $\alpha \in A$ and $\beta \in B$.
Then $\sup \|\alpha\#\beta\| \leq \|\alpha\| + \|\beta\| \leq \sup \|A\| + \sup \|B\|$ for all $\alpha \in A$ and $\beta \in B$.
Therefore $\sup_{\alpha \in A, \beta \in B} \sup \|\alpha\#\beta\| \leq \sup \|A\| + \sup \|B\|$.
Hence $\sup \|A\#B\| \leq \sup \|A\| + \sup \|B\|$.

Theorem 4.4 *Let* V/M *be a hyperquotient space over a real hyperfield* $\mathbf{R}$, *where* M *is a closed hypersubspace of* V. *Define* $\|\cdot\|_q : V/M \to \mathrm{R}$ *by* $\|\alpha\#M\|_q = \inf\{\sup \|\alpha\#m\| : m \in M\}$. *Then* $(V/M, +, \odot, \|\cdot\|_q)$ *is normed hyperquotient space.*

Proof: It is obvious that $\|\alpha\#M\|_q \geq 0$ for all $\alpha \in V$.
Now $\|\theta\#M\|_q = \inf\{\sup \|\theta\#m\| : m \in M\} = 0$, as $\theta \in M$.
That is $\|\alpha\#M\|_q = 0$ if $\alpha\#M = M$.
Conversely, let $\|\alpha\#M\|_q = 0$. Then there exists a sequence $\{m_k\}$ in M such that $\sup \|\alpha\#m_k\| \to 0$, as $k \to \infty$. So $\inf \|\alpha\#m_k\| \to 0$, as $k \to \infty$.
That is $\inf \|m_k\# - (-\alpha)\| \to 0$, as $k \to \infty$. Therefore $m_k \to -\alpha$, as $k \to \infty$.
Since M is a closed hypersubspace of V, $-\alpha \in M$. Then $\alpha \in M$. Therefore $\alpha\#M = M$.
Thus $\|\alpha\#M\|_q = 0$ implies that $\alpha\#M = M$.
Further, for $\alpha, \beta \in V$, we have

$\sup \|(\alpha\#M) + (\beta\#M)\|_q$
$= \sup \|\{x\#M : x \in \alpha\#\beta\}\|_q$
$= \sup\{\|x\#M\|_q : x \in \alpha\#\beta\}$
$= \sup_{x \in \alpha\#\beta} \|x\#M\|_q$
$= \sup_{x \in \alpha\#\beta} \inf\{\sup \|x\#m\| : m \in M\}$
$\leq \inf_{m \in M} \sup_{x \in \alpha\#\beta} \sup \|x\#m\|$
$= \inf_{m \in M} \sup \|\alpha\#\beta\#m\|$
$\leq \inf_{m \in M} \sup \|\alpha\#\beta\#m\#m\#(-m)\|$, as $\theta \in m\#(-m)$
$\leq \inf_{m \in M} \sup \|(\alpha\#m)\#(\beta\#m)\#(-1 * m)\|$, as $-m \in -1 * m$
$\leq \inf_{m \in M}[\sup \|\alpha\#m\| + \sup \|\beta\#m\| + \sup \| - 1 * m\|\,]$
$= \inf_{m \in M} \sup \|\alpha\#m\| + \inf_{m \in M} \sup \|\beta\#m\| + \inf_{m \in M} \|m\|$
$= \inf_{m \in M} \sup \|\alpha\#m\| + \inf_{m \in M} \sup \|\beta\#m\| + 0$, as $\theta \in M$
$= \|\alpha\#M\|_q + \|\beta\#M\|_q$.

Thus $\sup \|(\alpha\#M) + (\beta\#M)\|_q \leq \|\alpha\#M\|_q + \|\beta\#M\|_q$.
Further, for $a \in \mathbf{R}$ and $\alpha\#M \in V/M$, we have

$\sup \|a \odot (\alpha\#M)\|_q$
$= \sup \|\{x\#M : \ x \in a * \alpha\}\|_q$
$= \sup_{x\in a*\alpha} \|x\#M\|_q$
$= \sup_{x\in a*\alpha} \inf\{\sup \|x\#m\| : \ m \in M\}$
$\leq \inf_{m\in M} \sup_{x\in a*\alpha}\{\sup \|x\#m\|\}$
$= \inf_{m\in M} \sup \|(a*\alpha)\#m\|$
$\leq \inf_{m\in M} \sup \|a*(\alpha\#m\#(-m))\#m\|$
$\leq \inf_{m\in M} \sup \|a*(\alpha\#m)\#a*(-m)\#m\|$, by proposition 3.4
$\leq \inf_{m\in M}[\ \sup \|a*(\alpha\#m)\| + \sup \|a*(-m)\| + \|m\|\ ]$, by proposition 4.3
$\leq \inf_{m\in M}[|a| \sup \|\alpha\#m\| + |a|.\|m\| + \|m\|]$, by proposition 4.2
$= \inf_{m\in M} |a| sup\|\alpha\#m\| + \inf_{m\in M} |a|.\|m\| + \inf_{m\in M} \|m\|$
$= |a| \inf_{m\in M} \sup \|\alpha\#m\| + 0 + 0$
$= |a|\|\alpha\#M\|_q$.
Hence $(V/M, +, \odot, \|\cdot\|_q)$ is normed hyperquotient space.

Definition 4.5 *A sequence $\{\alpha_n\#M\}_n$ in a normed hyperquotient space V/M is said to converge to a point $\alpha\#M \in V/M$ if for any $\epsilon\,(>0)$, there exists a positive integer n_0 such that $\inf \|(\alpha_n\#M) + (-\alpha\#M)\|_q < \epsilon$, for all $n \geq n_0$, where M is a closed hypersubspace of V.*

Definition 4.6 *A sequence $\{\alpha_n\#M\}_n$ in a normed hyperquotient space V/M is said to be a Cauchy sequence if for any $\epsilon\,(>0)$, there exists a positive integer n_0 such that $\inf \|(\alpha_n\#M) + (-\alpha_m\#M)\|_q < \epsilon$ for all $m, n \geq n_0$, where M is a closed hypersubspace of V.*
The normed hyperquotient space V/M is said to be complete if every cauchy sequence in V/M converges to some point in V/M.

Theorem 4.7 *If V be a Banach space and M be a closed hypersubspace of V, then the hyperquotient space V/M with the norm as defined in theorem 4.4, is a Banach space.*

Proof: It is enough to show that V/M is complete. Let $\{\alpha_n\#M\}_n$ be a Cauchy sequence in V/M. Then for any given $\epsilon\,(>0)$, there exists a positive integer n_0 such that $\inf \|(\alpha_n\#M) + (-\alpha_m\#M)\|_q < \frac{\epsilon}{2}$, for all $m, n \geq n_0$.
Then $\inf \|\{x\#M : \ x \in \alpha_n\# - \alpha_m\}\|_q < \frac{\epsilon}{2}$.
That is, $\inf_{x\in\alpha_n\#-\alpha_m} \|x\#M\|_q < \frac{\epsilon}{2}$.
Therefore $\inf_{x\in\alpha_n\#-\alpha_m} \inf_{p\in M} \sup \|x\#p\| < \frac{\epsilon}{2}$.
Thus there exists $x \in \alpha_n\# - \alpha_m$ such that $\inf_{p\in M} \sup \|x\#p\| \leq \frac{\epsilon}{2} < \epsilon$.
Now $x \in \alpha_n\# - \alpha_m$. Then $\inf \|\alpha_n\# - \alpha_m\| \leq \|x\|$.
That is, $\inf \|\alpha_n\# - \alpha_m\| \leq \sup \|x\#p\# - p\|$ for any $p \in M$. Therefore
$\inf \|\alpha_n\# - \alpha_m\| \leq \inf_{p\in M} \sup \|x\#p\# - p\| \leq \inf_{p\in M}(\sup \|x\#p\| + \|-p\|)$.
So $\inf \|\alpha_n\# - \alpha_m\| \leq \inf_{p\in M} \sup \|x\#p\|$, as $\inf_{p\in M} \|-p\| = 0$.
since $\inf_{p\in M} \sup \|x\#p\| < \epsilon$, $\inf \|\alpha_n\# - \alpha_m\| < \epsilon$, for all $m, n \geq n_0$.

Therefore the sequence $\{\alpha_n\}_n$ is a Cauchy sequence in V.
Again since V is a Banach space, the sequence $\{\alpha_n\}$ is convergent and converges to some point α(say) in V.
We now show that $\{\alpha_n \# M\}_n$ converges to $\alpha \# M$.
$\inf \|(\alpha_n \# M) + (-\alpha \# M)\|_q$
$= \inf_{x \in \alpha_n \# -\alpha} \|x \# M\|_q$
$= \inf_{x \in \alpha_n \# -\alpha} \inf_{p \in M} \sup \|x \# p\|$
$\leq \inf_{x \in \alpha_n \# -\alpha} \sup \|x \# \theta\|$
$= \inf_{x \in \alpha_n \# -\alpha} \|x\|$
$= \inf \|\alpha_n \# -\alpha\|$.
Therefore $\inf \|(\alpha_n \# M) + (-\alpha \# M)\|_q \leq \inf \|\alpha_n \# -\alpha\|$.
Since α_n converges to α, $\alpha_n \# M$ converges to $\alpha \# M$.
Hence the normed hyperquotient space V/M is a Banach space.

5 Open Problem

In this paper [6], we have considered hypervector space having all the structures as hyper structure over a hyperfield. So, is the most general form for a hypervector space. With respect to this structure, here we have considered hyperquotient space and then a suitable norm is defined on this space and this normed linear space is shown to be a Banach space under a few sufficient conditions. Now one can further define operator on this Banach space and study properties of operator on this space.

ACKNOWLEDGEMENTS. The authors are grateful to Raghunath Pramanick for his valuable suggestions to rectify this paper in its present form.

References

[1] M. Krasner, *A class of hyperrings and hyperfields*, Intern. J.Math. and Math. Sci., 6(2) (1938), 307–312.

[2] F. Marty, *Sur une generalisation de la notion de groupe.*, In 8‘eme congr‘es des Mathematiciens Scandinaves, Stockholm, (1934) 45–49, .

[3] F. Marty, *Role de la notion dhypergroupe dans letude des groupes non abeliens*, Comptes Renclus Acad. Sci. Paris Math, 201 (1935), 636–638.

[4] A. Nakassis, *Expository and survey article recent results in hyperring and hyperfield theory*, internet. J. Math. and Math. Sci., 11(2) (1988), 209–220.

[5] P. Raja and S. M. Vaezpour, *Normed hypervector spaces* , Iranian Journal of Mathematical Sciences and Informatics, 2 (2007), 35–44.

[6] S. Roy and T. K. Samanta, *A note on hypervector spaces*, Discussiones Mathematicae - General Algebra and Applications, 31(1) (2011), 75–99.

[7] S. Roy and T. K. Samanta, *Innerproduct hyperspaces*, (accepted in Italian Journal of Pure and Applied Mathematics).

[8] S. Roy and T. K. Samanta, *A note on intuitionistic fuzzy hypervector spaces*, Global Journal of Computer Science and Technology, 10(10) (2010), 84–93.

[9] S. Roy and T. K. Samanta, *A note on convergence in normed hypervector spaces*, Gen. Math. Notes, 2(1) (2011), 149–158.