On some Hadamard type inequalities for MT-convex functions

Mevlüt TUNC, Yusuf SUBAŞ, Ibrahim KARABAYIR

Department of Mathematics, Faculty of Science and Arts, Kilis 7 Aralık University P.O.Box 79000 Kilis, Turkey
e-mail:mevluttunc@kilis.edu.tr
e-mail:ysubas@kilis.edu.tr
e-mail:ikarabayir@kilis.edu.tr

Abstract

In this paper, we establish some new Hadamard type inequalities for MT-convex functions and give few applications for some special means.

Keywords: Hadamard’s inequality, MT-convexity, means.

1 Introduction

The following inequality is known in the literature as Hermite-Hadamard inequality. This inequality has the update for a lot of different type convex functions in the mathematics literature.

Theorem 1.1 Let $f : I \subseteq R \rightarrow R$ be a convex function and $a < b$ with $a, b \in I$. Then

$$f \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_a^b f(x) \, dx \leq \frac{f(a) + f(b)}{2}$$

(1)

Definition 1.2 [3] We say that $f : I \rightarrow \mathbb{R}$ is Godunova-Levin function or that $f$ belongs to the class $Q(I)$ if $f$ is nonnegative and for all $x, y \in I$ and $t \in (0, 1)$ we have

$$f \left( tx + (1 - t)y \right) \leq \frac{f(x)}{t} + \frac{f(y)}{1 - t}$$

(2)
Definition 1.3 [2] We say that \( f : I \to \mathbb{R} \) is a \( P \)-function or that \( f \) belongs to the class \( P(I) \) if \( f \) is nonnegative and for all \( x, y \in I \) and \( t \in [0,1] \), we have

\[
f(tx + (1 - t)y) \leq f(x) + f(y).
\]  

Obviously, \( Q(I) \supset P(I) \) and for applications it is important to note that \( P(I) \) also consists only of nonnegative monotonic, convex and quasi-convex functions, i.e., nonnegative functions satisfying

\[
f(tx + (1 - t)y) \leq \max\{f(x), f(y)\}.
\]

In [9], Tunç and Yıldırım defined the following \( MT \)-convex functions class.

Definition 1.4 \( f : I \subseteq R \to R \) nonnegative function provides, with \( \forall x, y \in I \) and \( t \in (0,1) \),

\[
f(tx + (1 - t)y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y)
\]

\( f \) is \( MT \)-convex function. This class is shown as \( MT(I) \). If (4) change direction, \( f \) is \( MT \)-concave function.

Again in [10], the following inequalities are given by Tunç.

Theorem 1.5 Let \( f, g : [a,b] \subseteq R \to R \) two convex functions and \( f, g \in L_1 [a,b] \). Then,

\[
\frac{1}{(b-a)^2} \int_a^b (b-x) \left( f(a)g(x) + g(a)f(x) \right) dx \\
+ \frac{1}{(b-a)^2} \int_a^b (x-a) \left( f(b)g(x) + g(b)f(x) \right) dx \\
\leq \frac{M(a,b)}{3} + \frac{N(a,b)}{6} + \frac{1}{b-a} \int_a^b f(x)g(x) dx
\]

where \( M(a,b) = f(a)g(a) + f(b)g(b) \), \( N(a,b) = f(a)g(b) + f(b)g(a) \).
Theorem 1.6 Let $f, g : [a, b] \subseteq \mathbb{R} \rightarrow \mathbb{R}$ two convex functions and $f, g \in L^1[a, b]$. Then,

$$\frac{1}{(b-a)^2} \int_a^b \left( f \left( \frac{a+b}{2} \right) g(x) + g \left( \frac{a+b}{2} \right) f(x) \right) dx \leq \frac{1}{2} \int_a^b f(x) g(x) + \frac{M(a, b)}{12} + \frac{N(a, b)}{6}$$

where $M(a, b) = f(a) g(a) + f(b) g(b) - f(b) g(a)$.

Definition 1.7 [13] If $f, g : \mathbb{R} \rightarrow \mathbb{R}$ functions, for $x, y \in X$ provides the following inequality

$$(f(x) + f(y))(g(x) + g(y)) \geq 0$$

it is said that similar sequent (briefly s.o.) functions for them.

In accordance with the above studies, works performed in literature can be looked at no [1]-[13] for references given below.

The purpose of this work is to establish some new results on Hermite Hadamard inequality given (1) for $MT$-convex functions and to apply some basic inequalities for real numbers in numeric integration.

2 Results For $MT$-Convexity

Remark 2.1 i) $f, g : (1, \infty) \rightarrow \mathbb{R}$, $f(x) = x^p, g(x) = (1+x)^p, p \in (0, \frac{1}{1000})$,

ii) $h : [1, 3/2] \rightarrow \mathbb{R}$, $h(x) = (1+x^2)^m, m \in (0, \frac{1}{100})$,

are $MT$-convex functions, but they are not convex. All of the positive convex functions is also an $MT$-convex function, but the reverse is not always true. Since $f$ is $MT$-convex and $t \leq \frac{\sqrt{t}}{2\sqrt{1-t}}, (1-t) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}$, it is written

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y),$$

this indicates that each positive convex function is a $MT$-convex function.

Remark 2.2 Since $2\sqrt{t}\sqrt{1-t} \geq t(1-t)$, for $t \in (0, 1)$, it is written

$$f(ta + (1-t)b) \leq \frac{tf(a) + (1-t)f(b)}{2\sqrt{t}\sqrt{1-t}} \leq \frac{tf(a) + (1-t)f(b)}{t(1-t)}.$$
As can be seen from this inequality, each $MT$-convex function is $Q(I)$-Godunova-Levin function. However, $MT$-convex functions class allows us to obtain a better upper bound than $Q(I)$-Godunova-Levin function. Obviously, $Q(I) \supset MT(I)$. Moreover, for $t \in [0.2, 0.8]$, we have

$$f(ta + (1-t)b) \leq \frac{tf(a) + (1-t)f(b)}{2\sqrt{t\sqrt{1-t}}} \leq f(a) + f(b).$$

Then, we can say that each $MT$-convex function is $P(I)$ function, on $[0.2, 0.8]$. So, $P(I) \supset MT(I)$.

**Theorem 2.3** Let $f : [a, b] \subseteq R \rightarrow R$ a nonnegative $MT$-convex function and $f \in L_1[a, b]$. Then,

$$\frac{1}{b-a} \int_a^b f(x) \, dx \leq \frac{\pi}{4} (f(a) + f(b)) \tag{5}$$

**Proof:** i) Because $f$ is a $MT$-convex function, if we get the integral of

$$f(ta + (1-t)b) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(a) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(b)$$

inequality $(0, 1)$ for $t$, the proof is completed.

The above inequality can be proven by another way given below.

ii) Since $f$ is a $MT$-convex function, we can write

$$f(ta + (1-t)b) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(a) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(b) \tag{6}$$

$$f(tb + (1-t)a) \leq \frac{\sqrt{t}}{2\sqrt{1-t}} f(b) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(a) \tag{7}$$

By adding (6) and (7), we get

$$f(ta + (1-t)b) + f(tb + (1-t)a) \leq \left(\frac{1}{2\sqrt{t\sqrt{1-t}}}\right) (f(a) + f(b)) \tag{8}$$

By integrating above inequality (8), according to $t$ over $[0, 1]$, we obtain

$$\int_0^1 (f(ta + (1-t)b) + f(tb + (1-t)a)) \, dt \leq (f(a) + f(b)) \int_0^1 \frac{1}{2\sqrt{t\sqrt{1-t}}} \, dt$$
By taking into account
\[
\int_0^1 f (ta + (1 - t)b) \, dt = \int_0^1 f (tb + (1 - t)a) \, dt = \frac{1}{b - a} \int_a^b f (x) \, dx
\]
and
\[
\int_0^1 \frac{1}{2 \sqrt{t(1 - t)}} \, dt = \frac{1}{2\pi}
\]
the proof is completed.

**Theorem 2.4** Let \(f, g : [a, b] \subseteq R \rightarrow R\) two nonnegative MT-convex functions and \(f, g \in L_1 [a, b]\). Then, the following inequality holds:

\[
g (a) \frac{(b - x) \sqrt{(b - x)(x - a)}}{(b - a)^3} \int_a^b f (x) \, dx + g (b) \frac{(x - a) \sqrt{(b - x)(x - a)}}{(b - a)^3} \int_a^b f (x) \, dx + f (a) \frac{(b - x) \sqrt{(b - x)(x - a)}}{(b - a)^3} \int_a^b g (x) \, dx + f (b) \frac{(x - a) \sqrt{(b - x)(x - a)}}{(b - a)^3} \int_a^b g (x) \, dx
\]

\[
\leq \frac{1}{2} \left\{ \frac{1}{3} f (a) g (a) + \frac{1}{3} f (b) g (b) + \frac{1}{6} [f (a) g (b) + f (b) g (a)] + \int_a^b (b - x)(x - a) f (x) g (x) \, dx \right\}
\]

where \(M (a, b), N (a, b)\) are like above.

**Proof:** Since \(f\) and \(g\) are MT-convex, it is written

\[
f (ta + (1 - t)b) \leq \frac{\sqrt{t}}{2 \sqrt{1 - t}} f (a) + \frac{\sqrt{1 - t}}{2 \sqrt{t}} f (b)
\]

\[
g (ta + (1 - t)b) \leq \frac{\sqrt{t}}{2 \sqrt{1 - t}} g (a) + \frac{\sqrt{1 - t}}{2 \sqrt{t}} g (b)
\]

By using \(er + fp \leq ep + fr, e, f, r, p \in R^+\) basic inequality, it is written

\[
f (ta + (1 - t)b) \left( \frac{\sqrt{t}}{2 \sqrt{1 - t}} g (a) + \frac{\sqrt{1 - t}}{2 \sqrt{t}} g (b) \right) + g (ta + (1 - t)b) \left( \frac{\sqrt{t}}{2 \sqrt{1 - t}} f (a) + \frac{\sqrt{1 - t}}{2 \sqrt{t}} f (b) \right)
\]

\[
\leq \left[ \frac{\sqrt{t}}{2 \sqrt{1 - t}} f (a) + \frac{\sqrt{1 - t}}{2 \sqrt{t}} f (b) \right] \left[ \frac{\sqrt{t}}{2 \sqrt{1 - t}} g (a) + \frac{\sqrt{1 - t}}{2 \sqrt{t}} g (b) \right] + f (ta + (1 - t)b) g (ta + (1 - t)b)
\]
By substituting inequality (11) according to
If this statement is edited, we get
\[ g(a) \frac{\sqrt{t}}{2\sqrt{1-t}} f(ta + (1-t)b) + g(b) \frac{\sqrt{1-t}}{2\sqrt{t}} f(ta + (1-t)b) \]
\[ + f(a) \frac{\sqrt{t}}{2\sqrt{1-t}} g(ta + (1-t)b) + f(b) \frac{\sqrt{1-t}}{2\sqrt{t}} g(ta + (1-t)b) \]
\[ \leq \frac{t}{4(1-t)} f(a) g(a) + \frac{1-t}{4t} f(b) g(b) + \frac{1}{4} f(a) g(b) + \frac{1}{4} f(b) g(a) \]
\[ + f(ta + (1-t) b) g(ta + (1-t) b) \]
If both sides of (10) inequality are multiplied by \( t (1-t) \), we get
\[ g(a) t \sqrt{t} \sqrt{1-t} f(ta + (1-t) b) + g(b) (1-t) \sqrt{1-t} f(ta + (1-t) b) \]
\[ + f(a) t \sqrt{t} \sqrt{1-t} g(ta + (1-t) b) + f(b) (1-t) \sqrt{1-t} g(ta + (1-t) b) \]
\[ \leq \frac{1}{2} \left\{ t^2 f(a) g(a) + (1-t)^2 f(b) g(b) \right. \]
\[ + t (1-t) [f(a) g(b) + f(b) g(a)] \]
\[ + t (1-t) [f(ta + (1-t) b) + g(ta + (1-t) b)]] \}
By integrating inequality (11) according to \( t \) over \([0,1]\),
\[ g(a) \int_0^1 t \sqrt{t} \sqrt{1-t} f(ta + (1-t) b) dt \]
\[ + f(a) \int_0^1 t \sqrt{t} \sqrt{1-t} g(ta + (1-t) b) dt \]
\[ \leq \frac{1}{2} \left\{ f(a) g(a) \int_0^1 t^2 dt + f(b) g(b) \int_0^1 (1-t)^2 dt \right. \]
\[ + [f(a) g(b) + f(b) g(a)] \int_0^1 t (1-t) dt \]
\[ + \int_0^1 t (1-t) [f(ta + (1-t) b) + g(ta + (1-t) b)] dt \}
By substituting \( ta + (1-t) b = x, \ (a-b) dt = dx, \) we get
\[ \int_0^1 t \sqrt{t} \sqrt{1-t} f(ta + (1-t) b) dt = \frac{b-x}{b-a} \sqrt{(b-x)(x-a)} \int_a^b f(x) dx \]
\[
\int_0^1 (1-t) \sqrt{1-t} f \left( t a + (1-t) b \right) dt = \frac{x - a}{b - a} \sqrt{\frac{(b-x)(x-a)}{(b-a)}} \frac{1}{b-a} \int_a^b f(x) \, dx
\]

(14)

similarly

\[
\int_0^1 t \sqrt{1-t} g \left( t a + (1-t) b \right) dt = \frac{b - x}{b - a} \sqrt{\frac{(b-x)(x-a)}{(b-a)}} \frac{1}{b-a} \int_a^b g(x) \, dx
\]

(15)

\[
\int_0^1 (1-t) \sqrt{1-t} g \left( t a + (1-t) b \right) dt = \frac{x - a}{b - a} \sqrt{\frac{(b-x)(x-a)}{(b-a)}} \frac{1}{b-a} \int_a^b g(x) \, dx
\]

(16)

and

\[
\int_0^1 t^2 dt = \int_0^1 (1-t)^2 dt = \frac{1}{3}, \quad \int_0^1 t (1-t) dt = \frac{1}{6}
\]

(17)

If these statements (13)-(17) can be used in (12), the proof is completed.

**Theorem 2.5** Let \( f, g \in [a,b] \to R \) two nonnegative MT-convex functions and \( f, g \in L_1[a,b] \). Then,

\[
\frac{8}{3} f \left( \frac{a+b}{2} \right) \leq M(a,b) + N(a,b)
\]

(18)

where \( M \) and \( N \) are like above.

**Proof:** Since \( f \) and \( g \) are MT-convex functions, we can write

\[
f \left( \frac{a+b}{2} \right) = f \left( \frac{ta + (1-t) b}{2} + \frac{(1-t) a + tb}{2} \right)
\]

(19)

\[
\leq \frac{1}{2} \left[ f \left( ta + (1-t) b \right) + f \left( (1-t) a + tb \right) \right]
\]

\[
\leq \frac{1}{2} \left( \frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}} \right) \left( f(a) + f(b) \right)
\]

and

\[
g \left( \frac{a+b}{2} \right) = g \left( \frac{ta + (1-t) b}{2} + \frac{(1-t) a + tb}{2} \right)
\]

(20)

\[
\leq \frac{1}{2} \left( \frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}} \right) \left( g(a) + g(b) \right)
\]
By multiplying (19) and (20) we get
\[
fg\left(\frac{a + b}{2}\right) (a + b)^2 \leq \left[\frac{1}{4} \left(\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}}\right)\right]^2 (f(a) + f(b)) (g(a) + g(b))
\]
\[
= \frac{1}{16} \left(\frac{t}{1-t} + \frac{1-t}{t} + 2\right) (f(a) + f(b)) (g(a) + g(b))
\]
\[
= \frac{1}{16} \left(\frac{1}{t(1-t)}\right) (f(a) + f(b)) (g(a) + g(b))
\]
If both sides of (21) inequality are multiplied by \(t(1-t)\), we get
\[
t(1-t)fg\left(\frac{a + b}{2}\right) \leq \frac{1}{16} (f(a) + f(b)) (g(a) + g(b))
\] (22)

By integrating inequality (22) according to \(t\) over \([0, 1]\), the proof is completed.

**Corollary 2.6** In Theorem 2.5, if \(f\) and \(g\) are s.o., we obtain
\[
\frac{4}{3}fg\left(\frac{a + b}{2}\right) \leq M(a,b)
\]

**Theorem 2.7** Let \(f, g : [a, b] \subseteq R \rightarrow R\) two nonnegative MT-convex functions and \(f, g \in L_1[a, b]\). Then, we have
\[
f\left(\frac{a + b}{2}\right) (g(a) + g(b)) + g\left(\frac{a + b}{2}\right) (f(a) + f(b)) \leq \frac{16}{3\pi} fg\left(\frac{a + b}{2}\right) + 2 (f(a) + f(b)) (g(a) + g(b))
\] (23)

**Proof:** Since \(f\) and \(g\) are MT-convex functions, we can write
\[
f\left(\frac{a + b}{2}\right) \leq \frac{1}{2} \left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}\right) (f(a) + f(b))
\]
\[
g\left(\frac{a + b}{2}\right) \leq \frac{1}{2} \left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}\right) (g(a) + g(b))
\]
By using \(er + fp \leq ep + fr\), \(e, f, r, p \in R^+\) basic inequality, we obtain,
\[
\frac{1}{2} f\left(\frac{a + b}{2}\right) \left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}\right) (g(a) + g(b))
\]
\[
+ \frac{1}{2} g\left(\frac{a + b}{2}\right) \left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}\right) (f(a) + f(b))
\]
\[
\leq fg\left(\frac{a + b}{2}\right) + \left(\frac{1}{4} \left(\frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}}\right)\right)^2 (f(a) + f(b)) (g(a) + g(b))
\]
If this statement is edited, we get

\[
\frac{1}{4} f \left( \frac{a + b}{2} \right) (g(a) + g(b)) \left( \frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}} \right) + \frac{1}{4} g \left( \frac{a + b}{2} \right) (f(a) + f(b)) \left( \frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}} \right) \leq f g \left( \frac{a + b}{2} \right) \left( \frac{\sqrt{t}}{\sqrt{1-t}} + \frac{\sqrt{1-t}}{\sqrt{t}} \right) \] (24)

If both sides of (24) are multiplied by \((1-t)\), we get

\[
\frac{1}{4} f \left( \frac{a + b}{2} \right) (g(a) + g(b)) \left( t \sqrt{1-t} + (1-t) \sqrt{t} \right) + \frac{1}{4} g \left( \frac{a + b}{2} \right) (f(a) + f(b)) \left( t \sqrt{1-t} + (1-t) \sqrt{t} \right) \leq f g \left( \frac{a + b}{2} \right) t (1-t) + \frac{1}{16} (f(a) + f(b)) (g(a) + g(b)) \] (25)

By integrating of inequality (25), according to \(t\) over \([0,1]\), we get

\[
\frac{1}{4} f \left( \frac{a + b}{2} \right) (g(a) + g(b)) \left( \int_0^1 t \sqrt{1-t} dt + \int_0^1 (1-t) \sqrt{t} dt \right) + \frac{1}{4} g \left( \frac{a + b}{2} \right) (f(a) + f(b)) \left( \int_0^1 t \sqrt{1-t} dt + \int_0^1 (1-t) \sqrt{t} dt \right) \leq f g \left( \frac{a + b}{2} \right) \int_0^1 t (1-t) dt + \frac{1}{16} (f(a) + f(b)) (g(a) + g(b)) .
\]

By accounting the following equalities

\[
\int_0^1 t \sqrt{1-t} dt = \int_0^1 t^{\frac{3}{2}} (1-t)^{\frac{1}{2}} dt = \frac{\pi}{16}
\]

\[
\int_0^1 (1-t) \sqrt{1-t} dt = \int_0^1 t^{\frac{3}{2}} (1-t)^{\frac{3}{2}} dt = \frac{\pi}{16}
\]

the proof is completed.

3 Applications to some special means

We now consider the applications of our Theorems to the following special means
The arithmetic mean: $A = A(a, b) := \frac{a + b}{2}, \ a, b \geq 0,$

The geometric mean: $G = G(a, b) := \sqrt{ab}, \ a, b \geq 0,$

The identric mean: $I = I(a, b) := \begin{cases} a & \text{if } a = b \\ \frac{1}{e} \left( \frac{b}{a} \right)^{1/a} & \text{if } a \neq b \end{cases}, \ a, b \geq 0,$

The p-logarithmic mean: $L_p = L_p(a, b) := \begin{cases} b^{p+1} - a^{p+1} & \text{if } a \neq b \\ \frac{1}{p} & \text{if } a = b \end{cases}, \ p \in R \setminus \{-1, 0\}; \ a, b > 0.$

The following propositions holds:

**Proposition 3.1** Let $0 < a < b.$ Then one has the inequality

$$I(a, b) \geq G^{\pi/2}(a, b). \quad (26)$$

**Proof:** If we choose in Theorem 2.3, applied to the $MT$-convex function $f : (0, 1] \to R, \ f(x) = -\ln x,$ we obtain

$$\frac{-1}{b - a} \int_a^b \ln x dx \leq \frac{-\pi}{4} (\ln a + \ln b)$$

$$\leq \frac{-\pi}{4} \ln(ab)$$

$$\leq \frac{-\pi}{4} \ln \left( G^2(a, b) \right)$$

$$\leq \frac{-\ln \left( G^{\pi/2}(a, b) \right) }{}$$

which gives us

$$-\ln I(a, b) \leq -\ln \left( G^{\pi/2}(a, b) \right)$$

and the inequality is proved.

**Proposition 3.2** Let $1 < a < b, p \in (0, \frac{1}{1000})$, then we have

$$L_p(a, b) \leq \frac{\pi}{2} A(a^p, b^p). \quad (27)$$

**Proof:** The inequality follows from (5) applied to the $MT$-convex function $f : (1, \infty) \to R, \ f(x) = x^p, \ p \in (0, \frac{1}{1000}).$ The details are omitted.

**Proposition 3.3** Let $1 < a < b, p \in (0, \frac{1}{1000})$, then, we have

$$\frac{\sqrt{(b - x)(x - a)}}{(b - a)} L_p(a, b) \{ a^p (b - x) + b^p (x - a) \}$$

$$\leq \frac{1}{4} \left\{ \frac{1}{3} a^{2p} + \frac{1}{3} b^{2p} + \frac{1}{3} (ab)^p \right\}$$

$$- (b - a) L_{2p+2}^2(a, b) - L_{2p+1}^2(a, b) - ab (b - a) L_{2p}^2(a, b) \quad (28)$$
Proof: The inequality follows from (9) applied to the $MT$-convex functions $f, g : R \to R$, $f(x) = g(x) = x^p, p \in \left(0, \frac{1}{1000}\right)$. The details are omitted.

4 Open Problem

It is a well-known fact that if $f$ is a convex function on the interval $I \subset \mathbb{R}$, then the Hadamard’s inequality retains for the convex functions. As a matter of fact, it has been demonstrated lots of this type inequalities for several convex functions. Therefore, there is one questions as follows:

How can be set up the general versions of the inequalities (5), (9), (18) and (23) including several differentiable $MT$-convex function on $I$.

References


