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On some Hadamard type inequalities for MT-convex functions

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Abstract

In this paper, we establish some new Hadamard type inequalities for MT-convex functions and give few applications for some special means.
Keywords: *Hadamard's inequality, MT-convexity, means.*

1 Introduction

The following inequality is known in the literature as Hermite-Hadamard inequality. This inequality has the update for a lot of different type convex functions in the mathematics literature.

Theorem 1.1 *Let $f : I \subseteq R \to R$ be a convex function and $a < b$ with $a, b \in I$. Then*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

Definition 1.2 *[3] We say that $f : I \to \mathbb{R}$ is Godunova-Levin function or that f belongs to the class $Q(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in (0,1)$ we have*

$$f(tx+(1-t)y) \leq \frac{f(x)}{t} + \frac{f(y)}{1-t}. \tag{2}$$

Definition 1.3 *[2] We say that $f : I \to \mathbb{R}$ is a P-function or that f belongs to the class $P(I)$ if f is nonnegative and for all $x, y \in I$ and $t \in [0,1]$, we have*

$$f(tx + (1-t)y) \le f(x) + f(y). \tag{3}$$

Obviously, $Q(I) \supset P(I)$ and for applications it is important to note that $P(I)$ also consists only of nonnegative monotonic, convex and quasi-convex functions, i.e., nonnegative functions satisfying

$$f(tx + (1-t)y) \le \max\{f(x), f(y)\}.$$

In [9], Tunç and Yıldırım defined the following MT-convex functions class.

Definition 1.4 *$f : I \subseteq R \to R$ nonnegative function provides, with $\forall x, y \in I$ and $t \in (0,1)$,*

$$f(tx + (1-t)y) \le \frac{\sqrt{t}}{2\sqrt{1-t}} f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}} f(y) \tag{4}$$

f is MT-convex function. This class is shown as $MT(I)$. If (4) change direction, f is MT-concave function.

Again in [10], the following inequalities are given by Tunç.

Theorem 1.5 *Let $f, g : [a,b] \subseteq R \to R$ two convex functions and $f, g \in L_1[a,b]$. Then,*

$$\begin{aligned}
&\frac{1}{(b-a)^2}\int_a^b (b-x)\left(f(a)g(x) + g(a)f(x)\right)dx \\
&+\frac{1}{(b-a)^2}\int_a^b (x-a)\left(f(b)g(x) + g(b)f(x)\right)dx \\
\le\ &\frac{M(a,b)}{3} + \frac{N(a,b)}{6} + \frac{1}{b-a}\int_a^b f(x)g(x)\,dx
\end{aligned}$$

where $M(a,b) = f(a)g(a) + f(b)g(b)$, $N(a,b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.6 *Let $f, g : [a, b] \subseteq R \to R$ two convex functions and $f, g \in L_1[a, b]$. Then,*

$$\begin{aligned} & \frac{1}{(b-a)^2}\int_a^b \left(f\left(\frac{a+b}{2}\right)g(x)+g\left(\frac{a+b}{2}\right)f(x)\right)dx \\ \leq\ & \frac{1}{2(b-a)}\int_a^b f(x)\,g(x)+\frac{M(a,b)}{12}+\frac{N(a,b)}{6} \\ & +f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \end{aligned}$$

where $M(a,b) = f(a)\,g(a) + f(b)\,g(b),\quad N(a,b) = f(a)\,g(b) + f(b)\,g(a)$.

Definition 1.7 *[13]If $f, g : X \to R$ functions, for $x, y \in X$ provides the following inequality*

$$(f(x) + f(y))\,(g(x) + g(y)) \geq 0$$

it is said that similar sequent (briefly s.o.) functions for them.

In accordance with the above studies, works performed in literature can be looked at no [1]-[13] for references given below.

The purpose of this work is to establish some new results on Hermite Hadamard inequality given (1) for MT-convex functions and to apply some basic inequalities for real numbers in numeric integration.

2 Results For MT-Convexity

Remark 2.1 *i)* $f, g : (1, \infty) \to \mathbb{R},\ f(x) = x^p, g(x) = (1+x)^p,\ p \in \left(0, \frac{1}{1000}\right)$,

ii) $h : [1, 3/2] \to \mathbb{R},\ h(x) = (1+x^2)^m, m \in \left(0, \frac{1}{100}\right)$,

are MT-convex functions, but they are not convex. All of the positive convex functions is also an MT-convex function, but the reverse is not always true. Since f is MT-convex and $t \leq \frac{\sqrt{t}}{2\sqrt{1-t}}$, $(1-t) \leq \frac{\sqrt{1-t}}{2\sqrt{t}}$, it is written

$$f(tx + (1-t)\,y) \leq tf(x) + (1-t)\,f(y) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(x) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(y),$$

this indicates that each positive convex function is a MT-convex function.

Remark 2.2 *Since $2\sqrt{t}\sqrt{1-t} \geq t(1-t)$, for $t \in (0,1)$, it is written*

$$f(ta + (1-t)\,b) \leq \frac{tf(a) + (1-t)\,f(b)}{2\sqrt{t}\sqrt{1-t}} \leq \frac{tf(a) + (1-t)\,f(b)}{t(1-t)}.$$

As can be seen from this inequality ,each MT-convex function is $Q(I)$-Godunova-Levin function. However, MT-convex functions class allows us to obtain a better upper bound than $Q(I)$-Godunova-Levin function. Obviously, $Q(I) \supset MT(I)$. Moreover, for $t \in [0.2, 0.8]$, we have

$$f(ta + (1-t)b) \leq \frac{tf(a) + (1-t)f(b)}{2\sqrt{t}\sqrt{1-t}} \leq f(a) + f(b).$$

Then, we can say that each MT-convex function is $P(I)$ function, on $[0.2, 0.8]$. So, $P(I) \supset MT(I)$.

Theorem 2.3 *Let $f : [a, b] \subseteq R \to R$ a nonnegative MT-convex function and $f \in L_1[a, b]$. Then,*

$$\frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{\pi}{4}(f(a) + f(b)) \tag{5}$$

Proof: i) Because f is a MT-convex function, if we get the integral of

$$f(ta + (1-t)b) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(a) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(b)$$

inequality $(0, 1)$ for t, the proof is completed.

The above inequality can be proven by another way given below.

ii) Since f is a MT-convex function, we can write

$$f(ta + (1-t)b) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(a) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(b) \tag{6}$$

$$f(tb + (1-t)a) \leq \frac{\sqrt{t}}{2\sqrt{1-t}}f(b) + \frac{\sqrt{1-t}}{2\sqrt{t}}f(a) \tag{7}$$

By adding (6) and (7), we get

$$f(ta + (1-t)b) + f(tb + (1-t)a) \leq \left(\frac{1}{2\sqrt{t}\sqrt{1-t}}\right)(f(a) + f(b)) \tag{8}$$

By integrating above inequality (8), according to t over $[0, 1]$, we obtain

$$\int_0^1 (f(ta + (1-t)b) + f(tb + (1-t)a))\,dt \leq (f(a) + f(b))\int_0^1 \frac{1}{2\sqrt{t(1-t)}}dt$$

By taking into account

$$\int_0^1 f(ta+(1-t)b)\,dt=\int_0^1 f(tb+(1-t)a)\,dt=\frac{1}{b-a}\int_a^b f(x)\,dx$$

and

$$\int_0^1 \frac{1}{2\sqrt{t(1-t)}}dt=\frac{1}{2}\pi$$

the proof is completed.

Theorem 2.4 *Let $f,g:[a,b]\subseteq R\rightarrow R$ two nonnegative MT-convex functions and $f,g\in L_1[a,b]$. Then, the following inequality holds;*

$$\begin{aligned}
&g(a)\frac{(b-x)\sqrt{(b-x)(x-a)}}{(b-a)^3}\int_a^b f(x)\,dx+g(b)\frac{(x-a)\sqrt{(b-x)(x-a)}}{(b-a)^3}\int_a^b f(x)\,dx\\
&+f(a)\frac{(b-x)\sqrt{(b-x)(x-a)}}{(b-a)^3}\int_a^b g(x)\,dx+f(b)\frac{(x-a)\sqrt{(b-x)(x-a)}}{(b-a)^3}\int_a^b g(x)\,dx\\
\leq\;&\frac{1}{2}\left\{\frac{1}{3}f(a)g(a)+\frac{1}{3}f(b)g(b)+\frac{1}{6}[f(a)g(b)+f(b)g(a)]\right.\\
&\left.+\int_a^b (b-x)(x-a)f(x)g(x)\,dx\right\}
\end{aligned}\tag{9}$$

where $M(a,b)$, $N(a,b)$ are like above.

Proof: Since f and g are MT-convex, it is written

$$f(ta+(1-t)b)\leq\frac{\sqrt{t}}{2\sqrt{1-t}}f(a)+\frac{\sqrt{1-t}}{2\sqrt{t}}f(b)$$

$$g(ta+(1-t)b)\leq\frac{\sqrt{t}}{2\sqrt{1-t}}g(a)+\frac{\sqrt{1-t}}{2\sqrt{t}}g(b)$$

By using $er+fp\leq ep+fr$, $e,f,r,p\in R^+$ basic inequality, it is written

$$\begin{aligned}
&f(ta+(1-t)b)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}g(a)+\frac{\sqrt{1-t}}{2\sqrt{t}}g(b)\right)\\
&+g(ta+(1-t)b)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}f(a)+\frac{\sqrt{1-t}}{2\sqrt{t}}f(b)\right)\\
\leq\;&\left[\frac{\sqrt{t}}{2\sqrt{1-t}}f(a)+\frac{\sqrt{1-t}}{2\sqrt{t}}f(b)\right]\left[\frac{\sqrt{t}}{2\sqrt{1-t}}g(a)+\frac{\sqrt{1-t}}{2\sqrt{t}}g(b)\right]\\
&+f(ta+(1-t)b)\,g(ta+(1-t)b)
\end{aligned}$$

If this statement is edited, we get

$$
\begin{aligned}
& g(a)\frac{\sqrt{t}}{2\sqrt{1-t}}f(ta+(1-t)b)+g(b)\frac{\sqrt{1-t}}{2\sqrt{t}}f(ta+(1-t)b) \\
& +f(a)\frac{\sqrt{t}}{2\sqrt{1-t}}g(ta+(1-t)b)+f(b)\frac{\sqrt{1-t}}{2\sqrt{t}}g(ta+(1-t)b) \\
\leq\ & \frac{t}{4(1-t)}f(a)g(a)+\frac{1-t}{4t}f(b)g(b)+\frac{1}{4}f(a)g(b)+\frac{1}{4}f(b)g(a) \\
& +f(ta+(1-t)b)g(ta+(1-t)b)
\end{aligned}
\tag{10}
$$

If both sides of (10) inequality are multiplied by $t(1-t)$, we get

$$
\begin{aligned}
& g(a)t\sqrt{t}\sqrt{1-t}f(ta+(1-t)b)+g(b)(1-t)\sqrt{t}\sqrt{1-t}f(ta+(1-t)b) \\
& +f(a)t\sqrt{t}\sqrt{1-t}g(ta+(1-t)b)+f(b)(1-t)\sqrt{t}\sqrt{1-t}g(ta+(1-t)b) \\
\leq\ & \frac{1}{2}\left\{t^2f(a)g(a)+(1-t)^2f(b)g(b)\right. \\
& +t(1-t)[f(a)g(b)+f(b)g(a)] \\
& \left.+t(1-t)[f(ta+(1-t)b)+g(ta+(1-t)b)]\right\}
\end{aligned}
\tag{11}
$$

By integrating inequality (11) according to t over $[0,1]$,

$$
\begin{aligned}
& g(a)\int_0^1 t\sqrt{t}\sqrt{1-t}f(ta+(1-t)b)\,dt \\
& +g(b)\int_0^1 (1-t)\sqrt{t}\sqrt{1-t}f(ta+(1-t)b)\,dt \\
& +f(a)\int_0^1 t\sqrt{t}\sqrt{1-t}g(ta+(1-t)b)\,dt \\
& +f(b)\int_0^1 (1-t)\sqrt{t}\sqrt{1-t}g(ta+(1-t)b)\,dt \\
\leq\ & \frac{1}{2}\left\{f(a)g(a)\int_0^1 t^2dt+f(b)g(b)\int_0^1(1-t)^2dt\right. \\
& +[f(a)g(b)+f(b)g(a)]\int_0^1 t(1-t)\,dt \\
& \left.+\int_0^1 t(1-t)[f(ta+(1-t)b)+g(ta+(1-t)b)]\,dt\right\}
\end{aligned}
\tag{12}
$$

By substituting $ta+(1-t)b=x,\ \ (a-b)\,dt=dx,$ we get

$$
\int_0^1 t\sqrt{t}\sqrt{1-t}f(ta+(1-t)b)\,dt=\frac{b-x}{b-a}\frac{\sqrt{(b-x)(x-a)}}{(b-a)}\frac{1}{b-a}\int_a^b f(x)\,dx
\tag{13}
$$

$$\int_0^1 (1-t)\sqrt{t}\sqrt{1-t}f(ta+(1-t)b)\,dt = \frac{x-a}{b-a}\frac{\sqrt{(b-x)(x-a)}}{(b-a)}\frac{1}{b-a}\int_a^b f(x)\,dx \tag{14}$$

similarly

$$\int_0^1 t\sqrt{t}\sqrt{1-t}g(ta+(1-t)b)\,dt = \frac{b-x}{b-a}\frac{\sqrt{(b-x)(x-a)}}{(b-a)}\frac{1}{b-a}\int_a^b g(x)\,dx \tag{15}$$

$$\int_0^1 (1-t)\sqrt{t}\sqrt{1-t}g(ta+(1-t)b)\,dt = \frac{x-a}{b-a}\frac{\sqrt{(b-x)(x-a)}}{(b-a)}\frac{1}{b-a}\int_a^b g(x)\,dx \tag{16}$$

and

$$\int_0^1 t^2dt = \int_0^1 (1-t)^2\,dt = \frac{1}{3},\ \int_0^1 t(1-t)\,dt = \frac{1}{6} \tag{17}$$

If these statements (13)-(17)can be used in (12), the proof is completed.

Theorem 2.5 *Let $f, g \in [a,b] \to R$ two nonnegative MT-convex functions and $f, g \in L_1[a,b]$. Then,*

$$\frac{8}{3}f\left(\frac{a+b}{2}\right) \leq M(a,b) + N(a,b) \tag{18}$$

where M and N are like above.

Proof: Since f and g are MT-convex functions, we can write

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ &\leq \frac{1}{2}\left[f(ta+(1-t)b) + f((1-t)a+tb)\right] \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}\right)(f(a)+f(b)) \end{aligned} \tag{19}$$

and

$$\begin{aligned} g\left(\frac{a+b}{2}\right) &= g\left(\frac{ta+(1-t)b}{2} + \frac{(1-t)a+tb}{2}\right) \\ &\leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}} + \frac{\sqrt{1-t}}{2\sqrt{t}}\right)(g(a)+g(b)) \end{aligned} \tag{20}$$

By multiplying (19) and (20) we get

$$
\begin{aligned}
& fg\left(\frac{a+b}{2}\right) \qquad (21)\\
\leq\; & \left[\frac{1}{4}\left(\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right)\right]^2 (f(a)+f(b))(g(a)+g(b))\\
=\; & \frac{1}{16}\left(\frac{t}{1-t}+\frac{1-t}{t}+2\right)(f(a)+f(b))(g(a)+g(b))\\
=\; & \frac{1}{16}\left(\frac{1}{t(1-t)}\right)(f(a)+f(b))(g(a)+g(b))
\end{aligned}
$$

If both sides of (21) inequality are multiplied by $t(1-t)$, we get

$$
t(1-t)\,fg\left(\frac{a+b}{2}\right) \leq \frac{1}{16}(f(a)+f(b))(g(a)+g(b)) \qquad (22)
$$

By integrating inequality (22) according to t over $[0,1]$, the proof is completed.

Corollary 2.6 *In Theorem 2.5, if f and g are s.o., we obtain*

$$
\frac{4}{3}fg\left(\frac{a+b}{2}\right) \leq M(a,b)
$$

Theorem 2.7 *Let $f,g:[a,b]\subseteq R\to R$ two nonnegative MT-convex functions and $f,g\in L_1[a,b]$. Then, we have*

$$
\begin{aligned}
& f\left(\frac{a+b}{2}\right)(g(a)+g(b))+g\left(\frac{a+b}{2}\right)(f(a)+f(b)) \qquad (23)\\
\leq\; & \frac{16}{3\pi}fg\left(\frac{a+b}{2}\right)+2(f(a)+f(b))(g(a)+g(b))
\end{aligned}
$$

Proof: Since f and g are MT-convex functions, we can write

$$
f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{\sqrt{1-t}}{2\sqrt{t}}\right)(f(a)+f(b))
$$

$$
g\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{\sqrt{1-t}}{2\sqrt{t}}\right)(g(a)+g(b))
$$

By using $er+fp\leq ep+fr$, $e,f,r,p\in R^+$ basic inequality, we obtain,

$$
\begin{aligned}
& \frac{1}{2}f\left(\frac{a+b}{2}\right)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{\sqrt{1-t}}{2\sqrt{t}}\right)(g(a)+g(b))\\
& +\frac{1}{2}g\left(\frac{a+b}{2}\right)\left(\frac{\sqrt{t}}{2\sqrt{1-t}}+\frac{\sqrt{1-t}}{2\sqrt{t}}\right)(f(a)+f(b))\\
\leq\; & fg\left(\frac{a+b}{2}\right)+\left(\frac{1}{4}\left(\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right)\right)^2(f(a)+f(b))(g(a)+g(b))
\end{aligned}
$$

If this statement is edited, we get

$$\begin{aligned}
&\frac{1}{4}f\left(\frac{a+b}{2}\right)(g(a)+g(b))\left(\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right) \\
&+\frac{1}{4}g\left(\frac{a+b}{2}\right)(f(a)+f(b))\left(\frac{\sqrt{t}}{\sqrt{1-t}}+\frac{\sqrt{1-t}}{\sqrt{t}}\right) \\
\leq\ & fg\left(\frac{a+b}{2}\right)+\frac{1}{16}\frac{1}{t(1-t)}(f(a)+f(b))(g(a)+g(b))
\end{aligned} \tag{24}$$

If both sides of (24) are multiplied by $t(1-t)$, we get

$$\begin{aligned}
&\frac{1}{4}f\left(\frac{a+b}{2}\right)(g(a)+g(b))\left(t\sqrt{t}\sqrt{1-t}+(1-t)\sqrt{t}\sqrt{1-t}\right) \\
&+\frac{1}{4}g\left(\frac{a+b}{2}\right)(f(a)+f(b))\left(t\sqrt{t}\sqrt{1-t}+(1-t)\sqrt{t}\sqrt{1-t}\right) \\
\leq\ & fg\left(\frac{a+b}{2}\right)t(1-t)+\frac{1}{16}(f(a)+f(b))(g(a)+g(b))
\end{aligned} \tag{25}$$

By integrating of inequality (25), according to t over $[0,1]$, we get

$$\begin{aligned}
&\frac{1}{4}f\left(\frac{a+b}{2}\right)(g(a)+g(b))\left(\int_0^1 t\sqrt{t}\sqrt{1-t}dt+\int_0^1(1-t)\sqrt{t}\sqrt{1-t}dt\right) \\
&+\frac{1}{4}g\left(\frac{a+b}{2}\right)(f(a)+f(b))\left(\int_0^1 t\sqrt{t}\sqrt{1-t}dt+\int_0^1(1-t)\sqrt{t}\sqrt{1-t}dt\right) \\
\leq\ & fg\left(\frac{a+b}{2}\right)\int_0^1 t(1-t)\,dt+\frac{1}{16}(f(a)+f(b))(g(a)+g(b)).
\end{aligned}$$

By accounting the following equalities

$$\int_0^1 t\sqrt{t}\sqrt{1-t}dt=\int_0^1 t^{\frac{3}{2}}(1-t)^{\frac{1}{2}}\,dt=\frac{\pi}{16}$$

$$\int_0^1 (1-t)\sqrt{t}\sqrt{1-t}dt=\int_0^1 t^{\frac{1}{2}}(1-t)^{\frac{3}{2}}\,dt=\frac{\pi}{16}$$

the proof is completed.

3 Applications to some special means

We now consider the applications of our Theorems to the following special means

The arithmetic mean: $A = A(a,b) := \frac{a+b}{2}$, $a,b \geq 0$,
The geometric mean: $G = G(a,b) := \sqrt{ab}$, $a,b \geq 0$,
The identric mean: $I = I(a,b) := \begin{cases} a & \text{if } \quad a = b \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } \quad a \neq b \end{cases}$, $a,b \geq 0$,
The p-logarithmic mean: $L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{1/p} & \text{if } \quad a \neq b \\ a & \text{if } \quad a = b \end{cases}$,
$p \in R \backslash \{-1, 0\}$; $a, b > 0$.

The following propositions holds:

Proposition 3.1 *Let* $0 < a < b$. *Then one has the inequality*

$$I(a,b) \geq G^{\frac{\pi}{2}}(a,b). \tag{26}$$

Proof: If we choose in Theorem 2.3, applied to the MT-convex function $f : (0,1] \to R$, $f(x) = -\ln x$, we obtain

$$\begin{aligned} -\frac{1}{b-a}\int_a^b \ln x dx &\leq -\frac{\pi}{4}(\ln a + \ln b) \\ &\leq -\frac{\pi}{4}\ln(ab) \\ &\leq -\frac{\pi}{4}\ln\left(G^2(a,b)\right) \\ &\leq -\ln\left(G^{\frac{\pi}{2}}(a,b)\right) \end{aligned}$$

which gives us

$$-\ln I(a,b) \leq -\ln\left(G^{\frac{\pi}{2}}(a,b)\right)$$

and the inequality is proved.

Proposition 3.2 *Let* $1 < a < b$, $p \in \left(0, \frac{1}{1000}\right)$, *then we have*

$$L_p^p(a,b) \leq \frac{\pi}{2} A(a^p, b^p). \tag{27}$$

Proof: The inequality follows from (5) applied to the MT-convex function $f : (1,\infty) \to \mathbb{R}$, $f(x) = x^p$, $p \in \left(0, \frac{1}{1000}\right)$. The details are omitted.

Proposition 3.3 *Let* $1 < a < b$, $p \in \left(0, \frac{1}{1000}\right)$, *then, we have*

$$\begin{aligned} &\frac{\sqrt{(b-x)(x-a)}}{(b-a)^2} L_p^p(a,b)\left\{a^p(b-x) + b^p(x-a)\right\} \\ \leq\; &\frac{1}{4}\Big\{\frac{1}{3}a^{2p} + \frac{1}{3}b^{2p} + \frac{1}{3}(ab)^p \\ &-(b-a)L_{2p+2}^{2p+2}(a,b) - L_{2p+1}^{2p+1}(a,b) - ab(b-a)L_{2p}^{2p}(a,b)\Big\} \end{aligned} \tag{28}$$

Proof: The inequality follows from (9) applied to the MT-convex functions $f, g : R \to R$, $f(x) = g(x) = x^p, p \in \left(0, \frac{1}{1000}\right)$. The details are omitted.

4 Open Problem

It is a well-known fact that if f is a convex function on the interval $I \subset \mathbb{R}$, then the Hadamard's inequality retains for the convex functions. As a matter of fact, it has been demonstrated lots of this type inequalities for several convex functions. Therefore, there is one questions as follows:

How can be set up the general versions of the inequalities (5), (9), (18) and (23) including several differentiable MT-convex function on I.

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