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Some convergence results for non-linear maps in Banach spaces

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Abstract

A strong convergence theorem for asymptotically generalized $\Phi-$ hemicontractive map in real Banach space is proved using the iterative sequence generated by this map. The result of this paper extend and improve the very recent result of Kim et al., (2009) which itself is a generalization of many of the previous results.

Keywords: *Fixed point iteration schemes iteration; asymptotically pseudocontractive mappings; Banach spaces, asymptotically generalized $\Phi-$ hemicontractive map*
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1 Introduction

We denote by J the normalized duality mapping from X into 2^{X^*} by

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

where X^* denotes the dual space of real normed linear space X and $\langle ., .\rangle$ denotes the generalized duality pairing between elements of X and X^*. We first recall and define some concepts as follows (see, []):
Let C be a nonempty subset of real normed linear space X.
Definition 1.1. A mapping $T : C \to X$ is called strongly pseudocontractive

if for all $x, y \in C$, there exist $j(x-y) \in J(x-y)$ and a constant $k \in (0,1)$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq (1-k)\|x-y\|^2.$$
A mapping T is called strongly ϕ-pseudocontractive if for all $x, y \in C$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \phi(\|x-y\|)\|x-y\|$$
and is called generalized strongly Φ-pseudocontractive if for all $x, y \in C$, there exist $j(x-y) \in J(x-y)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that
$$\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2 - \Phi(\|x-y\|).$$
Every strongly ϕ- pseudocontractive operator is a generalized strongly Φ-pseudo
contractive operator with $\Phi : [0,\infty) \to [0,\infty)$ defined by $\Phi(s) = \phi(s)s$, and every strongly pseudocontractive operator is strongly ϕ- pseudocontractive operator where ϕ is defined by $\phi(s) = ks$ for $k \in (0,1)$ while the converses need not be true. An example by Hirano and Huang [6] showed that a strongly ϕ-pseudocontractive operator T is not always a strongly pseudocontractive operator.
A mapping T is called generalized $\Phi-$ hemicontractive if $F(T) = \{x \in C : x = Tx\} \neq \emptyset$ and for all $x \in C$ and $x^* \in F(T)$, there exist $j(x-x^*) \in J(x-x^*)$ and a strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that
$$\langle Tx - Tx^*, j(x-x^*)\rangle \leq \|x-x^*\|^2 - \Phi(\|x-x^*\|).$$

Definition 1.2. A mapping $T : C \to C$ is called asymptotically generalized $\Phi-$ pseudocontractive with sequence $\{k_n\}$ if for each $n \in N$ and $x, y \in C$, there exist constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$, strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that
$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2 - \Phi(\|x-y\|)$$
and is called asymptotically generalized $\Phi-$ hemicontractive with sequence $\{k_n\}$ if $F(T) \neq \emptyset$ and for each $n \in N$ and $x \in C$, $x^* \in F(T)$, there exist constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$, strictly increasing function $\Phi : [0,\infty) \to [0,\infty)$ with $\Phi(0) = 0$ such that
$$\langle T^n x - T^n x^*, j(x-x^*)\rangle \leq k_n\|x-x^*\|^2 - \Phi(\|x-x^*\|).$$
Clearly, the class of asymptotically generalized $\Phi-$ hemicontractive mappings is the most general among those defined above. (see, [2]).

Definition 1.3. A mapping $T : C \to X$ is called Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|,$$

for all $x, y \in C$ and is called generalized Lipschitzian if there exists a constant $L > 0$ such that

$$\|Tx - Ty\| \leq L(\|x - y\| + 1),$$

for all $x, y \in C$.
A mapping $T : C \to C$ is called uniformly $L-$ Lipschitzian if for each $n \in N$, there exists a constant $L > 0$ such that

$$\|T^n x - T^n y\| \leq L\|x - y\|,$$

for all $x, y \in C$.
It is obvious that the class of generalized Lipschitzian map includes the class of Lipschitz map. Moreover, every mapping with a bounded range is a generalized Lipschitzian mapping.
Sahu [6] introduced the following new class of nonlinear mappings which is more general than the class of generalized Lipschitzian mappings and the class of uniformly L- Lipschitzian mappings.
Fix a sequence $\{a_n\}$ in $[0, \infty]$ with $a_n \to 0$.
Definition 1.4. A mapping $T : C \to C$ is called nearly Lipschitzian with respect to $\{a_n\}$ if for each $n \in N$, there exists a constant $k_n > 0$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + a_n)$$

for all $x, y \in C$.
A nearly Lipschitzian mapping T with sequence $\{a_n\}$ is said to be nearly uniformly $L-$ Lipschitzian if $k_n = L$ for all $n \in N$.
Observe that the class of nearly uniformly $L-$ Lipschitzian mappings is more general than the class of uniformly $L-$ Lipschitzian mappings.
In recent years, many authors have given much attention to iterative methods for approximating fixed points of Lipschitz type pseudocontractive type nonlinear mappings (see, [1-4, 7, 9- 14]).
Ofoedu [9] used the modified Mann iteration process introduced by Schu [12] ,

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T_1^n x_n \quad n \geq 0, \tag{1.1}$$

to obtain a strong convergence theorem for uniformly Lipschitzian asymptotically pseudo-contractive mapping in real Banach space setting. This result itself is a generalization of many of the previous results (see [9] and the references therein).

Recently, Chang et al. [3] proved a strong convergence theorem for a pair of L-Lipschitzian mappings instead of a single map used in [9]. In fact, they proved the following theorem :
Theorem 1.1 ([3]). Let E be a real Banach space, K be a nonempty closed convex subset of E, $T_i : K \to K$, $(i = 1, 2)$ be two uniformly L_i-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ is the set of fixed points of T_i in K and ρ be a point in $F(T_1) \cap F(T_2)$. Let $k_n \subset [1, \infty)$ be a sequence with $k_n \to 1$. Let $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\beta_n\}_{n=1}^{\infty}$ be two sequences in $[0, 1]$ satisfying the following conditions:
(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$ (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ (iii) $\sum_{n=1}^{\infty} \beta_n < \infty$
(iv) $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.
For any $x_1 \in K$, let $\{x_n\}_{n=1}^{\infty}$ be the iterative sequence defined by

$$\begin{aligned} x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T_1^n y_n \\ y_n &= (1 - \beta_n)x_n + \beta_n T_2^n x_n. \end{aligned} \tag{1.2}$$

If there exists a strictly increasing function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T_1^n x_n - \rho, j(x_n - \rho) > \leq k_n \|x_n - \rho\|^2 - \Phi(\|x_n - \rho\|)$$

for all $j(x - \rho) \in J(x - \rho)$ and $x \in K$, (i = 1, 2), then $\{x_n\}_{n=1}^{\infty}$ converges strongly to ρ.
The result above extends and improves the corresponding results of [8] from one uniformly Lipschitzian asymptotically pseudocontractive mapping to two uniformly Lipschitzian asymptotically pseudocontractive mappings. In fact, if the iteration parameter $\{\beta_n\}_{n=0}^{\infty}$ in Theorem 1.1 above is equal to zero for all n and $T_1 = T_2 = T$ then, we have the main result of Ofoedu [9].
Very recently, Kim, Sahu and Nam [7] used the notion of nearly uniformly $L-$ Lipschitzian to established a strong convergence result for asymptotically generalized $\Phi-$ hemi contractive mappings in a general Banach space. This result itself is a generalization of many of the previous results. see, [7]. Indeed, they proved the following:
Theorem 1.2 ([7]). Let C be a nonempty closed convex subset of a real Banach space X and $T : C \to C$ a nearly uniformly L-Lipschitzian mappings with sequence $\{a_n\}$ and asymptotically generalized $\Phi-$hemicontractive mapping with sequence $\{k_n\}$ and $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the conditions:
(i)$\{\frac{a_n}{\alpha_n}\}$ is bounded,
(ii)$\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii)$\sum_{n=1}^{\infty} \alpha_n^2 < \infty$ and $\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$.
Then the sequence $\{x_n\}_{n=1}^{\infty}$ in C defined by (1.1) converges to a unique fixed point of T.

It is natural to ask, whether the result in Theorem 1.2 can be extend and improve upon in the same space setting?
It is the purpose of this paper to answer this question. For this, we need the following Lemmas.
Lemma 1.1 [3]. Let X be a real Banach space, then for all $x, y \in X$, there exists $j(x+y) \in J(x+y)$ so that

$$\|x+y\|^2 \leq \|x\|^2 + 2 < y, j(x+y) >$$

Lemma 1.2 [8]. Let (r_n) be a non- negative sequence which satisfies the following inequality

$$r_{n+1} \leq (1-t_n)r_n + s_n,$$

where $r_n \in (0,1)$, $\forall n \in N$, $\sum_{n=1}^{\infty} t_n = \infty$ and $s_n = o(t_n)$. Then $\lim_{n\to\infty} r_n = 0$.

2 Main results

Theorem 2.1. Let C be a nonempty closed convex subset of a real Banach space X and $T_1, T_2 : C \to C$ be two nearly uniformly L-Lipschitzian mappings with $F(T_1) \cap F(T_2) \neq \phi$, where $F(T_i)$ $(i = 1, 2)$ is the set of fixed points of T_1 and T_2 in C and, ρ be a point in $F(T_1) \cap F(T_2)$. Let T_1 be asymptotically generalized $\Phi-$hemicontractive mapping with sequence $\{k_n\}$ and $\{\alpha_n\}, \{\beta_n\}$ be two sequences in $[0,1]$ satisfying the conditions:
(i) $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} \beta_n = 0$
(ii)$\sum_{n=1}^{\infty} \alpha_n = \infty$,
(iii) there exists $\gamma_0 \in R$ such that if $\alpha_n + \beta_n < \gamma_0$, $\forall n \geq 0$.
Then, the sequence $\{x_n\}_{n=1}^{\infty}$ in C defined by (1.2) converges to a common fixed point of $T_1 \cap T_2$.
Proof. The uniqueness of the fixed point comes from the definition of asymptotically generalized $\Phi-$hemicontractive mapping. Since T_1 is nearly uniformly L-Lipschitzian asymptotically generalized $\Phi-$hemicontractive mapping. Applying Lemma 1.1, we have in view of definitions 1.2 and 1.4 that

$$\begin{aligned}
\|x_{n+1} - p\|^2 &= \langle x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| + \alpha_n\langle T_1^n y_n - p, j(x_{n+1} - p)\rangle \\
&= (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +\alpha_n\langle T_1^n x_{n+1} - p, j(x_{n+1} - p)\rangle \\
&\quad +\alpha_n\langle T_1^n y_n - T_1^n x_{n+1}, j(x_{n+1} - p)\rangle \\
&\leq (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +\alpha_n\{k_n\|x_n - \rho\| - \Phi(\|x_{n+1} - p\|)\} \\
&\quad +\alpha_n L(\|y_n - x_{n+1}\| + a_n)\|x_{n+1} - \rho\|
\end{aligned} \tag{2.1}$$

Observe that

$$\begin{aligned}
\|y_n - x_{n+1}\| &\le \|(1-\alpha_n)x_n + \alpha_n T_1^n y_n - y_n\| \\
&= \|(1-\alpha_n)(x_n - y_n) + \alpha_n(T_1^n y_n - y_n)\| \\
&\le (1-\alpha_n)\|x_n - y_n\| + \alpha_n(\|T_1^n y_n - \rho\| + \|y_n - \rho\|) \\
&\le (1-\alpha_n)\|x_n - y_n\| + \alpha_n\{L(\|y_n - \rho\| + a_n) + \|y_n - x_n\|\} \\
&\le (1-\alpha_n)\|x_n - y_n\| + \alpha_n(1+L)\|y_n - \rho\| + a_n\alpha_n L \\
&\le (1-\alpha_n)\|x_n - y_n\| \\
&\quad +\alpha_n(1+L)(\|y_n - x_n\| + \|x_n - \rho\|) + a_n\alpha_n L \\
&\le (1+\alpha_n L)\|y_n - x_n\| + \alpha_n(1+L)\|x_n - \rho\| + a_n\alpha_n L \\
&= (1+\alpha_n L)\|(1-\beta_n)x_n + \beta_n T_2^n x_n - x_n\| \\
&\quad +\alpha_n(1+L)\|x_n - \rho\| + a_n\alpha_n L \\
&= (1+\alpha_n L)\beta_n\|T_2^n x_n - x_n\| + \alpha_n(1+L)\|x_n - \rho\| + a_n\alpha_n L \\
&\le (1+\alpha_n L)\{\beta_n(\|x_n - \rho\|) + \|T_2^n x_n - \rho\|)\} \\
&\quad +\alpha_n(1+L)\|x_n - \rho\| + a_n\alpha_n L \\
&\le (1+\alpha_n L)\{\beta_n(\|x_n - \rho\| + L(\|x_n - \rho\| + a_n)\} \\
&\quad +\alpha_n(1+L)\|x_n - \rho\| + a_n\alpha_n L \\
&= (1+\alpha_n L)(1+L)\beta_n\|x_n - \rho\| + \beta_n a_n L + \alpha_n a_n L \\
&\quad +\alpha_n(1+L)\|x_n - \rho\| \\
&= d_n\|x_n - \rho\| + a_n L(\alpha_n + \beta_n) \\
&\le d_n\|x_n - \rho\| + a_n L\gamma_0,
\end{aligned} \tag{2.2}$$

where
$d_n = (1+\alpha_n L)(1+L)\beta_n + \alpha_n(1+L)$.
Substituting (2.2) into (2.1), we obtain

$$\begin{aligned}
\|x_{n+1} - p\|^2 &\le (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +\alpha_n\{k_n\|x_n - \rho\| - \Phi(\|x_{n+1} - p\|)\} \\
&\quad +\alpha_n L(d_n\|x_n - \rho\| + a_n L\gamma_0 + a_n)\|x_{n+1} - \rho\| \\
&= (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +\alpha_n\{k_n\|x_n - \rho\| - \Phi(\|x_{n+1} - p\|)\} \\
&\quad +\alpha_n L(d_n\|x_n - \rho\| + a_n(1 + L\gamma_0))\|x_{n+1} - \rho\| \\
&= (1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| \\
&\quad +\alpha_n\{k_n\|x_n - \rho\| - \Phi(\|x_{n+1} - p\|)\} \\
&\quad +\alpha_n d_n L(\|x_n - \rho\| + \tfrac{a_n(1+L\gamma_0)}{d_n})\|x_{n+1} - \rho\|
\end{aligned} \tag{2.3}$$

We note from the fact that $2AB \le A^2 + B^2$ that

$$(1-\alpha_n)\|x_n - p\|\|x_{n+1} - p\| \le \frac{1}{2}((1-\alpha_n)^2\|x_n - p\|^2 + \|x_{n+1} - p\|^2), \tag{2.4}$$

and

$$(\|x_n - \rho\| + \frac{a_n(1+L\gamma_0)}{d_n})\|x_{n+1} - \rho\| \le \frac{1}{2}((\|x_n - \rho\| + \frac{a_n(1+L\gamma_0)}{d_n})^2 + \|x_{n+1} - p\|^2). \tag{2.5}$$

Substituting (2.4) and (2.5) into (2.3), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \tfrac{1}{2}((1-\alpha_n)^2\|x_n-p\|^2+\|x_{n+1}-p\|^2)\\ &\quad+\alpha_n\{k_n\|x_{n+1}-p\|^2-\alpha_n\Phi(\|x_{n+1}-p\|\}\\ &\quad+\alpha_n d_n L\tfrac{1}{2}((\|x_n-\rho\|+\tfrac{a_n(1+L\gamma_0)}{d_n})^2+\|x_{n+1}-p\|^2)\\ &\le \tfrac{1}{2}((1-\alpha_n)^2\|x_n-p\|^2+\|x_{n+1}-p\|^2)\\ &\quad+\alpha_n\{k_n\|x_{n+1}-p\|^2-\alpha_n\Phi(\|x_{n+1}-p\|\}\\ &\quad+\alpha_n d_n L\tfrac{1}{2}(2\|x_n-\rho\|^2+2(\tfrac{a_n(1+L\gamma_0)}{d_n})^2+\|x_{n+1}-p\|^2)\end{aligned} \tag{2.6}$$

Simplying (2.6) completely, we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le (1-\alpha_n)^2\|x_n-p\|^2\\ &\quad+2\alpha_n\{k_n\|x_{n+1}-p\|^2-\alpha_n\Phi(\|x_{n+1}-p\|\}\\ &\quad+\alpha_n d_n L(2\|x_n-\rho\|^2+2(\tfrac{a_n(1+L\gamma_0)}{d_n})^2+\|x_{n+1}-p\|^2),\end{aligned} \tag{2.7}$$

implying

$$\begin{aligned}(1-2\alpha_nk_n-\alpha_nd_nL)\|x_{n+1}-p\|^2 &\le (1-\alpha_n)^2\|x_n-p\|^2\\ &\quad-2\alpha_n\Phi(\|x_{n+1}-p\|)\\ &\quad+2\alpha_nd_nL\|x_n-\rho\|^2\\ &\quad+2\alpha_nd_nL(\tfrac{a_n(1+L\gamma_0)}{d_n})^2.\end{aligned} \tag{2.8}$$

Since $\lim_{n\to\infty}\alpha_n=\lim_{n\to\infty}\beta_n=0$, there exists a natural number N_1 such that

$$\frac{1}{2}<1-2\alpha_nk_n-\alpha_nd_nL<1$$

for all $N_1>N$. Then, (2.8) implies that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \tfrac{(1-\alpha_n)^2+2\alpha_nd_nL}{(1-2\alpha_nk_n-\alpha_nd_nL))}\|x_n-p\|^2\\ &\quad-\tfrac{2\alpha_n}{(1-2\alpha_nk_n-\alpha_nd_nL)}\Phi(\|x_{n+1}-p\|)+\tfrac{2\alpha_nd_nL(\frac{a_n(1+L\gamma_0)}{d_n})^2}{1-2\alpha_nk_n-\alpha_nd_nL}.\end{aligned} \tag{2.9}$$

If we set $b_n=\|x_n-\rho\|$, then (2.9) can be re-written as

$$\begin{aligned}b_{n+1}^2 &\le \tfrac{(1-\alpha_n)^2+2\alpha_nd_nL}{(1-2\alpha_nk_n-\alpha_nd_nL}b_n^2\\ &\quad-\tfrac{2\alpha_n}{(1-2\alpha_nk_n-\alpha_nd_nL)}\Phi(b_{n+1})+\tfrac{2\alpha_nd_nL(\frac{a_n(1+L\gamma_0)}{d_n})^2}{1-2\alpha_nk_n-\alpha_nd_nL)}\\ &\le b_n^2+2\alpha_n\tfrac{(k_n-1)+\alpha_n+3\alpha_nd_nL}{1-2\alpha_nk_n-\alpha_nd_nL}b_n^2\\ &\quad-\tfrac{2\alpha_n}{(1-2\alpha_nk_n-\alpha_nd_nL)}\Phi(b_{n+1})+\tfrac{2\alpha_nd_nL(\frac{a_n(1+L\gamma_0)}{d_n})^2}{(1-2\alpha_nk_n-\alpha_nd_nL)}\\ &\le b_n^2+4\alpha_nB_nb_n+2\alpha_nC_n-2\alpha_n\Phi(b_{n+1}),\end{aligned} \tag{2.10}$$

where $B_n=(k_n-1)+\alpha_n+3\alpha_nd_nL,\ \ C_n=2d_nL(\frac{a_n(1+L\gamma_0)}{d_n})^2$.
Suppose we set $inf_{n\ge N}\frac{\Phi(b_{n+1})}{1+b_{n+1}^2}=\lambda$. Then $\lambda=0$. If it is not the case, we

assume that $\lambda > 0$. Let $0 < \mu < min\{1, \lambda\}$, then $\frac{\Phi(b_{n+1})}{1+b_{n+1}^2} \geq \mu$, i.e.,

$$\Phi(b_{n+1}) \geq \mu + \mu b_{n+1}^2 \geq \mu b_{n+1}^2. \tag{2.11}$$

Substituting (2.11) into (2.10) yields

$$b_{n+1}^2 \leq \frac{1+4\alpha_n B_n}{1+2\alpha_n\mu} b_n^2 + \frac{2\alpha_n C_n}{1+2\alpha_n\mu}, \tag{2.12}$$

Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = 0$, we choose $N_1 > N$ such that

$$\begin{aligned}(1 + 4\alpha_n B_n) - (1 - \alpha_n\mu)(1 + 2\alpha_n\mu) &= \alpha_n(4B_n - \mu + 2\alpha_n\mu)\\ &\leq 0.\end{aligned}$$

Therefore, we get

$$\frac{1+4\alpha_n B_n}{1+2\alpha_n\mu} \leq (1 - \alpha_n\mu) \tag{2.13}$$

for all $N_1 > N$.
It follows from (2.12) and (2.13) that

$$b_{n+1}^2 \leq (1 - \alpha_n\mu)b_n^2 + \frac{4\alpha_n C_n}{1+2\alpha_n\mu}, \tag{2.14}$$

for all $N_1 > N$.
Using Lemma 1.2 , we have that $\lim_{n\to\infty} b_n = 0$, which is a contradiction and so $\lambda = 0$. Thus, there exists an infinite subsequence $\{b_{n_j+1}\} \subset \{b_n\}$ such that $\lim_{n\to\infty} b_{n_j+1} = 0$. Since $\lim_{n\to\infty} \alpha_n = \lim_{n\to\infty} B_n = \lim_{n\to\infty} C_n = 0$. $\forall\, \epsilon \in (0, 1)$, choose $N_{j_0} > N$ such that $b_{n_j+1} < \epsilon$, $C_{n_j+1} < \frac{\Phi(\epsilon)}{8}$, $B_{n_j+1} < \frac{\Phi(\epsilon)}{8(1+\epsilon^2)}$ Next, we prove that $\lim_{n\to\infty} b_{n_j+m} = 0$ by induction. It is obvious that the conclusion holds for $m = 1$. First, we want to prove that $b_{n_j+2} < \epsilon$. Suppose it is not the case for $m = 2$. Then $b_{n_j+2} \geq \epsilon$, this implies $\Phi(b_{n_j+2}) \geq \Phi(\epsilon)$. Using (2.11) we now obtain the following

$$\begin{aligned} b_{n_j+2}^2 &\leq b_{n_j+1}^2 + 4\alpha_{n_j+1}B_{n_j+1}b_{n_j+1}^2\\ &\quad +2\alpha_{n_j+1}C_{n_j+1} - 2\alpha_{n_j+1}\Phi(b_{n_j+2})\\ &\leq \epsilon^2 - \alpha_{n_j+1}\Phi(\epsilon)\\ &< \epsilon^2 \end{aligned} \tag{2.15}$$

which is a contracdiction. Hence $b_{n_j+2} < \epsilon$. Assume that it holds for $m = k$. Then by the argument above, we can easily prove that it holds for $m = k + 1$. Thus, we obtain that $\lim_{n\to\infty} b_n = 0$, i.e., $\lim_{n\to\infty} \|x_n - \rho\| = 0$.

Remark 2.2. Our Theorem 2.1 removes the conditions in Theorem 2.1 of [7] and that of Theorem 2.1 in [3], by replacing them with weaker conditions i.e., from $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$,$\sum_{n=1}^{\infty} \alpha_n(k_n - 1) < \infty$ to $\lim_{n\to\infty} \alpha_n = 0$. We equally extend their single map to pair of maps. Furthermore, we use a more general iteration procedure. Therefore our results extend and improve the very recent results of Kim et al. [7] which in turn is a correction, improvement and generalization of several results.

Example 2.3 Let $X = R$, $C = [0, 1]$ and $T_i(i = 1, 2) : C \to C$ be a map defined by

$$T_i x = \{ \begin{array}{c} \frac{x}{3}, \; x \in [0,1], \\ 0, \; x = 1. \end{array}$$

Clearly, $T_i(i = 1, 2)$ is nearly uniformly Lipschitzian and asymptotically generalized $\Phi-$ hemicontractive map. Now suppose we take $\alpha_n = \beta_n = \frac{1}{\sqrt{n}}$ for all $n \geq 1$. For arbitrary $x_1 \in C$, the sequence $\{x_n\}_{n=1}^{\infty} \subset C$ defined by (1.2) converges strongly to the unique fixed point $\rho \in T_1 \cap T_2$.

3 Open Problem

Question 3.1. In the above proof, we used $b_{n_j+m} < \epsilon$ for all $m \geq 1$ to deduce $\lim_{n\to\infty} b_n = 0$. Can this be modify to improve Theorem 2.1?

Question 3.2. Can we extend Theorem 2.1 to a class of map which is more general than the class of nearly uniformly Lipschitzian asymptotically generalized $\Phi-$hemicontractive mapping?

Question 3.3. Will a sequence $\{x_n\}_{n=1}^{\infty}$ involving three maps converge strongly to a unique fixed point of more general maps than the class of nearly uniformly Lipschitzian asymptotically generalized $\Phi-$hemicontractive mapping?

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