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On sandwich theorems for some subclasses of analytic functions

Lifeng Guo[1] and Gejun Bao[2]

1.School of Mathematical Science and Technology
Northeast Petroleum University, Daqing 163318, China.
(e-mail: hitglf@yahoo.com.cn)

2. Department of Mathematics, Harbin Institute of Technology.
Harbin 150001, China.(e-mail: baogj@hit.edu.cn)

Abstract

The purpose of this present paper is to derive some subordination and superordination results for certain normalized analytic functions in the open unit disk. Relevant connections of the results, which are presented in the paper, with various known results are also considered.

Keywords: *univalent functions; starlike functions; subordination; superordination.*

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1 Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{H}[a, n]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots. \tag{1}$$

Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \cdots. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$ of starlike functions in $\mathbb{U}$, if it satisfies the inequality $\mathrm{Re}\big(\frac{zf'(z)}{f(z)}\big) > 0,\ z \in \mathbb{U}$. Furthermore, a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{C}$ of convex functions in $\mathbb{U}$, if it satisfies the inequality $\mathrm{Re}\big(1 + \frac{zf''(z)}{f'(z)}\big) > 0,\ z \in \mathbb{U}$.

Let $f(z)$ and $F(z)$ be analytic in $\mathbb{U}$, then we say that the function $f(z)$ is subordinate to $F(z)$ in $\mathbb{U}$, if there exists an analytic function $w(z)$ in $\mathbb{U}$ such that $|w(z)| \le |z|$, and $f(z) \equiv F(w(z))$, denoted $f \prec F$ or $f(z) \prec F(z)$. If $F(z)$ is univalent in $\mathbb{U}$, then the subordination is equivalent to $f(0) = F(0)$ and $f(\mathbb{U}) \subset F(\mathbb{U})$.

Let $p, h \in \mathcal{H}$ and let $\phi(r, s, t; z) : \mathbb{C}^3 \times \mathbb{U} \to \mathbb{C}$. If p and $\phi(p(z), zp'(z), z^2p''(z); z)$ are univalent and if p satisfies the second-order superordination

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z), \tag{3}$$

then p is a solution of the differential superordination (1.2). (If f is subordinate to F, then F is superordinate to f.) An analytic function q is called a subordinant if $q \prec p$ for all p satisfying (1.2). A univalent subordinant Q that satisfies $q \prec Q$ for all subordinants q of (1.2) is said to be the best subordinant. Recently Miller and Mocanu [1] obtained conditions on h, q and ϕ for which the following implication holds:

$$h(z) \prec \phi(p(z), zp'(z), z^2p''(z); z) \Rightarrow q(z) \prec p(z). \tag{4}$$

Using the results of Miller and Mocanu [1], Bulboacă [2] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [3]. Ali et al. [4] have used the results of Bulboacă [2] and obtained sufficient conditions for certain normalized analytic functions $f(z)$ to satisfy

$$q(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z), \tag{5}$$

where q_1 and q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$. Shanmugam et al. [5] obtained sufficient conditions for normalized analytic functions $f(z)$ to satisfy

$$q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z) \text{ and } q_1(z) \prec \frac{z^2f'(z)}{f^2(z)} \prec q_2(z) \tag{6}$$

where q_1 and q_2 are given univalent functions in $\mathbb{U}$ with $q_1(0) = 1$ and $q_2(0) = 1$, while Obradović and Owa [6] obtained subordination results with the quantity $(f(z)/z)^\mu$ (see also [7]).

For $0 < \alpha < 1$, a function $f(z) \in N(\alpha)$ if and only if $f(z) \in \mathcal{A}$ and

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\right\} > 0,\ z \in \mathbb{U}. \tag{7}$$

$N(\alpha)$ was introduced by M.Obradović [8] recently, and he called this class of functions to be non-Bazilevič type. Tuneski and Darus [9] obtained Fekete-Szegö inequality for the non-Bazilevic class of functions. Using this non-Bazilevič class, Wang et al. [10] studied many subordination results for the class $N(\alpha,\lambda,A,B)$ defined as

$$N(\alpha,\lambda,A,B)=\left\{f(z)\in\mathcal{A}:(1+\lambda)\left(\frac{z}{f(z)}\right)^{\alpha}-\lambda\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha}\prec\frac{1+Az}{1+Bz},z\in\mathbb{U}\right\}. \tag{8}$$

where $0<\alpha<1,\lambda\in\mathbb{C},-1\leq B\leq 1,A\neq B,A\in\mathbb{R}$.

The main object of the present sequel to the aforementioned works is to apply a method based on the differential subordination in order to derive several subordination results. Furthermore, we obtain the previous results of Srivastava and Lashin [7], Singh [11], Shanmugam et al. [12] and Obradović andOwa [6] as special cases of some of the results presented here.

2 Some lemmas

To prove our main result, we will need the following lemmas:

Definition 2.1. [1] *Denote by Σ the set of all functions $f(z)$ that are analytic and injective on $\bar{\mathbb{U}}-E(f)$, where*

$$E(f)=\left\{\xi\in\partial\mathbb{U}:\lim_{z\to\xi}f(z)=\infty\right\}, \tag{9}$$

and are such that $f'(\xi)\neq 0$ for $\xi\in\partial\mathbb{U}-E(f)$.

Lemma 2.1. [5] *Let q be univalent in $\mathbb{U}$ and let $\beta,\gamma\in\mathbb{C}$ with $\mathfrak{Re}\big(1+\frac{zq''(z)}{q'(z)}\big)>\max\big\{0,-\mathfrak{Re}\frac{\beta}{\gamma}\big\}$. If $p(z)$ is analytic in $\mathbb{U}$ and*

$$\beta p(z)+\gamma zp'(z)\prec\beta q(z)+\gamma zq'(z), \tag{10}$$

then $p(z)\prec q(z)$ and q is the best dominant.

Lemma 2.2. [13] *Let q be univalent in $\mathbb{U}$ and let θ,ρ be analytic in a domain Ω containing $q(\mathbb{U})$ with $\rho(w)\neq 0$ when $w\in q(\mathbb{U})$. Set $h(z)=zq'(z)\rho(q(z))$, $F(z)=\theta(q(z))+h(z)$. Suppose that*

(1) $h(z)$ is starlike univalent in $\mathbb{U}$;

(2) $\mathfrak{Re}\big(\frac{zF'(z)}{h(z)}\big)>0$ for $z\in\mathbb{U}$.

If

$$\theta(p(z))+zp'(z)\rho(F(z))\prec\theta(q(z))+zq'(z)\rho(q(z)), \tag{11}$$

then $p(z)\prec q(z)$ and q(z) is the best dominant.

Lemma 2.3. [1] *Let q be convex univalent in $\mathbb{U}$ and let $\gamma \in \mathbb{C}$ with $\mathfrak{Re}(\gamma) > 0$. If $p(z) \in \mathcal{H}[q(0), 1] \cap \Sigma$ and $p(z) + \gamma z p'(z)$ is univalent in $\mathbb{U}$, and*

$$q(z) + \gamma z q'(z) \prec p(z) + \gamma z p'(z), \tag{12}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

Lemma 2.4. [3] *Let q be convex univalent in $\mathbb{U}$, and let θ, ρ be analytic in a domain Ω containing $q(\mathbb{U})$. Suppose that*

(1) $zq'(z)\rho(q(z))$ is starlike univalent in $\mathbb{U}$;

(2) $\mathfrak{Re}\big(\frac{\theta'(q(z)}{\rho(q(z))}\big) > 0$ for $z \in \mathbb{U}$.

If $p(z) \in \mathcal{H}[q(0), 1] \subseteq \Sigma$, with $p(\mathbb{U}) \subset \Omega$ and $\theta(p(z)) + zp'(z)\rho(p(z))$ is univalent in $\mathbb{U}$ and

$$\theta(q(z)) + zq'(z)\rho(q(z)) \prec \theta(p(z)) + zp'(z)\rho(p(z)), \tag{13}$$

then $q(z) \prec p(z)$ and q is the best subordinant.

3 Subordination for analytic functions

By using Lemma 2.1, we first prove the following Theorem.

Theorem 3.1. *Let q be univalent in $\mathbb{U}$, $0 < \alpha < 1$ and $\gamma \in \mathbb{C}$. Suppose q satisfies*

$$\mathfrak{Re}\Big\{1 + \frac{zq''(z)}{q'(z)}\Big\} > \max\Big\{0, -\mathfrak{Re}\frac{\gamma}{\alpha}\Big\}. \tag{14}$$

If $f(z) \in \mathcal{A}, g(z) \in \mathcal{S}^*$, and satisfies the subordination

$$\left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec q(z) + \frac{\gamma}{\alpha}zq'(z), \tag{15}$$

then

$$\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec q(z) \tag{16}$$

and q is the best dominant.

Proof. Let $F(z) = (\frac{g(z)}{f(z)})^{\alpha}$, then $F(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in $\mathbb{U}$. Then a computation shows that

$$\left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} = F(z) + \frac{\gamma}{\alpha}zF'(z). \tag{17}$$

By the hypothesis (15), we obtain that

$$F(z) + \frac{\gamma}{\alpha}zF'(z) \prec q(z) + \frac{\gamma}{\alpha}zq'(z). \tag{18}$$

The assertion of Theorem 3.1 now follows by an application of Lemma 2.1 with $\gamma = \frac{\gamma}{\alpha}$ and $\beta = 1$.

Taking $q(z) = (1 + Az)/(1 + Bz)$ in Theorem 3.1, we have the following corollary.

Corollary 3.1. *Let $-1 \leq B < A \leq 1$ and (15) hold. If $f(z) \in \mathcal{A}, g(z) \in \mathcal{S}^*$, and satisfies the subordination*

$$\left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec \frac{1 + \mathrm{A}z}{1 + \mathrm{B}z} + \frac{\gamma(\mathrm{A} - \mathrm{B})z}{\alpha(1 + \mathrm{B}z)^2}, \quad (19)$$

then

$$\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec \frac{1 + \mathrm{A}z}{1 + \mathrm{B}z}, \quad (20)$$

and $\frac{1+\mathrm{A}z}{1+\mathrm{B}z}$ is the best dominant.

Taking $g(z) = z, \gamma = -1$ in Theorem 3.1, we have the following corollary.

Corollary 3.2. *Let q be univalent in $\mathbb{U}$ and $0 < \alpha < 1$. Suppose q satisfies*

$$\mathfrak{Re}\{1 + \frac{zq''(z)}{q'(z)}\} > \frac{1}{\alpha}. \quad (21)$$

If $f(z) \in \mathcal{A}$ and satisfies the subordination

$$\frac{zf'(z)}{f(z)}\left(\frac{z}{f(z)}\right)^{\alpha} \prec q(z) - \frac{1}{\alpha}zq'(z), \quad (22)$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec q(z) \quad (23)$$

and q is the best dominant.

Taking $\gamma = 1$ and $g(z) = z$ in Theorem 3.1, we get the following corollary.

Corollary 3.3. *Let q be univalent in $\mathbb{U}$ and $0 < \alpha < 1$. Suppose q satisfies*

$$\mathfrak{Re}\{1 + \frac{zq''(z)}{q'(z)}\} > 0. \quad (24)$$

If $f(z) \in \mathcal{A}$ and satisfies the subordination

$$\left(2 - \frac{zf'(z)}{f(z)}\right)\left(\frac{z}{f(z)}\right)^{\alpha} \prec q(z) + \frac{1}{\alpha}zq'(z), \quad (25)$$

then

$$\left(\frac{z}{f(z)}\right)^{\alpha} \prec q(z) \quad (26)$$

and q is the best dominant.

Theorem 3.2. *Let q be univalent in $\mathbb{U}$, $\gamma(\neq 0), \varepsilon, \kappa \in \mathbb{C}$, $0 \leq \beta \leq 1$, $f(z) \in \mathcal{A}$ and $g(z) \in \mathcal{S}^*$. Suppose q satisfies*

$$\mathfrak{Re}\big\{1 + \frac{zq''(z)}{q'(z)}\big\} > \max\big\{0, -\mathfrak{Re}\frac{\kappa}{\gamma}\big\}. \tag{27}$$

Let

$$G(z) := \left(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\right)^{\alpha} \left(\kappa + \gamma\alpha\left(\frac{zf'(z) + \beta z^2 f''}{(1-\beta)f(z) + \beta z f'(z)} - \frac{zg'(z)}{g(z)}\right)\right) + \varepsilon.$$

If

$$G(z) \prec \kappa q(z) + \varepsilon + \gamma z q'(z), \tag{28}$$

then

$$\left(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\right)^{\alpha} \prec q(z) \tag{29}$$

and $q(z)$ is the best dominant.

Proof. Define the function $F(z)$ by

$$F(z) = \left(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\right)^{\alpha}. \tag{30}$$

Then a computation shows that

$$\alpha\left(\frac{zf'(z) + \beta z^2 f''}{(1-\beta)f(z) + \beta z f'(z)} - \frac{zg'(z)}{g(z)}\right) = \frac{zF'(z)}{F(z)}, \tag{31}$$

and hence

$$\alpha F(z)\left(\frac{zf'(z) + \beta z^2 f''}{(1-\beta)f(z) + \beta z f'(z)} - \frac{zg'(z)}{g(z)}\right) = zF'(z). \tag{32}$$

By the hypothesis (28), we obtain that

$$\kappa F(z) + \varepsilon + \gamma z F'(z) \prec \kappa q(z) + \varepsilon + \gamma z q'(z). \tag{33}$$

By setting $\theta(w) = \kappa w + \varepsilon$, $\rho(w) = \gamma$, it can be easily observed that $\theta(w)$ and $\rho(w)$ are analytic in $\mathbb{C}$. Also, we let

$$h(z) = zq'(z)\rho(q(z)) = \gamma z q'(z) \text{ and } p(z) = \theta(q(z)) + h(z) = \kappa q(z) + \varepsilon + \gamma z q'(z). \tag{34}$$

From (27), we find that $h(z)$ is starlike univalent in $\mathbb{U}$, and that

$$\mathfrak{Re}\left(\frac{zp'(z)}{h(z)}\right) = \mathfrak{Re}\left(\frac{\kappa}{\gamma} + 1 + \frac{zq''(z)}{q'(z)}\right) > 0, \tag{35}$$

by the hypothesis (27). Thus, by applying Lemma 2.2, our proof of Theorem 3.2 is completed.

For $\beta = 1, \varepsilon = 0$, $\kappa = 1$ and $g(z) = z$, we get the following corollary.

Corollary 3.4. *Let q be univalent in $\mathbb{U}$ and $f(z) \in \mathcal{A}$. Suppose q satisfies*

$$\mathfrak{Re}\big\{1 + \frac{zq''(z)}{q'(z)}\big\} > \max\big\{0, -\mathfrak{Re}\frac{1}{\gamma}\big\}. \tag{36}$$

If

$$\big(f'(z)\big)^{\alpha}\big(1 + \frac{\gamma\alpha z f''}{f'(z)}\big) \prec q(z) + \gamma z q'(z), \tag{37}$$

then

$$\big(f'(z)\big)^{\alpha} \prec q(z) \tag{38}$$

and $q(z)$ is the best dominant.

For $\beta = 0, \varepsilon = 0$, $\kappa = 1$ and $g(z) = z$, we get the following corollary.

Corollary 3.5. *Let q be univalent in $\mathbb{U}$, $\gamma(\neq 0) \in \mathbb{C}$ and $f(z) \in \mathcal{A}$. Suppose q satisfies*

$$\mathfrak{Re}\big\{1 + \frac{zq''(z)}{q'(z)}\big\} > \max\big\{0, -\mathfrak{Re}\frac{1}{\gamma}\big\}. \tag{39}$$

If

$$(1 - \alpha\gamma)\big(\frac{f(z)}{z}\big)^{\alpha} + \alpha\gamma\frac{zf'(z)}{f(z)}\big(\frac{f(z)}{z}\big)^{\alpha} \prec q(z) + \gamma z q'(z), \tag{40}$$

then

$$\big(\frac{(f(z)}{z}\big)^{\alpha} \prec q(z) \tag{41}$$

and $q(z)$ is the best dominant.

4 Superordination for analytic functions

Theorem 4.1. *Let q be convex univalent in $\mathbb{U}$, $0 < \alpha < 1$, $\gamma \in \mathbb{C}$ with $\mathfrak{Re}\big(\gamma\big) > 0$. Suppose q satisfies $\big(\frac{g(z)}{f(z)}\big)^{\alpha} \in \mathcal{H}[q(0), 1] \cap \Sigma$. Let*

$$\left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} \tag{42}$$

be univalent in $\mathbb{U}$. If

$$q(z) + \frac{\gamma}{\alpha}zq'(z) \prec \left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}, \tag{43}$$

then

$$q(z) \prec \left(\frac{g(z)}{f(z)}\right)^{\alpha} \tag{44}$$

and q is the best subordinant.

Proof. Let $F(z) = (\frac{g(z)}{f(z)})^{\alpha}$. Then a computation shows that

$$\left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} = F(z) + \frac{\gamma}{\alpha}zF'(z). \tag{45}$$

By the hypothesis (43), we obtain that

$$q(z) + \frac{\gamma}{\alpha}zq'(z) \prec F(z) + \frac{\gamma}{\alpha}zF'(z). \tag{46}$$

Theorem 4.1 follows as an application of Lemma 2.3.

For $\gamma = 1$ and $g(z) = z$, we get the following corollary.

Corollary 4.1. *Let q be convex univalent in $\mathbb{U}$, $0 < \alpha < 1$. Suppose q satisfies $\left(\frac{z}{f(z)}\right)^{\alpha} \in \mathcal{H}[q(0), 1] \cap \Sigma$. Let*

$$\left(2 - \frac{zf'(z)}{f(z)}\right)\left(\frac{z}{f(z)}\right)^{\alpha} \tag{47}$$

be univalent in $\mathbb{U}$. If

$$q(z) + \frac{1}{\alpha}zq'(z) \prec \left(2 - \frac{zf'(z)}{f(z)}\right)\left(\frac{z}{f(z)}\right)^{\alpha}, \tag{48}$$

then

$$q(z) \prec \left(\frac{z}{f(z)}\right)^{\alpha} \tag{49}$$

and q is the best subordinant.

Theorem 4.2. *Let q be convex univalent in $\mathbb{U}$, $\gamma(\neq 0), \varepsilon, \kappa \in \mathbb{C}$ and $0 \leq \beta \leq 1$. Suppose q satisfies*

$$\mathfrak{Re}\left(\frac{\kappa}{\gamma}q'(z)\right) > 0. \tag{50}$$

and

$$\left(\frac{(1-\beta)f(z) + \beta zf'(z)}{g(z)}\right)^{\alpha} \in \mathcal{H}[q(0), 1] \cap \Sigma, \tag{51}$$

and

$$H(z) := \left(\frac{(1-\beta)f(z) + \beta zf'(z)}{g(z)}\right)^{\alpha}\left(\kappa + \gamma\alpha\left(\frac{zf'(z) + \beta z^2 f''}{(1-\beta)f(z) + \beta zf'(z)} - \frac{zg'(z)}{g(z)}\right)\right) + \varepsilon, \tag{52}$$

is univalent in $\mathbb{U}$. *If*

$$\kappa q(z)+\varepsilon+\gamma zq'(z)\prec H(z), \tag{53}$$

then

$$q(z)\prec\left(\frac{(1-\beta)f(z)+\beta zf'(z)}{g(z)}\right)^{\alpha} \tag{54}$$

and q *is the best subordinant.*

Proof. Define the function $F(z)$ by

$$F(z)=\left(\frac{(1-\beta)f(z)+\beta zf'(z)}{g(z)}\right)^{\alpha}. \tag{55}$$

Then a computation shows that

$$\alpha\left(\frac{zf'(z)+\beta z^2f''}{(1-\beta)f(z)+\beta zf'(z)}-\frac{zg'(z)}{g(z)}\right)=\frac{zF'(z)}{F(z)}. \tag{56}$$

and hence

$$\alpha F(z)\left(\frac{zf'(z)+\beta z^2f''}{(1-\beta)f(z)+\beta zf'(z)}-\frac{zg'(z)}{g(z)}\right)=zF'(z). \tag{57}$$

By the hypothesis (53), we obtain that

$$\kappa q(z)+\varepsilon+\gamma zq'(z)\prec\kappa F(z)+\varepsilon+\gamma zF'(z). \tag{58}$$

By setting $\theta(w)=\kappa w+\varepsilon,\ \rho(w)=\gamma$, it can be easily observed that $\theta(w)$ and $\rho(w)$ are analytic in $\mathbb{C}$. Now,

$$\mathfrak{Re}\left(\frac{\theta'(q(z))}{\rho(q(z))}\right)=\mathfrak{Re}\left(\frac{\kappa}{\gamma}q'(z)\right)>0, \tag{59}$$

by the hypothesis (50). Thus, by applying Lemma 2.4, our proof of Theorem 4.2 is completed.

For $\beta=1,\varepsilon=0$, $\kappa=1$ and $g(z)=z$, we get the following corollary.

Corollary 4.2. *Let* q *be convex univalent in* $\mathbb{U}$, $\gamma(\neq 0)\in\mathbb{C}$. *Suppose* q *satisfies*

$$\mathfrak{Re}\left(\frac{q'(z)}{\gamma}\right)>0. \tag{60}$$

and

$$\big(f'(z)\big)^{\alpha}\in\mathcal{H}[q(0),1]\cap\Sigma, \tag{61}$$

and

$$\big(f'(z)\big)^{\alpha}\left(1+\gamma\alpha\left(\frac{zf''}{f'(z)}\right)\right), \tag{62}$$

is univalent in $\mathbb{U}$. *If*

$$q(z) + \gamma z q'(z) \prec \big(f'(z)\big)^{\alpha}\left(1 + \gamma\alpha\left(\frac{zf''}{f'(z)}\right)\right), \tag{63}$$

then

$$q(z) \prec \big(f'(z)\big)^{\alpha} \tag{64}$$

and q *is the best subordinant.*

For $\beta = 0, \varepsilon = 0$, $\kappa = 1$ and $g(z) = z$, we get the following corollary.

Corollary 4.3. *Let* q *be convex univalent in* $\mathbb{U}$, $\gamma(\neq 0), \varepsilon, \kappa \in \mathbb{C}$ *and* $0 \leq \beta \leq 1$. *Suppose* q *satisfies*

$$\mathfrak{Re}\left(\frac{q'(z)}{\gamma}\right) > 0. \tag{65}$$

and

$$\left(\frac{f(z)}{z}\right)^{\alpha} \in \mathcal{H}[q(0), 1] \cap \Sigma, \tag{66}$$

and

$$(1 - \gamma\alpha)\left(\frac{f(z)}{z}\right)^{\alpha} + \gamma\alpha\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}, \tag{67}$$

is univalent in $\mathbb{U}$. *If*

$$q(z) + \gamma z q'(z) \prec (1 - \gamma\alpha)\left(\frac{f(z)}{z}\right)^{\alpha} + \gamma\alpha\frac{zf'(z)}{f(z)}\left(\frac{f(z)}{z}\right)^{\alpha}, \tag{68}$$

then

$$q(z) \prec \left(\frac{f(z)}{z}\right)^{\alpha} \tag{69}$$

and q *is the best subordinant.*

5 Sandwich results

Combining the results of differential subordination and superordination, we state the following sandwich results.

Theorem 5.1. *Let* q_1 *be univalent and let* q_2 *be convex univalent in* $\mathbb{U}$, $0 < \alpha < 1$ *and* $\gamma \in \mathbb{C}$ *with* $\mathfrak{Re}(\gamma) > 0$. *Suppose* q_2 *satisfies (14). If* $\left(\frac{g(z)}{f(z)}\right)^{\alpha} \in \mathcal{H}[q_1(0), 1] \cap \Sigma$, $\left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha}$ *is univalent in* $\mathbb{U}$, *and*

$$q_1(z) + \frac{\gamma}{\alpha} z q_1'(z) \prec \left(1 + \gamma\frac{zg'(z)}{g(z)}\right)\left(\frac{g(z)}{f(z)}\right)^{\alpha} - \gamma\frac{zf'(z)}{f(z)}\left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec q_2(z) + \frac{\gamma}{\alpha} z q_2'(z), \tag{70}$$

then

$$q_1(z) \prec \left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec q_2(z) \tag{71}$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

For $\gamma = 1$ and $g(z) = z$, we get the following corollary.

Corollary 5.1. *Let q_1 be univalent and let q_2 be convex univalent in $\mathbb{U}$, $0 < \alpha < 1$. Suppose q_1 satisfies (4.1) and q_2 satisfies (15). If $\left(\frac{g(z)}{f(z)}\right)^{\alpha} \in \mathcal{H}[q_1(0), 1] \cap \Sigma$, $\left(2 - \frac{zf'(z)}{f(z)}\right)\left(\frac{z}{f(z)}\right)^{\alpha}$ is univalent in $\mathbb{U}$, and*

$$q_1(z) + \frac{1}{\alpha} z q_1'(z) \prec \left(2 - \frac{zf'(z)}{f(z)}\right)\left(\frac{z}{f(z)}\right)^{\alpha} \prec q_2(z) + \frac{1}{\alpha} z q_2'(z), \tag{72}$$

then

$$q_1(z) \prec \left(\frac{z}{f(z)}\right)^{\alpha} \prec q_2(z) \tag{73}$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

Theorem 5.2. *Let q_1 be convex univalent and let q_2 be convex univalent in $\mathbb{U}$, $\gamma(\neq 0), \varepsilon, \kappa \in \mathbb{C}$, $0 \le \beta \le 1$, and q_1 satisfies (50), q_2 satisfies (27). Suppose*

$$\left(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\right)^{\alpha} \in \mathcal{H}[q(0), 1] \cap \Sigma. \tag{74}$$

Let

$$\left(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\right)^{\alpha} \left(\kappa + \gamma\alpha\left(\frac{zf'(z) + \beta z^2 f''}{(1-\beta)f(z) + \beta z f'(z)} - \frac{zg'(z)}{g(z)}\right)\right) + \varepsilon, \tag{75}$$

is univalent in $\mathbb{U}$. If

$$\begin{aligned} & \kappa q_1(z) + \varepsilon + \gamma z q_1'(z) \\ \prec\ & \left(\tfrac{(1-\beta)f(z)+\beta z f'(z)}{g(z)}\right)^{\alpha} \left(\kappa + \gamma\alpha\left(\tfrac{zf'(z)+\beta z^2 f''}{(1-\beta)f(z)+\beta z f'(z)} - \tfrac{zg'(z)}{g(z)}\right)\right) + \varepsilon \\ \prec\ & \kappa q_2(z) + \varepsilon + \gamma z q_2'(z) \end{aligned} \tag{76}$$

then

$$q_1(z) \prec \left(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\right)^{\alpha} \prec q_2(z) \tag{77}$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

For $\beta = 1, \varepsilon = 0$, $\kappa = 1$ and $g(z) = z$, we get the following corollary.

Corollary 5.2. *Let q_1 be convex univalent and let q_2 be convex univalent in $\mathbb{U}$, $\gamma(\neq 0) \in \mathbb{C}$, $0 \leq \beta \leq 1$, and q_1 satisfies (50), q_2 satisfies (27). Suppose*

$$\big(f'(z)\big)^{\alpha} \in \mathcal{H}[q(0), 1] \cap \Sigma. \tag{78}$$

Let

$$\big(f'(z)\big)^{\alpha}\bigg(1 + \gamma\alpha\bigg(\frac{zf''}{f'(z)}\bigg)\bigg), \tag{79}$$

is univalent in $\mathbb{U}$. If

$$\kappa q_1(z) + \varepsilon + \gamma z q_1'(z) \prec \big(f'(z)\big)^{\alpha}\bigg(1 + \gamma\alpha\bigg(\frac{zf''}{f'(z)}\bigg)\bigg) \prec \kappa q_2(z) + \varepsilon + \gamma z q_2'(z) \tag{80}$$

then

$$q_1(z) \prec \big(f'(z)\big)^{\alpha} \prec q_2(z) \tag{81}$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

For $\beta = 0, \varepsilon = 0$, $\kappa = 1$ and $g(z) = z$, we get the following corollary.

Corollary 5.3. *Let q_1 be convex univalent and let q_2 be convex univalent in $\mathbb{U}$, $\gamma(\neq 0) \in \mathbb{C}$ and q_1 satisfies (50), q_2 satisfies (27). Suppose*

$$(\frac{f(z)}{z}\bigg)^{\alpha} \in \mathcal{H}[q(0), 1] \cap \Sigma. \tag{82}$$

Let

$$(1 - \gamma\alpha)\bigg(\frac{f(z)}{z}\bigg)^{\alpha} + \gamma\alpha\frac{zf'(z)}{f(z)}\bigg(\frac{f(z)}{z}\bigg)^{\alpha}, \tag{83}$$

is univalent in $\mathbb{U}$. If

$$q_1(z) + \gamma z q_1'(z) \prec \quad (1 - \gamma\alpha)\bigg(\frac{f(z)}{z}\bigg)^{\alpha} + \gamma\alpha\frac{zf'(z)}{f(z)}\bigg(\frac{f(z)}{z}\bigg)^{\alpha} \prec q_2(z) + \gamma z q_2'(z) \tag{84}$$

then

$$q_1(z) \prec \bigg(\frac{f(z)}{z}\bigg)^{\alpha} \prec q_2(z) \tag{85}$$

and $q_1(z)$ and $q_2(z)$ are, respectively, the best subordinant and the best dominant.

6 Open Problem

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, and $\mathcal{H}[a,p]$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots. \tag{86}$$

Let $\mathcal{A}(p)$ be the subclass of $\mathcal{H}$ consisting of functions of the form

$$f(z) = z^p + a_{p+1} z^{p+1} + a_{p+2} z^{p+2} + \cdots. \tag{87}$$

A function $f \in \mathcal{A}(p)$ is said to be in the class $\mathcal{S}^*(p)$ of p-valent starlike functions in $\mathbb{U}$, if it satisfies the inequality $\mathrm{Re}\big(\frac{zf'(z)}{pf(z)}\big) > 0,\ z \in \mathbb{U}$.

Let $f(z) \in \mathcal{A}(p)$ and $g(z \in \mathcal{S}^*(p)$. We can consider sufficient conditions on h, q_1, q_2 and ϕ for which the following implication holds:

$$q_1(z) \prec \left(\frac{g(z)}{f(z)}\right)^{\alpha} \prec q_2(z), \tag{88}$$

or

$$q_1(z) \prec \left(\frac{(1-\beta)f(z) + \beta z f'(z)}{g(z)}\right)^{\alpha} \prec q_2(z), \tag{89}$$

where $0 < \alpha < 1$ and $0 \le \beta \le 1$.

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