Int. J. Open Problems Comput. Math. , Vol. 6, No. 1, March, 2013
ISSN 1998-6262;
www.i-csrs.org

Some New Sequence Spaces on Seminormed Spaces

R. Çolak, Ç. A. Bektaş and Y. Altın

Department of Mathematics,
Firat University, Elazig, 23119, TURKEY.
e-mail: rcolak@firat.edu.tr

Abstract

In this paper, we define the sequence spaces $[V, \lambda, M, p, s]_1 (\phi, q)$, $[V, \lambda, M, p, s]_0 (\phi, q)$ and $[V, \lambda, M, p, s]_\infty (\phi, q)$ and introduce some properties of these spaces.

Keywords : *Difference sequence spaces, Orlicz function, seminorm.*
2000 AMS: *40A05, 40C05, 46A45.*

1 Introduction

Let ℓ_∞, c and c_0 be the linear spaces of bounded, convergent and null sequences $x = (x_k)$ with complex terms, respectively, normed by $\|x\|_\infty = \sup_k |x_k|$.

Throughout the paper w denotes the set of all sequences of complex numbers.

Let σ be a one to one mapping of the set $\mathbb{N}$ of positive integers into itself such that $\sigma^k(n) = \sigma(\sigma^{k-1}(n))$, $k = 1, 2, ...$ A continuous linear functional φ on ℓ_∞ is said to be an invariant mean or a σ- mean if and only if

(i) $\varphi(x) \geq 0$ when the sequence $x = (x_n)$ has $x_n \geq 0$ for all n,

(ii) $\varphi(e) = 1$, where $e = (1, 1, 1, ...)$,

(iii) $\varphi\left(\left\{x_{\sigma(n)}\right\}\right) = \varphi\left(\{x_n\}\right)$ for all $x \in \ell_\infty$.

If $x = (x_n)$, set $Tx = (Tx_n) = \left(x_{\sigma(n)}\right)$. It can be shown that (see Schaefer [?]).

$$V\sigma = \{x = (x_n) : \lim_k t_{kn}(x) = Le \text{ uniformly in } n,\ L = \sigma - \lim x\} \qquad (1)$$

where

$$t_{kn}(x) = \frac{1}{k+1}\sum_{j=0}^{k} T^j x_n.$$

The special case of (1), in which $\sigma(n) = n+1$ was given by Lorentz [?].

Subsequently invariant means have been studied by Ahmad and Mursaleen [?], Mursaleen [?, ?], Raimi [?] and many others. The space

$$BV\sigma = \{x \in \ell_\infty :_k |\phi_{kn}(x)| < \infty, \text{ uniformly in } n\}$$

was defined by Mursaleen [?], where

$$\phi_{kn}(x) = t_{kn}(x) - t_{k-1,n}(x)$$

assuming that $t_{kn}(x) = 0$, for $k = -1$.

Note that for any sequences x, y and scalar λ, we have $\phi_{kn}(x+y) = \phi_{kn}(x) + \phi_{kn}(y)$ and $\phi_{kn}(\lambda x) = \lambda\phi_{kn}(x)$.

An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$.

It is well known that if M is a convex function and $M(0) = 0$, then $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 \leq \lambda \leq 1$.

An Orlicz function M is said to satisfy Δ_2-condition for all values of y, if there exists a constant $K > 0$, such that $M(2y) \leq KM(y)$ $(y > 0)$.

Lindenstrauss and Tzafriri [?] used the idea of Orlicz function to construct the sequence space

$$\ell_M = \{x \in w : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) < \infty \text{ for some } \rho > 0\}.$$

The space ℓ_M is a Banach spaces with the norm

$$\|x\| = \inf\{\rho > 0 : \sum_{k=1}^{\infty} M(\frac{|x_k|}{\rho}) \leq 1\}$$

and this space is called an Orlicz sequence space. For $M(t) = t^p$, $1 \leq p < \infty$, the space ℓ_M coincides with the classical sequence space ℓ_p.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that

$$\lambda_{n+1} \leq \lambda_n + 1, \ \lambda_1 = 1$$

$\lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$.

Let $x \in w$ and $X, Y \subset w$. Then we shall write

$$M(X,Y) =_{x\in X} x^{-1} * Y = \{a \in w : ax \in Y \text{ for all } x \in X\}.$$

The set $X^{\alpha} = M(X, \ell_1)$ is called Köthe-Toeplitz dual space or α-dual of X. Let X be a sequence space. Then X is called:

(i) Solid (or normal) if $(\alpha_k x_k) \in X$ whenever $(x_k) \in X$, for all sequences (α_k) of scalars with $|\alpha_k| \leq 1$.

(ii) Monotone provided X contains the canonical preimages of all its stepspaces.

(iii) Perfect if $X = X^{\alpha\alpha}$.

It is well known that X is perfect $\Rightarrow X$ is normal $\Rightarrow X$ is monotone.

Let X be a complex linear space with zero element θ and $X = (X, q)$ be a seminormed space with the seminorm q. By $S(X)$ we denote the linear space of all sequences $x = (x_k)$ with $(x_k) \in X$ and the usual coordinatewise operations: $\alpha x = (\alpha x_k)$ and $x + y = (x_k + y_k)$, for each $\alpha \in \mathbb{C}$ where $\mathbb{C}$ denotes the set of complex numbers.

The following inequality will be used throughout the paper. Let $p = (p_k)$ be a bounded sequence of strictly positive real numbers with $0 < p_k \leq \sup_k p_k = G$, $D = \max\left(1, 2^{G-1}\right)$, then

$$|a_k + b_k|^{p_k} \leq D\left\{|a_k|^{p_k} + |b_k|^{p_k}\right\} \tag{2}$$

where $a_k, b_k \in \mathbb{C}$. Also for any complex λ,

$$|\lambda|^{p_k} \leq \max\left(1, |\lambda|^G\right). \tag{3}$$

2 Main results

In this section we will define the sequence spaces $[V, \lambda, M, p, s]_1(\phi, q)$, $[V, \lambda, M, p, s]_0(\phi, q)$ and $[V, \lambda, M, p, s]_\infty(\phi, q)$.

Definition 2.1 *Let M be an Orlicz function, X be a seminormed space with seminorm q and $s \geq 0$ a real number. Then we define*

$$[V, \lambda, M, p, s]_1(\phi, q) = \left\{ \begin{array}{c} x \in S(X) : \lim\limits_{n\to\infty} \frac{1}{\lambda_n} \sum\limits_{k \in I_n} k^{-s} \left[M\left(q\left(\frac{\phi_{kn}(x) - L}{\rho}\right)\right)\right]^{p_k} = 0 \\ \text{for some } L \text{ and } \rho > 0 \end{array} \right\},$$

$$[V, \lambda, M, p, s]_0(\phi, q) = \left\{ \begin{array}{c} x \in S(X) : \lim\limits_{n\to\infty} \frac{1}{\lambda_n} \sum\limits_{k \in I_n} k^{-s} \left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)\right]^{p_k} = 0 \\ \text{for some } \rho > 0 \end{array} \right\},$$

$$[V, \lambda, M, p, s]_\infty(\phi, q) = \left\{ \begin{array}{c} x \in S(X) : \sup\limits_{n} \frac{1}{\lambda_n} \sum\limits_{k \in I_n} k^{-s} \left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)\right]^{p_k} < \infty \\ \text{for some } \rho > 0 \end{array} \right\}.$$

Throughout the paper Z will denote any one of the notation $0, 1$ or ∞.

If we take $M(x) = x$, then we write $[V, \lambda, p, s]_Z(\phi, q)$ instead of $[V, \lambda, M, p, s]_Z(\phi, q)$.

If we take $s = 0$, then $[V, \lambda, M, p, s]_Z(\phi, q) = [V, \lambda, M, p]_Z(\phi, q)$.

Theorem 2.2 *For any Orlicz function M, $[V, \lambda, M, p, s]_Z(\phi, q)$ are linear space over the complex field $\mathbb{C}$.*

The proof is a routine verification by using standard techniques and hence is omitted.

Theorem 2.3 *For any Orlicz function M and a bounded sequence $p = (p_k)$ of strictly positive real numbers, $[V, \lambda, M, p, s]_0(\phi, q)$ is a paranormed space (not necessarily total paranormed) with*

$$g(x) = \inf\left\{\rho^{\frac{p_n}{H}} : \sup_k \left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)\right] \leq 1,\ \rho > 0,\ n \in \mathbb{N}\right\}$$

where $H = \max(1, \sup_k p_k)$.

Proof. Clearly $g(x) = g(-x)$. Since $M(0) = 0$, we get $\inf\left\{\rho^{\frac{p_n}{H}}\right\} = 0$ for $x = \theta$. Now let $x, y \in [V, \lambda, M, p, s]_0(\phi, q)$ and let us choose $\rho_1 > 0$ and $\rho_2 > 0$ such that

$$\sup_k \left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho_1}\right)\right)\right] \leq 1$$

and

$$\sup_k \left[M\left(q\left(\frac{\phi_{kn}(y)}{\rho_2}\right)\right)\right] \leq 1.$$

Let $\rho = \rho_1 + \rho_2$. Then we get the triangle inequality from the following inequality and since ϕ_{kn} is linear

$$\sup_k \left[M\left(q\left(\frac{\phi_{kn}(x+y)}{\rho}\right)\right)\right] \leq \frac{\rho_1}{\rho}\sup_k \left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho_1}\right)\right)\right] + \frac{\rho_2}{\rho}\sup_k \left[M\left(q\left(\frac{\phi_{kn}(y)}{\rho_2}\right)\right)\right] \leq 1.$$

Finally let λ be a given non-zero scalar, then the continuity of the scalar multiplication follows from the following equality

$$\begin{aligned} g(\lambda x) &= \inf\left\{\rho^{\frac{p_n}{H}} : \sup_k \left[M\left(q\left(\frac{\phi_{kn}(\lambda x)}{\rho}\right)\right)\right] \leq 1,\ n \in \mathbb{N}\right\} \\ &= \inf\left\{(|\lambda| .s)^{\frac{p_n}{H}} : \sup_k \left[M\left(q\left(\frac{\phi_{kn}(x)}{s}\right)\right)\right] \leq 1,\ n \in \mathbb{N}\right\}, \end{aligned}$$

where $s = \frac{\rho}{|\lambda|}$. This completes the proof.

Theorem 2.4 *Let M, M_1, M_2 be Orlicz functions, then*

(i) *If there is a positive constant β such that $M(t) \leq \beta.t$ for all $t \geq 0$, then $[V, \lambda, M_1, p, s]_Z(\phi, q) \subseteq [V, \lambda, M \circ M_1, p, s]_Z(\phi, q)$,*

(ii) *$[V, \lambda, M_1, p, s]_Z(\phi, q) \cap [V, \lambda, M_2, p, s]_Z(\phi, q) \subseteq [V, \lambda, M_1 + M_2, p, s]_Z(\phi, q)$ where $Z = 0$ or 1 or ∞.*

Proof. **(i)** We give the proof for $Z = 0$. Let $x \in [V, \lambda, M_1, p, s]_0(\phi, q)$ so that

$$\frac{1}{\lambda_n}\sum_{k \in I_n} k^{-s}\left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)\right]^{p_k} \to 0$$

for some $\rho > 0$ and $n \to \infty$. Since $M(t) \leq \beta.t$ for all $t \geq 0$, we have by inequality (3).

$$\frac{1}{\lambda_n}\sum_{k \in I_n} k^{-s}\left[M(u_k)\right]^{p_k} \leq \max\left(1, \beta^G\right)\frac{1}{\lambda_n}\sum_{k \in I_n} k^{-s}\left[u_k\right]^{p_k}$$

where $u_k = M_1\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)$ and hence $x \in [V, \lambda, M \circ M_1, p, s]_0(\phi, q)$.

(ii) The proof is immediate using (2).

Theorem 2.5 *For any Orlicz function M, if $\lim\limits_{u \to \infty} \frac{M(u/\rho)}{(u/\rho)} > 0$ for some $\rho > 0$, then $[V, \lambda, M, p, s]_Z(\phi, q) \subseteq [V, \lambda, p, s]_Z(\phi, q)$.*

Proof. If $\lim\limits_{u \to \infty} \frac{M(u/\rho)}{(u/\rho)} > 0$ then there exists a number $\alpha > 0$ such that $M(u/\rho) \geq \alpha(u/\rho)$ for all $u > 0$ and some $\rho > 0$. Let $x \in [V, \lambda, M, p, s]_\infty(\phi, q)$ so that

$$\frac{1}{\lambda_n}\sum_{k \in I_n} k^{-s}\left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)\right]^{p_k} < \infty, \forall n \in \mathbb{N}.$$

Then

$$\frac{1}{\lambda_n}\sum_{k \in I_n} k^{-s}\left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)\right]^{p_k} \geq \max\left(1, (\frac{\alpha}{\rho})^G\right).\frac{1}{\lambda_n}\sum_{k \in I_n} k^{-s}\left[q\left(\phi_{kn}(x)\right)\right]^{p_k}.$$

Hence $x \in [V, \lambda, p, s]_\infty(\phi, q)$.

The other cases can be proved similarly.

Theorem 2.6 *Let M be an Orlicz function which satisfies Δ_2-condition, q, q_1, q_2 be seminorms and s, s_1, s_2 be non-negative real numbers. Then*

(i) *$[V, \lambda, M, p, s]_Z(\phi, q_1) \cap [V, \lambda, M, p, s]_Z(\phi, q_2) \subseteq [V, \lambda, M, p, s]_Z(\phi, q_1 + q_2)$,*

(ii) *If there exists a constant $L > 1$ such that $q_2(x) \leq Lq_1(x)$ for all $x \in X$, then $[V, \lambda, M, p, s]_Z(\phi, q_1) \subseteq [V, \lambda, M, p, s]_Z(\phi, q_2)$,*

(iii) *If $s_1 \leq s_2$, then $[V, \lambda, M, p, s_1]_Z(\phi, q) \subseteq [V, \lambda, M, p, s_2]_Z(\phi, q)$,*

(iv) *$[V, \lambda, M, p]_Z(\phi, q) \subseteq [V, \lambda, M, p, s]_Z(\phi, q)$.*

Proof is easy and thus omitted.

Theorem 2.7 *Suppose that* $0 < p_k \le t_k < \infty$ *for each* $k \in \mathbb{N}$. *Then* $[V, \lambda, M, p, s]_Z(\phi, q) \subseteq [V, \lambda, M, t, s]_Z(\phi, q)$ *where* $Z = 0$ *or* 1 *or* ∞.

Proof is easy.

Theorem 2.8 *The sequence spaces* $[V, \lambda, M, p, s]_0(\phi, q)$ *and* $[V, \lambda, M, p, s]_\infty(\phi, q)$ *are solid.*

Proof. We give the proof for $[V, \lambda, M, p, s]_0(\phi, q)$. Let $x \in [V, \lambda, M, p, s]_0(\phi, q)$ and (α_n) be any sequence of scalars such that $|\alpha_n| \le 1$ for $n \in \mathbb{N}$. Then we have

$$\begin{aligned} \lambda_n^{-1}{}_{k\in I_n} k^{-s}\left[M\left(q\left(\frac{\phi_{kn}(\alpha_n x)}{\rho}\right)\right)\right]^{p_k} &= \lambda_n^{-1}{}_{k\in I_n} k^{-s}\left[M\left(q\left(\frac{\alpha_n \phi_{kn}(x)}{\rho}\right)\right)\right]^{p_k} \\ &\le \lambda_n^{-1}{}_{k\in I_n} k^{-s}\left[M\left(q\left(\frac{\phi_{kn}(x)}{\rho}\right)\right)\right]^{p_k} \end{aligned}$$

and the right hand side tends to 0 as $n \to \infty$. Hence $(\alpha_n x_n) \in [V, \lambda, M, p, s]_0(\phi, q)$.

Corollary 2.9 *The sequence spaces* $[V, \lambda, M, p, s]_0(\phi, q)$ *and* $[V, \lambda, M, p, s]_\infty(\phi, q)$ *are monotone.*

3 Open Problem

The aim of this paper is to introduce and study the new sequence spaces $[V, \lambda, M, p, s]_1(\phi, q)$, $[V, \lambda, M, p, s]_0(\phi, q)$ and $[V, \lambda, M, p, s]_\infty(\phi, q)$. We also examine some topological properties and establish some inclusion relations between these spaces.

Is the sequence space $[V, \lambda, M, p, s]_1(\phi, q)$ solid or monotone? Therefore it is left as an open problem.

References

[1] Z. U. Ahmad and M. Mursaleen, *An application of Banach limits*, Proc. Amer. Math. Soc., 103, (1988), 244-246.

[2] S. Banach, *Theorie des Operations Linearies*, Warszawa, 1932.

[3] B. Choudhary and S. D. Parashar, *A sequence space defined by Orlicz function*, Approx Theory Appl. 18(4), (2002), 70-75.

[4] P. K. Kamthan and M. Gupta, *Sequence spaces and series*, Lecture Notes in Pure and Applied Mathematics, 65, Marcel Dekker Inc., New York, 1981.

[5] M. A. Krasnoselskii and Y. B. Rutickii, *Convex Function and Orlicz spaces*, Groningen, Netherlands, 1961.

[6] J. Lindenstrauss and L. Tzafriri, *On Orlicz sequence spaces*, Israel J. Math., 10 (1971), 379-390.

[7] G. G. Lorentz, *A contribution the theory of divergent series*, Acta Mat., 80(1948), 167-190.

[8] M. Mursaleen, *On some new invariant matrix methods of summability*, Quart. J. Math., Oxford 34(2) (1983), 77-86.

[9] M. Mursaleen, *Matrix transformations between some new sequence spaces*, Houston J. Math., 9(1983), 505-509.

[10] R. A. Raimi, *Invariant means and invariant matrix method of summability*, Duke Math. J., (1963), 81-94.

[11] P. Schaefer, *Infinite matrices and invariant means*, Proc. Amer. Math. Soc., 36(1972), 104-110.

[12] V. Karakaya and H. Dutta, *On some vector valued generalized difference modular sequence spaces*, Filomat 25(3) (2011), 15-27.